**Laboratoire d'Arithmétique,
Calcul formel et d'Optimisation**

UMR CNRS 6090

On the Semicontinuity of Vector-Valued Mappings

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Rapport de recherche n° 2004-09
Déposé le 15 septembre 2004

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ON THE SEMICONTINUITY OF VECTOR-VALUED MAPPINGS

M. AIT MANSOUR, C. MALIVERT AND M. THÉRA

ABSTRACT. We first provide, for a given vector-valued mapping f , a lower semicontinuous (l.s.c) regularization. Further, we present a new characterization of lower semicontinuity, which allows to derive another type of l.s.c regularization. A quasi-l.s.c regularization is thereafter presented.

1. INTRODUCTION

In this paper, we present a new characterization of the lower semicontinuity of vector-valued mappings in terms of the hypographical profile mapping. Let us describe the contents of the paper. In Section 2, we recall in details the necessary ingredients needed in this work. In Section 3, we establish a lower semicontinuous regularization for mappings with values in an order complete Daniell space. In the following section we consider, for a given vector-valued mapping, a kind of lower level set which is unusually a subset of the target space. We prove that it enjoys the nice property of having an ideal efficient point under two different situations. Section 5 contains our main results: After presenting a characterization of semicontinuity of vector-valued mappings, Theorem 5.2, we present another point of view to regard the regularization under more weaker assumptions on the target space, Theorem 5.5. As well, we furnish a quasi-lower semicontinuous regularization that preserves convexity. The regularizations we present here offer a strong property of semicontinuity. We end the paper by a comment.

2. PRELIMINARIES AND BASIC FACTS

Throughout this paper E is supposed to be a metrizable space and unless otherwise specified, F is a metrizable vector space ordered by a closed convex cone C with nonempty interior and pointed ($C \cap -C = \{0\}$). We assume that F is metrizable and we denote by $B(x, r)$ the open ball in F with center at x and radius r . For $x, y \in F$ we write

The research of the first author was supported by "Agence universitaire de la francophonie" and by Laco laboratory of the university of Limoges.

$x \leq_C y$ to denote the relation $y - x \in C$. If $+\infty$ denotes a greatest element with respect to the order $\leq_C$ and $-\infty$ denotes a least element, we use the notation $F^\bullet$ for $F \cup \{+\infty, -\infty\}$. As usual, given a subset $S \subset F$, we use the notations $\text{Int } S$ and $\text{cl } S$ for the topological interior and the closure of S , respectively. For $x, y \in E$, we will write $x <_C y$, if $y - x \in \text{Int } C$. For a subset $A \subset F$, whenever it exists, a point $a \in F$ is called a *least upper bound* of A when for every $d \in F$ the equivalence

$$a \leq_C d \text{ if and only if } b \leq_C d \text{ for every } b \in A.$$

is true. As usual, we denote by $\sup A$ the least upper bound of A . Similarly, the *greatest lower bound* of A , whenever it exists, $\inf A$, is defined by

$$\text{for every } c \in F, c \leq_C \inf A \text{ if and only if } c \leq_C b \text{ for every } b \in A.$$

F is called *order complete* if every nonempty upper bounded (resp. lower bounded) subset $D \subset F$ has a least upper bound (resp. a greatest lower bound). F is said to be a *semilattice*, if every finite set of F admits a least upper bound and a greatest lower bound. We need equally to recall that a subset $D \subset F$ is said to be:

- *directed*, if any finite subset of D has an upper bound in D ;
- *free disposal*, if $A - C = A$.

Now, recall that a net $\{x_\alpha \mid \alpha \in I\}$ from E is said to be *decreasing* (resp. *increasing*) with respect to C if $x_\beta \leq_C x_\alpha$ and $x_\beta \neq x_\alpha$ (resp. $x_\alpha \leq_C x_\beta$ and $x_\beta \neq x_\alpha$) for each $\alpha, \beta \in I, \beta > \alpha$. F will hence be said *Daniell*, if every increasing and upper bounded net has a limit (in F). It is worth recalling as well that a set $A \subset F$ is said to be:

- *C-complete* if it has no covers of the form $\{(x_\alpha - C)^c \mid \alpha \in I\}$ with $\{x_\alpha \mid \alpha \in I\}$ being a decreasing net in A , where B^c denotes the complement of a subset B ;
- *C-compact* if any cover of A of the form $\{U_\alpha + C \mid \alpha \in I, U_\alpha \text{ is open}\}$ admits a finite subcover;
- *C-semicompact* if any cover of A of the form $\{(x_\alpha - C)^c \mid \alpha \in I\}$ admits a finite subcover.

