**Laboratoire d'Arithmétique,
Calcul formel et d'Optimisation**

UMR CNRS 6090

Discretization of a Nonlinear Oscillator under Dry Friction via an Inertial Proximal Method

Alexandre Cabot

Rapport de recherche n° 2004-11
Déposé le 15 décembre 2004

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex

Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22 - laco@unilim.fr
http://www.unilim.fr/laco/

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex

Tél. (33) 5 55 45 73 23 - Fax. (33) 5 55 45 73 22 - laco@unilim.fr
http://www.unilim.fr/laco/

DISCRETIZATION OF A NONLINEAR OSCILLATOR UNDER DRY FRICTION VIA AN INERTIAL PROXIMAL METHOD

ALEXANDRE CABOT

Abstract. Let H a Hilbert space and A, $B : H \rightrightarrows H$ two maximal monotone operators. In this paper, we investigate the properties of the following proximal type algorithm:

$$(\mathcal{A}) \qquad (x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) + B(x_{n+2}) \ni 0,$$

where (λ_n) is a sequence of positive steps. Algorithm $(\mathcal{A})$ may be viewed as the discretized equation of a nonlinear oscillator subject to friction (represented by the operator A) and to external forces (described by B). We prove that, if $0 \in \operatorname{int}(A(0))$ (condition of dry friction), then the sequence (x_n) generated by $(\mathcal{A})$ is strongly convergent and its limit x_∞ satisfies $0 \in A(0) + B(x_\infty)$. Moreover, the limit x_∞ is achieved in a finite number of iterations as soon as $0 \in \operatorname{int}(A(0) + B(x_\infty))$.

1. Introduction

In a recent paper, Adly-Attouch-Cabot [2] have studied the dynamics of a nonlinear oscillator subject to dry friction via the following differential inclusion:

$$(1) \qquad \ddot{x}(t) + \partial\Phi(\dot{x}(t)) + \nabla f(x(t)) \ni 0, \qquad t \geq 0, \quad x(t) \in \mathbb{R}^n$$

(∂ denotes the subdifferential operator in the sense of convex analysis and ∇ is the classical gradient operator). Equation (1) describes the finite dimensional motion of a rigid body (of unit mass) sliding on a surface and subject to external forces deriving from the potential f. The term $-\partial\Phi(\dot{x})$ represents the dry frictional contact of the mass on the surface. When the friction is dry, *i.e.* $0 \in \operatorname{int}\partial\Phi(0)$ the convergence of the trajectories automatically holds. It is shown in [2] that the limit is achieved in a finite time. Such a limit x_∞ is a solution of the inclusion: $-\nabla f(x_\infty) \in \partial\Phi(0)$, *i.e.* x_∞ is a stationary solution of (1).
For numerical purposes, it is natural to deal with a discretized version of (1). In this paper, we will be especially interested in the following implicit discretization of (1):

$$(x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + \partial\Phi\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) + \nabla f(x_{n+2}) \ni 0,$$

where (λ_n) is a sequence of positive steps. The previous algorithm falls into the framework of proximal-like methods. In the whole paper, we extend the study to the general setting of maximal monotone operators in infinite dimensional Hilbert spaces. For that purpose, we consider a Hilbert space H and two maximal monotone

1991 *Mathematics Subject Classification.* 65K10, 49M25, 70F40.
Key words and phrases. Proximal point algorithm, maximal monotone operator, dry friction, finite convergence.

operators A, $B : H \rightrightarrows H$. We define the sequence (x_n) by the following recursive algorithm:

$$(\mathcal{A}) \qquad (x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) + B(x_{n+2}) \ni 0.$$

When $A(x) = \gamma\, x$ for some positive γ (which corresponds to a linear damping), the algorithm $(\mathcal{A})$ can be rewritten as:

$$(2) \qquad x_{n+2} - x_{n+1} - \alpha_n\,(x_{n+1} - x_n) + \beta_n\, B(x_{n+2}) \ni 0,$$

where the coefficients α_n and β_n are respectively defined by $\alpha_n := 1/(1 + \gamma\,\lambda_n)$ and $\beta_n := \lambda_n^2/(1 + \gamma\,\lambda_n)$. When $\alpha_n \equiv 0$, we recover the standard proximal point algorithm, for which we refer the reader to the abundant literature on this subject [11, 14, 15, 16]. On the other hand, when $\alpha_n > 0$ for some $n \in \mathbb{N}$, the extrapolation term $\alpha_n\,(x_{n+1} - x_n)$ takes into account a kind of inertia associated with the sequence. This situation has been analysed in details in [4] (and also [3] when B is a subdifferential operator), where the algorithm (2) is called the "inertial proximal point" method. The authors prove that, under adequate conditions, the sequence (x_n) defined by (2) weakly converges to some zero of B. This asymptotic behaviour is strongly related to the fact that $A(0) = \{0\}$.