Now, let A be a subset of F . A point a is said to be

- an *efficient (maximal)* point of A with respect to C if

$$(a + C) \cap A = \{a\}.$$

- *ideal efficient* of A with respect to C if $x \leq a$ for each $x \in A$.

$\text{Eff}(A|C)$ and $\text{IEff}(A|C)$ denote respectively the set of efficient and ideal efficient points of A .

Remark 2.1. Note that $\text{IEff}(A|C) \subset \text{Eff}(A|C)$. If, $\text{IEff}(A|C)$ is nonempty then $\text{IEff}(A|C) = \text{Eff}(A|C)$ (see [5]), and since the cone C is supposed to be pointed, $\text{IEff}(A)$ reduces to a singleton.

Definition 2.2. Let $f : E \rightarrow F$. A point x_0 is a C -minimal of f , if

$$f(E) \cap (f(x_0) - C) = \{f(x_0)\}.$$

Remark 2.3. The cone C being pointed, x_0 is a C -minimal of f if and only if

$$f(E) \cap (f(x_0) - C) \subset \{f(x_0)\} + C.$$

We conclude these preliminaries by pointing out some basic definitions: Given a map $f : E \rightarrow F^\bullet$, its *domain* is denoted by $\text{Dom } f$ and defined by

$$\text{Dom } f = \{x \in E \mid f(x) \in F\},$$

its *graph* by

$$\text{gph } f = \{(x, y) \in E \times F \mid x \in \text{Dom } f, y = f(x)\}$$

and its *hypograph* by

$$(2.1) \quad \text{hypo} = \{(x, \alpha) \in E \times F \mid \alpha \leq_C f(x)\}.$$

The *hypographical mapping* of f is given by $\mathcal{H}_f : E \rightrightarrows F$ and defined for $x \in E$ by

$$\mathcal{H}_f(\bar{x}) = \{\alpha \in F \mid \alpha \leq_C f(\bar{x})\}.$$

For a set-valued map $S : E \rightrightarrows F$ the domain and the graph are defined respectively by

$$\text{Dom } S = \{x \in E \mid S(x) \neq \emptyset\}$$

and

$$\text{gph } S = \{(x, y) \in E \times F \mid x \in \text{Dom } S, y \in S(x)\},$$

we notice that $\text{gph } \mathcal{H}_f = \text{hypo } f$ and that $\mathcal{H}_f^{-1}(\alpha) = \text{lev}_{\geq_C \alpha} f = \{x \in E \mid \alpha \leq_C f(x)\}$. Accordingly, the *strict hypographical* profile denoted by $\mathcal{H}_f^s$ is given by

$$\mathcal{H}_f^s(x) = \{\alpha \in F \mid \alpha <_C f(x)\}.$$

The *epigraphical* (resp. *strict*) profile and similar remarks hold in parallel. Given a mapping $f : E \rightarrow F^\bullet$ and $\bar{x} \in \text{Dom } f$, we consider the subset

$$\mathcal{L}_f(x) = \liminf_{z \rightarrow x} \mathcal{H}(z),$$

where

$$\liminf_{x \rightarrow \bar{x}} S(x) = \{y \in F \mid \forall (x_n) \rightarrow \bar{x}, \exists (y_n) \rightarrow y \text{ with } y_n \in S(x_n)\},$$

for a given set-valued map $S : E \rightrightarrows F$ and $\bar{x} \in \text{Dom } S$. Since E and F are supposed to be metrizable, observe that

$$\mathcal{L}_f(x) = \{\beta \in F \mid \forall (x_n) \rightarrow x, \exists (\beta_n) \rightarrow \beta, f(x_n) \geq_C \beta_n, \forall n.\}$$

This subset was introduced in [2] as a kind of lower level set of f at a point x . We first recall some basic properties of $\mathcal{L}_f$ from [2]:

Proposition 2.4. For every $x \in E$, the $\mathcal{L}_f(x)$ is

- (1) convex;
- (2) free disposal;
- (3) directed if in addition F is semilattice.

Proof. This proposition has been presented in [2] under a more restrictive assumption on the space F (F has been supposed there to be a complete lattice). For the convenience of the reader we present the proof:

(1) Convexity. Take $y_1, y_2 \in \mathcal{L}_f(x)$, $\lambda \in [0, 1]$ and (x_n) a sequence in E that converges to $\bar{x}$. Then, there exist two sequences (b'_n) and (b''_n) in F such that

$$\lim_{n \rightarrow +\infty} b'_n = y_1 \quad \text{and} \quad b'_n \leq_C f(x_n), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow +\infty} b''_n = y_2 \quad \text{and} \quad b''_n \leq_C f(x_n), \quad \forall n \in \mathbb{N}.$$

Take $b_n = \lambda b'_n + (1 - \lambda) b''_n$. On one hand, we have

$$(2.2) \quad \lim_{n \rightarrow +\infty} b_n = \lambda y_1 + (1 - \lambda) y_2.$$

On the other hand,

$$\begin{aligned} f(x_n) - b_n &= f(x_n) - \lambda b'_n - (1 - \lambda) b''_n \\ &= \lambda (f(x_n) - b'_n) + (1 - \lambda) (f(x_n) - b''_n). \end{aligned}$$

As $f(x_n) - b'_n \in C \cup \{+\infty\}$ and $f(x_n) - b''_n \in C \cup \{+\infty\}$,

$$(2.3) \quad b_n \leq_C f(x_n), \quad \forall n \in \mathbb{N}$$

we deduce from (2.2) and (2.3) that

$$\lambda y_1 + (1 - \lambda) y_2 \in \mathcal{L}_f(x).$$

(2) $\mathcal{L}_f(x)$ is free disposal: This comes directly from the definition of $\mathcal{L}_f(x)$.

(3) Directness: Take $y_1, y_2 \in \mathcal{L}_f(x)$ and a sequence (x_n) , converging to x . Then, there exist two sequences (b_n) and (b'_n) in F such that

$$(2.4) \quad \lim_{n \rightarrow +\infty} b_n = y_1 \quad \text{and} \quad b_n \leq_C f(x_n), \quad \forall n \in \mathbb{N}$$

and

$$(2.5) \quad \lim_{n \rightarrow +\infty} b'_n = y_2 \quad \text{and} \quad b'_n \leq_C f(x_n), \quad \forall n \in \mathbb{N}.$$

Since F is supposed to be a semilattice by [Proposition 5.2 in [11]], the map

$$\begin{aligned} \sup : F \times F &\rightarrow F \\ (x, y) &\mapsto \sup(x, y) \end{aligned}$$

is uniformly continuous. Taking into consideration (2.4) and (2.5), we obtain

$$\lim_{n \rightarrow +\infty} \sup(b_n, b'_n) = \sup(y_1, y_2)$$

On the other hand, we have

$$\sup(b_n, b'_n) \leq_C f(x_n), \quad \forall n \in \mathbb{N}.$$

It follows that $\sup(y_1, y_2) \in \mathcal{L}_f(x)$. □

Definition 2.5. f is said to be C -convex, if for every $\alpha \in [0, 1]$ and $x_1, x_2 \in E$

$$(2.6) \quad \alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha) x_2) + C.$$

Definition 2.6. [10] A mapping $f : E \rightarrow F^\bullet$ is said to be *lower semicontinuous* (l.s.c) at $x \in E$, if for any neighborhood V of zero (in F) and for any $b \in F$ satisfying $b \leq_C f(x)$, there exists a neighborhood U of $\bar{x}$ in E such that

$$f(U) \subset b + V + C \cup \{+\infty\}.$$

Remark 2.7. Following [10] if $f(x) \in F$, then Definition 2.6 amounts to saying that for all neighborhood V of zero, there exists a neighborhood U of $\bar{x}$ such that

$$(2.7) \quad f(U) \subset f(x) + V + C \cup \{+\infty\}.$$

Definition 2.8. [3] A mapping $f : E \rightarrow F$ is said to be *sequentially lower semicontinuous* (s-l.s.c) at $x \in E$, if for any $b \in F$ satisfying $b \leq_C f(x)$ and for any sequence (x_n) of E which converges to x , there exists a sequence (b_n) of F converging to b and satisfying $b_n \leq_C f(x_n)$, for every $n \in \mathbb{N}$.