In this paper, we focus on the situation corresponding to $0 \in \mathrm{int}\,(A(0))$, which can be interpreted as a condition of dry friction. The effect of such an assumption is to force the strong convergence of the sequence (x_n). The counterpart lies in the fact that the limit $x_\infty := s - \lim_{n\to+\infty} x_n$ is not (in general) a zero of B. The point x_∞ satisfies the inclusion $0 \in A(0) + B(x_\infty)$, *i.e.* is an equilibrium point of $(\mathcal{A})$. The main result of the paper consists in showing that the limit is achieved in a finite number of iterations as soon as the condition $0 \in \mathrm{int}\,(A(0) + B(x_\infty))$ is fulfilled (see Theorem 4.1). Since the boundary of the convex set $A(0) + B(x_\infty)$ has an empty interior, it is clear that the previous condition is not stringent. We may conjecture that, generically with respect to the initial data $(x_0, x_1) \in H^2$, the limit x_∞ is achieved in a finite number of steps. The question of finite convergence for proximal methods has been widely studied in the literature (see for example [10, 11, 12, 13]). It is interesting to note that we need no assumption of linear conditioning as it is often required in this type of results.

The paper is organized as follows. In Section 2, we start with a general result of existence and uniqueness of the iterates generated by $(\mathcal{A})$. Section 3 is devoted to the asymptotic properties of $(\mathcal{A})$ in case of convergence. These results are aimed at preparing the next section, where effective convergence results are proved. In Section 4 we assume that the operator B is the subdifferential of a convex function and that $0 \in \mathrm{int}\,(A(0))$ (condition of dry friction). We then arrive to the main result of the paper (Theorem 4.1), which states the strong convergence of the algorithm $(\mathcal{A})$ and gives a sufficient condition ensuring the finite convergence.

2. Algorithm $(\mathcal{A})$. Preliminary results

In the whole paper, H is a Hilbert space endowed with scalar product $\langle\,,\,\rangle$ and corresponding norm $|\,.\,|$. We consider two maximal monotone operators $A : H \rightrightarrows H$ and $B : H \rightrightarrows H$. Given initial data x_0, $x_1 \in H$ and a sequence (λ_n) of positive steps, let us define the sequence (x_n) by the following algorithm:

$$(\mathcal{A}) \qquad (x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) + B(x_{n+2}) \ni 0.$$

We start with a general result of existence and uniqueness of the iterates generated by $(\mathcal{A})$. Let us recall that, for every maximal monotone operator $C : H \rightrightarrows H$, the resolvent function $J_\lambda^C := (I + \lambda\, C)^{-1}$ is single-valued on H. For a complete account of the theory of maximal monotone operators in Hilbert spaces, we refer the reader to [6, 17].

Proposition 2.1. *Assume that $A : H \rightrightarrows H$ and $B : H \rightrightarrows H$ are maximal monotone operators satisfying respectively dom $A = H$ and* $\operatorname{int}(\mathit{dom}\ B) \neq \emptyset$. *Given x_0, $x_1 \in H$ and a sequence (λ_n) of positive numbers, the sequence (x_n) is uniquely defined by the algorithm $(\mathcal{A})$ and the initial data x_0, x_1.*

Proof. Let us define the maximal monotone operator $B_n : H \rightrightarrows H$ by $B_n(x) := B(\lambda_n\, x + x_{n+1})$. The algorithm $(\mathcal{A})$ can be rewritten as

$$\frac{1}{\lambda_n}\,(x_{n+2} - x_{n+1}) - \frac{1}{\lambda_n}\,(x_{n+1} - x_n) + \lambda_n\,(A + B_n)\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) \ni 0,$$

or equivalently:

$$\Big(I + \lambda_n\,(A + B_n)\Big)\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) \ni (x_{n+1} - x_n)/\lambda_n.$$

Since the sets $\operatorname{int}(\operatorname{dom}\ A) = H$ and $\operatorname{int}(\operatorname{dom}\ B_n)$ have a nonempty intersection, it is classical that the operator $A + B_n$ is maximal monotone. As a consequence, the previous inclusion is equivalent to the following equality:

$$(x_{n+2} - x_{n+1})/\lambda_n = J_{\lambda_n}^{A+B_n}\Big((x_{n+1} - x_n)/\lambda_n\Big), \tag{3}$$

which clearly shows that x_{n+2} is uniquely defined as a function of x_n and x_{n+1}. □

A classical example of monotone operators is given by the subdifferential of convex functions (*cf.* [6, 17]). Consider a closed convex function $f : H \to \mathbb{R}$, so that the subdifferential ∂f is maximal monotone and dom ∂f = dom $f = H$. If one takes $A := \partial f$ in algorithm $(\mathcal{A})$, the term $\partial f(x_{n+2} - x_{n+1})$ may have a mechanical interpretation in terms of friction. The case $f = |\,.\,|$ corresponds to the Coulomb friction whereas $f = |\,.\,|^2$ is associated to a viscous friction. Notice that the intermediate situation $f = |\,.\,|^\alpha$ with $\alpha \in]1, 2[$ has been studied by Amann-Díaz [5] and Díaz-Liñán [8] in the context of continuous dynamical systems.
More generally, given a convex function $\theta : \mathbb{R}_+ \to \mathbb{R}$, the following proposition shows that the operator $A := \partial[\theta(|\,.\,|)]$ is singlevalued on $H \setminus \{0\}$ and that the set $A(0)$ is determined by the value $\theta'(0)$.