Remark 2.9. For $x \in \text{Dom } f$, Definition 2.8 amounts to saying that for each sequence (x_n) converging to x , there exists a sequence (b_n) converging to $f(\bar{x})$ such that $b_n \leq_C f(x_n)$ for all $n \in \mathbb{N}$.

Note, from [3], it has been proved that Definitions 2.6 and 2.8 coincide whenever E and F are metrizable.

Definition 2.10. A mapping $f : E \rightarrow F$ is called *quasi-lower semicontinuous* or *level closed* if for every $y \in F$, $\text{lev}_{\leq_C}(y) := \{x \in E \mid f(x) \leq_C y\}$ is closed.

Definition 2.11. A mapping $f : E \rightarrow F$ is called *epi-closed* if its epigraph is closed in the product space $E \times F$.

Remark 2.12. Notice that every epi-closed map is quasi-l.s.c., while the converse is true if $\text{Int } C \neq \emptyset$. Also, every l.s.c. map is epi-closed but the converse may fail.

Remark 2.13. The quasi-lower semicontinuity is used under the terminology of *C-lower semicontinuity* in [4].

Definition 2.14. Let $S : E \rightrightarrows F$ be a set-valued map and $\bar{x} \in \text{Dom } S$. S is *lower semicontinuous* at $\bar{x}$ if, and only if $S(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} S(x)$.

3. THE RESULTS

3.1. A first lower semicontinuous regularization. In addition to hypotheses of section 2, we assume in this subsection that F is an order complete vector topological space and if $f : E \rightarrow F^\bullet$ is a given map, for each $x \in E$, we denote by $\mathcal{V}(x)$ the family of neighborhoods of x and introduce the following

$$\bar{f}(x) = \sup_{U \in \mathcal{V}(x)} \inf f(U).$$

Clearly, $\bar{f}$ is

- well defined as F is an order complete;
- a minorant of f , i.e., $\bar{f}(x) \leq_C f(x)$, for all $x \in E$.

Proposition 3.1. If $f : E \rightarrow F^\bullet$ is lower semicontinuous at $a \in E$, then $\bar{f}(a) = f(a)$.

Proof. The result is immediate if $f(a) = -\infty$ and can be easily seen if $f(a) = +\infty$. Suppose that $f(a) \in F$. For every $e \in \text{Int } C$, there exists a neighborhood U of a such that $f(U) \subset f(a) - e + C^\bullet$. This yields, $f(a) - e \leq_C \inf f(U)$ and $f(a) - e \leq_C \bar{f}(a)$. Since e is arbitrary in $\text{Int } C$, it follows that $f(a) \leq \bar{f}(a)$. Hence, $\bar{f}(a) = f(a)$. The proof is complete. $\square$

Corollary 3.2. Let $f : E \rightarrow F$ be a map. Suppose that $\bar{f}$ is lower semicontinuous. If $g : E \rightarrow F^\bullet$ is l.s.c and satisfies $g(a) \leq_C f(a)$, $a \in E$, then $g(a) \leq_C \bar{f}(a)$. In other words, $\bar{f}$ is the greatest lower semicontinuous minorant of f .

Proposition 3.3. For every map $f : E \rightarrow F$, we have $\bar{\bar{f}} = \bar{f}$.

Proof. It suffices to prove that for every $a \in E$ we have $\bar{f}(a) \leq_C \bar{\bar{f}}(a)$. Observe that for every neighborhood U of a and for every $x \in U$ we have

$$\inf f(U) \leq_C \bar{f}(x).$$

Therefore

$$\inf f(U) \leq_C \inf \bar{f}(U),$$

and consequently

$$\bar{f}(a) \leq_C \bar{\bar{f}}(a) = \sup_{U \in \mathcal{V}(a)} \inf \bar{f}(U).$$

The proof is therefore established. $\square$

Proposition 3.4. Let $f : E \rightarrow F$ and $a \in E$. If $\bar{f}(a) = f(a)$, then f is quasi-lower semicontinuous at a .

Proof. Let $b \in F$ such that $b - f(a) \notin C$. Suppose on the contrary that for every neighborhood U of a , there exists some $x \in U$ such that $b - f(x) \in C$. Then,

$$\sup_{U \in \mathcal{V}(a)} \inf f(U) \leq_C b,$$

hence $f(a) = \bar{f}(a) \leq_C b$, a contradiction. $\square$

Proposition 3.5. Assume that F is order complete Daniell. If, for $a \in E$, $\bar{f}(a) = f(a)$, then f is lower semicontinuous at a .