Proposition 2.2. *Consider a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ of class $\mathcal{C}^1$ which is convex, non decreasing and define the operator $A := \partial[\theta(|\,.\,|)]$. Then the operator A is maximal monotone and satisfies dom $A = H$. Moreover, the expression of A is given by:*

$$A(x) = \begin{cases} \frac{\theta'(|x|)}{|x|}\, x & \text{if} \quad x \neq 0, \\ \mathbb{B}(0, \theta'(0)) & \text{if} \quad x = 0. \end{cases}$$

Proof. Let us define the function $f := \theta(|\,.\,|)$. The function f is convex as the composition of the convex function $|\,.\,|$ with the convex non decreasing function θ. Since f is also continuous on H, we deduce that ∂f is a maximal monotone operator with dom $\partial f = H$. The function f is of class $\mathcal{C}^1$ on $H \setminus \{0\}$ as a composition of $\mathcal{C}^1$ functions and an immediate computation gives, for every $x \neq 0$

$$\nabla f(x) = \frac{\theta'(|x|)}{|x|}\, x. \tag{4}$$

Let us determine the set $\partial f(0)$; from its definition, we have

$$\begin{aligned} \xi \in \partial f(0) \quad &\Longleftrightarrow \quad \forall x \in H, \quad \theta(|x|) \geq \theta(0) + \langle \xi, x\rangle \\ &\Longleftrightarrow \quad \forall x \in H \setminus \{0\}, \quad (\theta(|x|) - \theta(0))/|x| \geq \langle \xi, x/|x|\rangle. \end{aligned} \tag{5}$$

First assume that inequality (5) is satisfied. Taking the upper limit when $x \to 0$ in both members, we find $\theta'(0) \geq |\xi|$, *i.e.* $\xi \in \mathbb{B}(0, \theta'(0))$. Conversely assume that $|\xi| \leq \theta'(0)$. From the Cauchy-Schwarz inequality, we have $\langle \xi, x/|x|\rangle \leq |\xi| \leq \theta'(0)$. On the other hand, the convexity of θ gives $\theta'(0) \leq (\theta(|x|) - \theta(0))/|x|$, which combined with the previous inequality finally yields (5). □

3. Properties of the limit in case of convergence

In the whole section, we keep the general setting introduced in Section 2: H is a Hilbert space and A, $B : H \rightrightarrows H$ are general maximal monotone operators. We will assume systematically that the sequence (x_n) defined by $(\mathcal{A})$ is strongly convergent. The results of the present section will be applied in Section 4 where effective convergence results will be derived.

3.1. Localization of the limit.

The next result asserts that, if the algorithm $(\mathcal{A})$ converges, the limit point x_∞ verifies $0 \in A(0) + B(x_\infty)$, *i.e.* x_∞ is a "stationary point" of the algorithm $(\mathcal{A})$.

Proposition 3.1. *Under the hypotheses of Proposition 2.1, assume moreover that the real sequence (λ_n) is minorized by some positive $\underline{\lambda}$. If the sequence (x_n) defined by algorithm $(\mathcal{A})$ is strongly convergent, then then the limit point $x_\infty := s - \lim_{n\to+\infty} x_n$ satisfies: $0 \in A(0) + B(x_\infty)$.*

Proof. Let $u_{n+2} \in A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big)$ and $v_{n+2} \in B(x_{n+2})$ satisfy:

$$(x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + u_{n+2} + v_{n+2} = 0. \tag{6}$$

Since $s - \lim_{n\to+\infty} x_n = x_\infty$ and since $\lambda_n \geq \underline{\lambda} > 0$ for every $n \in \mathbb{N}$, we have

$$s - \lim_{n\to+\infty} (x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 = 0. \tag{7}$$

On the other hand, since dom $A = H$, the operator A is locally bounded on H, which combined with $s - \lim_{n\to+\infty}(x_{n+2} - x_{n+1})/\lambda_n = 0$ implies that the sequence (u_n) is bounded in H. Therefore, there exist $\bar{u} \in H$ and a subsequence of (u_n), still denoted by (u_n) such that

$$w - \lim_{n\to+\infty} u_n = \bar{u}. \tag{8}$$

Since the graph of A is closed for the product topology $s - H \times w - H$, we have $\bar{u} \in A(0)$. On the other hand, in view of (6), (7) and (8), we deduce that $w - \lim_{n\to+\infty} v_n = -\bar{u}$. The same graph closedness property as above shows that $-\bar{u} \in B(x_\infty)$. Since $\bar{u} \in A(0) \cap -B(x_\infty)$, we conclude that $0 \in A(0) + B(x_\infty)$. □

When $A(0) = \{0\}$, Proposition 3.1 implies that $0 \in B(x_\infty)$, *i.e.* x_∞ is a zero of B. Let us now apply Proposition 3.1 to the case where $A := \partial[\theta(|\,.\,|)]$.

Corollary 3.1 (Case $A = \partial(\theta(|\,.\,|))$)**.** *Under the hypotheses of Proposition 3.1, consider a function* $\theta : \mathbb{R}_+ \to \mathbb{R}$ *of class* $\mathcal{C}^1$ *which is convex, non decreasing and take* $A := \partial[\theta(|\,.\,|)]$*. If* $x_\infty := s - \lim_{n\to+\infty} x_n$ *exists, then* $d(0, B(x_\infty)) \leq \theta'(0)$*.*

Proof. From Propositions 2.2 and 3.1, we have $0 \in \mathbb{B}(0, \theta'(0)) + B(x_\infty)$. This inclusion is equivalent to the existence of $\xi \in B(x_\infty)$ satisfying $|\xi| \leq \theta'(0)$, which is in turn equivalent to $d(0, B(x_\infty)) \leq \theta'(0)$. □

3.2. **Cases of "finite convergence".** According to the classical terminology, we will say that a sequence (x_n) is finitely convergent if it converges in a finite number of iterations; more precisely:

Definition 3.1. *A sequence* (x_n) *finitely converges to* x_∞ *if there exists* $n_0 \in \mathbb{N}$ *such that* $x_n = x_\infty$ *for every* $n \geq n_0$*.*

In case of convergence, Proposition 3.1 shows that the limit x_∞ of the sequence (x_n) satisfies $0 \in A(0) + B(x_\infty)$. The following result states that the convergence is finite as soon as $0 \in \operatorname{int}\Big(A(0) + B(x_\infty)\Big)$.

Proposition 3.2. *Under the hypotheses of Proposition 2.1, assume moreover that the real sequence* (λ_n) *is minorized by some positive* $\underline{\lambda}$*. If the sequence* (x_n) *defined by algorithm* $(\mathcal{A})$ *is strongly convergent and if the limit point* $x_\infty := s-\lim_{n\to+\infty} x_n$ *satisfies* $0 \in \operatorname{int}\Big(A(0) + B(x_\infty)\Big)$*, then* (x_n) *finitely converges to* x_∞*.*

Proof. We decompose the proof into three steps (i), (ii) and (iii).
(i) Let us first prove that the assumption $0 \in \operatorname{int}\Big(A(0) + B(x_\infty)\Big)$ implies the existence of $\varepsilon > 0$ such that

$$\forall x \in H \setminus \{x_\infty\}, \quad \forall \lambda \in \mathbb{R}_+^*, \qquad d\Big(0, A\left((x - x_\infty)/\lambda\right) + B(x)\Big) \geq \varepsilon. \tag{9}$$

Since $0 \in \operatorname{int}\Big(A(0)+B(x_\infty)\Big)$, there exists $\varepsilon > 0$ such that $\mathbb{B}(0, \varepsilon) \subset A(0)+B(x_\infty)$. For any $\lambda > 0$, consider the monotone operator C defined by

$$C(x) := A\left((x - x_\infty)/\lambda\right) + B(x).$$

The previous inclusion can be rewritten as $\mathbb{B}(0, \varepsilon) \subset C(x_\infty)$. Let $x \in H \setminus \{x_\infty\}$ and consider $y \in C(x)$. From the monotonicity of C, we have:

$$\forall \xi \in \mathbb{B}(0, \varepsilon), \quad \langle y - \xi, x - x_\infty\rangle \geq 0.$$

Taking the supremum when $\xi \in \mathbb{B}(0, \varepsilon)$ and using the Cauchy-Schwarz inequality, we deduce that

$$\varepsilon\,|x - x_\infty| \leq \langle y, x - x_\infty\rangle \leq |y|\,|x - x_\infty|,$$

so that we have $|y| \geq \varepsilon$. This being true for every $y \in C(x)$, we conclude that $d(0, C(x)) \geq \varepsilon$, which is exactly (9).
(ii) Let us now establish that

$$\lim_{n\to+\infty} d\Big(0, A\left((x_{n+2} - x_\infty)/\lambda_n\right) + B(x_{n+2})\Big) = 0. \tag{10}$$

Let $\varepsilon > 0$. Since $s - \lim_{n\to+\infty} x_n = x_\infty$ and since $\lambda_n \geq \underline{\lambda} > 0$ for every $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \Longrightarrow \quad (x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 \in \mathbb{B}(0, \varepsilon). \tag{11}$$

On the other hand, from the outer semicontinuity of A applied at $(x_{n+2}-x_\infty)/\lambda_n$, there exists $n_1 \geq n_0$ such that

$$n \geq n_1 \quad \Longrightarrow \quad A\Big((x_{n+2}-x_{n+1})/\lambda_n\Big) \subset A\Big((x_{n+2}-x_\infty)/\lambda_n\Big) + \mathbb{B}(0,\varepsilon). \tag{12}$$

Taking into account (11), (12) and the definition of the sequence (x_n), we deduce that

$$0 \in \mathbb{B}(0, 2\,\varepsilon) + A((x_{n+2}-x_\infty)/\lambda_n) + B(x_{n+2}). \tag{13}$$

It ensues that for every $n \geq n_1$,

$$d\Big(0, A\left((x_{n+2}-x_\infty)/\lambda_n\right) + B(x_{n+2})\Big) \leq 2\,\varepsilon,$$

which establishes (10).
(iii) We now prove the finite convergence of the sequence (x_n). Let us argue by contradiction and assume that there exists a subsequence of (x_n), still denoted by (x_n) such that $x_n \neq x_\infty$ for every $n \in \mathbb{N}$. From (9), we deduce that

$$d\Big(0, A\left((x_{n+2}-x_\infty)/\lambda_n\right) + B(x_{n+2})\Big) \geq \varepsilon,$$

which contradicts (10). □

If the assumption $0 \in \operatorname{int}\Big(A(0) + B(x_\infty)\Big)$ is not satisfied, the sequence (x_n) may not finitely converge toward x_∞. A counterexample is provided after the proof of Theorem 4.1. Let us now particularize Proposition 3.2 to the case where $A := \partial[\theta(|\,.\,|)]$.