Proof. Since F is Daniell and $\bar{f}(a) = f(a)$, the net $(\inf f(U))_{U \in \mathcal{V}(a)}$ converges to $f(a)$. Accordingly, for every neighborhood V of $f(a)$, there exists a neighborhood U of a with $\inf f(U) \in V$ such that $f(U) \subset V + C$. The proof is complete. $\square$

Corollary 3.6. Assume F is Daniell and order complete. Then $\bar{f}$ is lower semicontinuous.

4. EXISTENCE OF AN EFFICIENT POINT OF $\mathcal{L}_f(\bar{x})$

Having in mind that our goal is to define a l.s.c regularization, for a given vector-valued mapping $f : E \rightarrow F^\bullet$ and some $x \in \text{Dom } f$, it is of interest to analyze the set $\mathcal{L}_f(x)$ from the point of view of the existence of efficient points. The target space F will be supposed (in this section) a semilattice.

At first, we notice that $\mathcal{L}_f(\bar{x})$ may be empty in some particular situations when, for instance, one impose some restrictions on the space F like C -completeness (C -semicompactness, compactness, respectively). Indeed, by Proposition 2.4, the set $\mathcal{L}_f(\bar{x})$ is free disposal which is not possible in these cases. More generally, we establish the following:

Proposition 4.1. Assume that F is C -complete. Then, there is no nonempty free disposal subset in F .

Proof. Assume for contradiction that there exists $\emptyset \neq A \subset F$ such that $A - C = A$. We claim that A is C -complete. Indeed, if we suppose that A has some cover of the form $\{(x_\alpha - C)^c \mid \alpha \in I\}$ where $\{x_\alpha \mid \alpha \in I\}$ is some decreasing net in A , it follows that

$$(4.1) \quad A \subset \bigcup_{\alpha \in I} (x_\alpha - C)^c.$$

On the other hand, for all $\alpha \in I$ we have $x_\alpha - C \subset A - C = A$, which yields

$$\bigcap_{\alpha \in I} (x_\alpha - C) \subset A,$$

and equivalently

$$(4.2) \quad A^c \subset \bigcup_{\alpha \in I} (x_\alpha - C)^c.$$

Combining (4.1) and (4.2), we see that $F \subset \bigcup_{\alpha \in I} (x_\alpha - C)^c$, a contradiction because F is C -complete. Therefore, we conclude that A is C -complete.

Since $A \neq \emptyset$ we can choose a point $z \in A$ and consider the section A_z of A ($A_z = A \cap (z - C) = z - C$). A_z is free disposal. By similar arguments as before we deduce that A_z is C -complete. Hence, by [Theorem 3.3, p. 46, [5]] we have $\text{Eff}(A \mid -C) \neq \emptyset$. Let $a_0 \in \text{Eff}(A \mid -C)$. By definition of a_0 we have $A \cap (a_0 - C) = \{a_0\}$, but $A \cap (a_0 - C) = a_0 - C$ (because $A = A - C$). This leads to $a_0 = a_0 - C$ which is impossible since $\{0\} \neq C$, a contradiction. $\square$

Remark 4.2. (1) The conclusion is similar if F is compact or C -semicompact, since these two cases imply the C -completeness, see [6]. (2) The condition that f is locally C -lower bounded is sufficient for $\mathcal{L}_f(x)$ to be nonempty. It is also necessary as we will see later.

Corollary 4.3. If F is C -complete, then $C = \{0\}$.

Now, we examine the existence of efficient points:

4.1. First case: F is Daniell. The next proposition ensures the existence of efficient points of $\text{cl } \mathcal{L}_f(x)$ for every $x \in \text{Dom } f$.

Proposition 4.4. Assume that F is semilattice and Daniell. If $\mathcal{L}_f(x)$ is nonempty, then, $\text{Eff}(\text{cl } \mathcal{L}_f(x) \mid C) \neq \emptyset$.

Proof. Given that F is Daniell and $\text{cl } \mathcal{L}_f(x)$ is closed and upper bounded, the result follows from [9], quote also [5] (Theorem 3.3 and Lemma 3.5, pp. 46). $\square$

Proposition 4.5. Let $A \subset F$ and assume that F is semilattice and Daniell. If $\text{Eff}(A|C) \neq \emptyset$ and A is directed, then $\text{IEff}(A|C) \neq \emptyset$ and therefore $\text{IEff}(A|C) = \text{Eff}(A|C)$. Moreover for some $a \in A$, $\text{IEff}(A|C) = \text{Eff}(A|C) = \{a\}$.