Corollary 3.2 (Case $A = \partial(\theta(|\,.\,|))$)**.** *Under the hypotheses of Proposition 3.2, consider a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ of class $\mathcal{C}^1$ which is convex, non decreasing and take $A := \partial[\theta(|\,.\,|)]$. If $d(0, B(x_\infty)) < \theta'(0)$, then (x_n) finitely converges to x_∞.*

Proof. The assumption $d(0, B(x_\infty)) < \theta'(0)$ is equivalent to the existence of $\xi \in B(x_\infty)$ satisfying $|\xi| < \theta'(0)$, which is in turn equivalent to the inclusion $0 \in \operatorname{int}\Big(\mathbb{B}(0,\theta'(0))\Big) + B(x_\infty)$. On the other hand, remark that

$$\operatorname{int}\Big(\mathbb{B}(0,\theta'(0))\Big) + B(x_\infty) \,\subset\, \operatorname{int}\Big(\mathbb{B}(0,\theta'(0)) + B(x_\infty)\Big) = \operatorname{int}\Big(A(0) + B(x_\infty)\Big),$$

and we then apply Proposition 3.2. □

4. Strong convergence results under $0 \in \operatorname{int}(A(0))$

4.1. **Energy decay.** Let us now assume that the operator B is the subdifferential of some convex function $g : H \to \mathbb{R} \cup \{+\infty\}$. In this case, the algorithm $(\mathcal{A})$ may appear as an implicit discretization of a nonlinear oscillator subject to forces deriving from the potential g. Due to this mechanical interpretation, let us define the energy-like sequence (E_n) by

$$E_n = \frac{1}{2\,\lambda_n^2}\,|x_{n+1}-x_n|^2 + g(x_{n+1}). \tag{14}$$

It is easy to check that the sequence (E_n) is nonincreasing. More precisely, we will prove next that its decay rate is given by the function Φ_A defined as follows:

$$\Phi_A(x) := \inf_{y \in A(x)} \langle y, x\rangle. \tag{15}$$

The next lemma summarizes the main properties of Φ_A.

Lemma 4.1. *Let $A : H \rightrightarrows H$ a monotone operator such that dom $A = H$ and $0 \in A(0)$. The function Φ_A defined by (15) satisfies:*
(i) $\forall x \in H, \quad \Phi_A(x) \geq \sigma_{A(0)}(x) \geq 0$.
(ii) $0 \in \operatorname{int} A(0) \quad \Longrightarrow \quad \exists \alpha > 0, \forall x \in H, \quad \Phi_A(x) \geq \alpha\,|x|$.

Proof. (*i*) For any $x \in H$, let $y \in A(x)$ and $z \in A(0)$. The monotonicity of A yields $\langle y - z, x - 0\rangle \geq 0$, and hence $\langle y, x\rangle \geq \langle z, x\rangle$. Since this is true for every $y \in A(x)$ and $z \in A(0)$, we infer that:

$$\Phi_A(x) = \inf_{y \in A(x)} \langle y, x\rangle \geq \sup_{z \in A(0)} \langle z, x\rangle = \sigma_{A(0)}(x).$$

On the other hand, since $0 \in A(0)$, it is clear that $\sigma_{A(0)}(x) \geq 0$.
(*ii*) According to a classical result, the assertion $0 \in \operatorname{int} C$ (where $C \subset H$) is equivalent to the existence of $\alpha > 0$ such that $\sigma_C \geq \alpha\,|\,.\,|$. The conclusion is then an immediate consequence of (*i*). □

Remark 4.1. In the particular case where $A := \partial f$, conclusion (*i*) of Lemma 4.1 can be slightly improved. Indeed, the subdifferential inequality yields:

$$\forall x \in H, \qquad \Phi_{\partial f}(x) \geq f(x) - f(0) \geq \sigma_{\partial f(0)}(x).$$

When $A := \partial[\theta(|\,.\,|)]$, we deduce from Proposition 2.2 that the expression of $\Phi_{\partial[\theta(|\,.\,|)]}$ is given by:

$$\forall x \in H, \qquad \Phi_{\partial[\theta(|\,.\,|)]}(x) = |x|\,\theta'(|x|). \tag{16}$$

The following proposition shows that the sequence (E_n) is nonincreasing and expresses its decay rate as a function of Φ_A.

Proposition 4.1. *Let $A : H \rightrightarrows H$ a maximal monotone operator such that dom $A = H$ and $0 \in A(0)$. The operator B is supposed to be equal to ∂g for a closed convex function $g : H \to \mathbb{R} \cup \{+\infty\}$ satisfying $\operatorname{int}(\text{dom } g) \neq \emptyset$ and $\inf g > -\infty$. The positive sequence (λ_n) is assumed to be non decreasing. Then, the sequence (x_n) defined by algorithm $(\mathcal{A})$ satisfies:*
(i) $E_{n+1} - E_n \leq -\lambda_n\,\Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big)$*, where the sequence (E_n) and the function Φ_A are respectively defined by (14) and (15).*
(ii) $\sum_{n=0}^{+\infty} \lambda_n\,\Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) < +\infty$.

Proof. (*i*) Let $u_{n+2} \in A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big)$ and $v_{n+2} \in \partial g(x_{n+2})$ satisfy:

$$(x_{n+2} - 2\,x_{n+1} + x_n)/\lambda_n^2 + u_{n+2} + v_{n+2} = 0.$$

Take the scalar product with $x_{n+2} - x_{n+1}$ to obtain

$$\frac{1}{\lambda_n^2}\,\langle x_{n+2} - 2\,x_{n+1} + x_n, x_{n+2} - x_{n+1}\rangle + \langle u_{n+2}, x_{n+2} - x_{n+1}\rangle + \langle v_{n+2}, x_{n+2} - x_{n+1}\rangle = 0.$$

Since $x_{n+2} - x_{n+1} = \frac{1}{2}(x_{n+2} - 2\,x_{n+1} + x_n) + \frac{1}{2}(x_{n+2} - x_n)$ and since $|x_{n+2} - 2\,x_{n+1} + x_n|^2 \geq 0$, the previous equality yields:

$$\frac{1}{2\,\lambda_n^2}\,|x_{n+2} - x_{n+1}|^2 - \frac{1}{2\,\lambda_n^2}\,|x_{n+1} - x_n|^2 + \langle u_{n+2}, x_{n+2} - x_{n+1}\rangle + \langle v_{n+2}, x_{n+2} - x_{n+1}\rangle \leq 0. \tag{17}$$

From the definition of Φ_A, we have

$$\langle u_{n+2}, x_{n+2} - x_{n+1}\rangle \geq \lambda_n\,\Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big). \tag{18}$$

On the other hand, the convexity of g gives:

$$\langle v_{n+2}, x_{n+2} - x_{n+1}\rangle \geq g(x_{n+2}) - g(x_{n+1}). \tag{19}$$

By combining (17), (18), (19), together with $\lambda_{n+1} \geq \lambda_n$, we obtain:

$$E_{n+1} - E_n \leq -\lambda_n\, \Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big). \tag{20}$$

(ii) Let us add inequalities (20) for $n = 0, 1, ..., N$

$$\sum_{n=0}^{N} \lambda_n\, \Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) \leq E_0 - E_{N+1} \leq E_0 - \inf g.$$

Taking the limit when $N \to +\infty$ gives $\sum_{n=0}^{+\infty} \lambda_n\, \Phi_A\Big((x_{n+2} - x_{n+1})/\lambda_n\Big) < +\infty$. □

Let us now apply the result of Proposition 4.1 to the case $A := \partial\,|\,.\,|^\alpha$, for some $\alpha \geq 1$. In the sequel, l^α denotes the set of non-negative sequences (a_n) satisfying $\sum_{n=0}^{+\infty} a_n^\alpha < +\infty$.

Corollary 4.1. *Let $g : H \to \mathbb{R} \cup \{+\infty\}$ a closed convex function satisfying* $\operatorname{int}(\mathit{dom}\ g) \neq \emptyset$ *and* $\inf g > -\infty$. *The operators A and B are supposed to be respectively defined by $A := \partial\,|\,.\,|^\alpha$ ($\alpha \geq 1$) and $B := \partial g$. Consider a non decreasing sequence (λ_n) of positive numbers. Then, the sequence (x_n) defined by algorithm $(\mathcal{A})$ satisfies:*

$$|(x_{n+2} - x_{n+1})/\lambda_n| \in l^\alpha.$$

Proof. From Proposition 4.1 (ii) and expression (16), we obtain

$$\sum_{n=0}^{+\infty} \lambda_n\, |(x_{n+2} - x_{n+1})/\lambda_n|^\alpha < +\infty.$$

The conclusion follows from the fact that $\lambda_n \geq \lambda_0 > 0$ for every $n \in \mathbb{N}$. □

The case $\alpha = 1$ is specially interesting in the sense that $(|x_{n+1} - x_n|) \in l^1$ implies the strong convergence of the sequence (x_n). This remark is more widely exploited in the next paragraph, giving rise to the main result of the paper.

4.2. **Main result.** In what follows, the condition $0 \in \operatorname{int}(A(0))$ plays an essential role since it forces the strong convergence of the algorithm $(\mathcal{A})$ (see Theorem 4.1 below). From the mechanical point of view, the assumption $0 \in \operatorname{int}(A(0))$ can be interpreted as a condition of dry friction. Examples of nonlinear oscillators subject to Coulomb friction are given in [1, 7, 9].
The other important aspect of the next result is the finite convergence of $(\mathcal{A})$. The question of finite convergence for proximal-like algorithms has given rise to an abundant literature (see for example [10, 11, 12, 13]). Let us recall that the function $g : H \to \mathbb{R} \cup \{+\infty\}$ is linearly conditioned if $\operatorname{argmin} g \neq \emptyset$ and there exists $\gamma > 0$ such that

$$\forall x \in H, \quad g(x) \geq \min g + \gamma\, d(x, \operatorname{argmin} g).$$

It is well-known that the standard proximal method is finitely convergent for linearly conditioned convex functions. Our result does not require this kind of assumption. Here, the finite convergence of $(\mathcal{A})$ toward x_∞ is obtained under a very mild condition on x_∞. This result is strongly related to the fact that oscillators subject to dry friction are stabilized in a finite time. We refer the reader to [2] where a continuous version of $(\mathcal{A})$ is studied in the particular case $A = \partial f$ and $B = \nabla g$ (with $f, g : H \to \mathbb{R}$ respectively convex and differentiable). Let us now precisely state the main result of the paper:

Theorem 4.1. *Let $A : H \rightrightarrows H$ a maximal monotone operator such that dom $A = H$ and $0 \in \operatorname{int}(A(0))$. The operator B is supposed to be equal to ∂g for a closed convex function $g : H \to \mathbb{R} \cup \{+\infty\}$ satisfying $\operatorname{int}(dom\ g) \neq \emptyset$ and $\inf g > -\infty$. The positive sequence (λ_n) is assumed to be non decreasing. Then, the sequence (x_n) defined by algorithm $(\mathcal{A})$ satisfies:*
(i) $(|x_{n+1} - x_n|) \in l^1$ and therefore $x_\infty := s - \lim_{n\to+\infty} x_n$ exists.
(ii) The limit point x_∞ satisfies: $0 \in A(0) + \partial g(x_\infty)$.
(iii) If moreover $0 \in \operatorname{int}\Big(A(0) + \partial g(x_\infty)\Big)$, then the sequence (x_n) finitely converges to x_∞.

Proof. From Lemma 4.1 (*ii*), the assumption $0 \in \operatorname{int}(A(0))$ implies that $\Phi_A \geq \alpha\,|\,.\,|$ for some $\alpha > 0$. Assertion (*i*) is then an immediate consequence of Proposition 4.1. Items (*ii*) and (*iii*) result respectively from Proposition 3.1 and Proposition 3.2. □

Let us illustrate Theorem 4.1 by a few one-dimensional examples. For that purpose, we define the maximal monotone operator $\operatorname{Sgn} : \mathbb{R} \rightrightarrows \mathbb{R}$ by $\operatorname{Sgn}(x) = 1$ if $x > 0$, $\operatorname{Sgn}(x) = -1$ if $x < 0$ and $\operatorname{Sgn}(0) = [-1, 1]$.

Example 4.1. Take $H = \mathbb{R}$ and define the mappings A and g respectively by $A(x) = \alpha \operatorname{Sgn}(x)$ and $g(x) = \beta\, x^2/2$ where α and β are positive coefficients. The condition $0 \in \operatorname{int}(A(0))$ is clearly satisfied. The limit x_∞ must fulfil the inclusion $0 \in \alpha\,[-1, 1] + \beta\, x_\infty$, or equivalently $x_\infty \in [-\alpha/\beta, \alpha/\beta]$. If moreover $x_\infty \in]-\alpha/\beta, \alpha/\beta[$, Theorem 4.1 (*iii*) asserts that the sequence (x_n) finitely converges to x_∞.

Example 4.2. Take $H = \mathbb{R}$ and define the mappings A and g respectively by $A(x) = \alpha \operatorname{Sgn}(x)$ and $g(x) = \beta\,|x|$ where α and β are positive coefficients. The condition $0 \in A(0) + \partial g(x_\infty)$ can be rewritten as: $0 \in \alpha\,[-1, 1] + \beta \operatorname{Sgn}(x_\infty)$. If $\beta > \alpha$, the unique solution of the previous inclusion is $x_\infty = 0$ so that the sequence (x_n) converges toward 0. Since $0 \in \operatorname{int}(\alpha \operatorname{Sgn}(0) + \beta \operatorname{Sgn}(0)) =]-\alpha-\beta, \alpha+\beta[$, the convergence toward 0 is finite. Now assume that $\beta \leq \alpha$. It is then elementary to check that every $x_\infty \in \mathbb{R}$ is solution of $0 \in A(0) + \partial g(x_\infty)$: the sequence (x_n) may converge toward any element of $\mathbb{R}$. If moreover $\beta < \alpha$, the convergence is finite.

In view of Theorem 4.1 (*ii*) and (*iii*), finite convergence holds as soon as $0 \notin \operatorname{bd}\Big(A(0) + \partial g(x_\infty)\Big)$. Since the boundary of the convex set $A(0) + \partial g(x_\infty)$ has an empty interior, it is reasonable to think that the circumstances leading to $0 \in \operatorname{bd}\Big(A(0) + \partial g(x_\infty)\Big)$ are "exceptional". More precisely, we conjecture that generically with respect to the initial data $(x_0, x_1) \in H^2$, the point $x_\infty = s - \lim_{n\to+\infty} x_n$ satisfies the condition $0 \in \operatorname{int}\Big(A(0) + \partial g(x_\infty)\Big)$.

Let us now give a counterexample to finite convergence when $0 \notin \operatorname{int}\Big(A(0) + \partial g(x_\infty)\Big)$.

Example 4.3. Take $H = \mathbb{R}$ and define the operator A (resp. the function g) by $A(x) = \mathrm{Sgn}\,(x) + 2\,x$ (resp. $g(x) = x^2/2$). If the sequence (λ_n) is constant and equal to 1, the algorithm $(\mathcal{A})$ reduces to:

$$x_{n+2} - 2\,x_{n+1} + x_n + \mathrm{Sgn}\,(x_{n+2} - x_{n+1}) + 2\,(x_{n+2} - x_{n+1}) + x_{n+2} \ni 0,$$

or equivalently

$$4\,x_{n+2} - 4\,x_{n+1} + x_n \in -\mathrm{Sgn}\,(x_{n+2} - x_{n+1}).$$

Take as initial data $x_0 = x_1 = 2$. We let the reader check that the expression of x_n is given by $x_n = 1 + (1+n)/2^n$, so that (x_n) converges toward $x_\infty := 1$. There is no finite convergence but simply exponential convergence, because of $0 \notin \mathrm{int}\Big(A(0) + \partial g(1)\Big) =]0, 2[$.

When $A := \partial[\theta(|\,.\,|)]$, the statement of Theorem 4.1 can be simplified as follows:

Corollary 4.2. *Consider a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ of class $\mathcal{C}^1$ which is convex and such that $\theta'(0) > 0$. Let $g : H \to \mathbb{R} \cup \{+\infty\}$ a closed convex function satisfying $\mathrm{int}\,(dom\ g) \neq \emptyset$ and $\inf g > -\infty$. Define the operators A and B respectively by $A := \partial[\theta(|\,.\,|)]$ and $B := \partial g$. The positive sequence (λ_n) is assumed to be non decreasing. Then, the sequence (x_n) defined by algorithm $(\mathcal{A})$ satisfies:*
(i) $(|x_{n+1} - x_n|) \in l^1$ and therefore $x_\infty := s - \lim_{n\to+\infty} x_n$ exists.
(ii) The limit point x_∞ satisfies: $d\Big(0, \partial g(x_\infty)\Big) \leq \theta'(0)$.
(iii) If moreover $d\Big(0, \partial g(x_\infty)\Big) < \theta'(0)$, then the sequence (x_n) finitely converges to x_∞.

Proof. Proposition 2.2 shows that the assumption $\theta'(0) > 0$ implies $0 \in \mathrm{int}\,(A(0))$. Assertions (i) (resp. (ii), (iii)) are respectively consequences of Theorem 4.1 (resp. Corollaries 3.1, 3.2). □

Further remarks. Other possible developments
Theorem 4.1 asserts that, under adequate conditions, the sequence (x_n) defined by $(\mathcal{A})$ finitely converges. For numerical and theoretical purposes, it would be interesting to determine the minimal number of iterations which is necessary to attain convergence. This number is clearly related to the distance $d(0, \mathrm{bd}\,(A(0)))$ and it must tend to $+\infty$ when $d(0, \mathrm{bd}\,(A(0)))$ tends to 0.
On the other hand, in the whole paper we have considered the algorithm $(\mathcal{A})$, which is an exact discretization of the associated continuous dynamical system. For numerical applications, it would be important to deal with approximate discretizations $(\mathcal{A}_\varepsilon)$, for example by considering ε-enlargments of the operators A and B. These remarks certainly indicate directions for future investigation.

Acknowledgements

The author would like to thank Felipe Alvarez for fruitful discussions and pertinent remarks about the paper.

References

[1] S. Adly, D. Goeleven, A stability theory for second order nonsmooth dynamical systems with application to friction problems, *Journal de Mathématiques Pures et Appliquées*, vol. 83, (2004), 17-51.

[2] S. Adly, H. Attouch, A. Cabot, Finite time stabilization of nonlinear oscillators subject to dry friction, to appear in Progresses in Nonsmooth Mechanics and Analysis (edited by P. Alart, O. Maisonneuve and R.T. Rockafellar), Advances in Mathematics and Mechanics, Kluwer.

[3] F. Alvarez, On the minimizing property of a second order dissipative system in Hilbert space, *SIAM J. on Control and Optimization*, vol. 38, 4 (2000), 1102-1119.

[4] F. Alvarez, H. Attouch, An inertial proximal method for maximal monotone operators via discretization of a nonlinear oscillator with damping, *Set-Valued Analysis*, vol. 9 (2001), 3-11.

[5] H. Amann, J. I. Díaz, A note on the dynamics of an oscillator in the presence of strong friction, *Nonlinear Analysis*, vol. 55 (2003), 209-216.

[6] H. Brézis, Opérateurs maximaux monotones dans les espaces de Hilbert et équations d'évolution, Lecture Notes 5, North Holland, 1972.

[7] M. Cadivel, D. Goeleven and M. Shillor, Study of a unilateral oscillator with friction, *Math. Comput. Modelling*, vol. 32, 2/3 (2000), 381-391.

[8] J. I. Díaz, A. Liñán, On the asymptotic behavior of a damped oscillator under a sublinear friction term, *Rev. R. Acad. Cien. Serie A. Mat.*, vol. 95, 1 (2001), 155-160.

[9] Y. Dumont, D. Goeleven, M. Rochdi and M. Shillor, Frictional contact of a nonlinear spring, *Math. Comput. Modelling*, vol. 31, 2/3 (2000), 83-97.

[10] O. Lefebvre, C. Michelot, About the finite convergence of the proximal point algorithm, Trends in mathematical optimization, International Series of Numerical Mathematics, vol. 84 (1988), 153-161.

[11] B. Lemaire, About the convergence of the proximal method, Advances in optimization, *Lecture Notes in Econom. and Math. Systems*, 382, Springer, Berlin, (1992), 39-51.

[12] B. Lemaire, Bounded diagonally stationary sequences in convex optimization, *J. Convex Anal.* 1 (1994), 75-86.

[13] B. Lemaire, On the convergence of some iterative methods for convex minimization, Recent developments in optimization (Dijon, 1994), *Lecture Notes in Econom. and Math. Systems*, 429, Springer, Berlin, 1995, 252-268.

[14] B. Martinet, Régularisation d'inéquations variationnelles par approximations successives, *Revue Française d'Informatique et Recherche Opérationnelle*, 4 (1970), 154-159.

[15] B. Martinet, Détermination approchée d'un point fixe d'une application pseudo-contractante, *C.R. Acad. Sci.*, Paris, 274 (1972), 163-165.

[16] R.T. Rockafellar, Monotone operators and the proximal point algorithm, *SIAM Journal on Control and Optimization*, 14 (1976), pp. 877-898.

[17] R.T. Rockafellar and R. Wets, Variational analysis, Springer, Berlin, 1998.

Laboratoire LACO, Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex, France (alexandre.cabot@unilim.fr).