Proof. Let $a \in \text{Eff}(A|C)$. We shall show that $a \in \text{IEff}(A|C)$. Let $b \in A$, since A directed, take $d = \sup(a, b)$. Of course, $a \leq_C d$. As $a \in \text{Eff}(A|C)$, it follows that $d \leq_C a$, hence by transitivity $b \leq_C a$, which means that $a \in \text{IEff}(A|C)$. Remark 2.1 completes the proof. $\square$

Proposition 4.6. Assume that F is a semilattice and Daniell. If $\text{Eff}(\text{cl } \mathcal{L}_f(x)|C)$ is nonempty, then $\text{Eff}(\text{cl } \mathcal{L}_f(x)|C)$ coincides with

$$\text{IEff}(\text{cl } \mathcal{L}_f(x)|C)$$

and reduces to a singleton.

Proof. Following Proposition 2.4, $\text{cl } \mathcal{L}_f(x)$ is directed. Then the result follows from Proposition 4.5. $\square$

We denote by a_x the unique (ideal) efficient point of $\text{cl } \mathcal{L}_f(x)$.

For a given map $f : E \rightarrow F^\bullet$ such that for every $x \in \text{Dom } f$ the subset $\mathcal{L}_f(x)$ is nonempty, let us introduce the map $J_f : E \rightarrow F$ defined for $x \in \text{Dom } f$ by $J_f(x) = a_x$. We observe that $\text{Dom } J_f = \text{Dom } f$.

Remark 4.7. The Daniell property is essential to prove the existence of ideal efficient point for $\mathcal{L}_f(x)$. We would like to point out that the class of topological vector spaces with the Daniell property contains for examples:

- Every vector topological space for which the order-intervals are compacts;
- Let (E, K) and (F, C) two topological vector spaces ordered respectively by the cones K and C . If C has a nonempty interior, then E is Daniell with respect to the strong topology $\beta(E, F)$;
- Every semi-reflexive and normal space;
- Every vector topological space ordered by cone generated by a compact base;
- The space $\mathcal{C}([a, b], \mathbb{R})$ of real-valued functions continuous on an interval $[a, b]$, where $a, b \in \mathbb{R}$;
- Let E be a Hilbert space and H be the space of all hermitian operators on E , ordered by the cone H_+ defined by

$$H_+ = \{A \in H \mid \langle Ax, x \rangle \geq 0, \forall x \in E\}.$$

(H, H_+) is Daniell for the pointwise convergence.

4.2. Second case: F is order complete. In this case the supremum of $\mathcal{L}_f(\bar{x})$, for every $\bar{x} \in \text{Dom } f$, exists. We prove at first that it is preserved by any element of the continuous dual F^* :

Proposition 4.8. For every $\bar{x} \in \text{Dom } f$ and every $\xi \in F^*$, $\xi(\sup \mathcal{L}_f(\bar{x})) = \sup \xi(\mathcal{L}_f(\bar{x}))$.

Proof. Let $\xi \in Y^*$. $\mathcal{L}_f(\bar{x})$ being directed, thanks to Proposition 4.1 of [1], it suffices to check that ξ is continuous with respect to the Scott topology. Since $\mathbb{R}$ is continuous (in the sense of [1] see example 1.1 in [1] for further details) and ξ is lower semicontinuous continuous (because continuous), it follows from [Theorem 4.2, [1]] that ξ is Scott-continuous. The proof is then complete. $\square$

Proposition 4.9. Let $f : E \rightarrow F^\bullet$ such that for every $x \in \text{Dom } f$, $\text{cl } \mathcal{L}_f(x)$ is nonempty. Assume that F is an order complete. Then, $\text{cl } \mathcal{L}_f(x)$ has a unique ideal efficient point.

Proof. $\text{cl } \mathcal{L}_f(x)$ being nonempty and upper bounded, as F is order complete, $\sup \text{cl } \mathcal{L}_f(x)$ exists. Let us denote this supremum again by $J_f(x)$. Since $\text{cl } \mathcal{L}_f(x)$ is convex, it can be rewritten as the intersection of all closed half spaces containing $\mathcal{L}_f(\bar{x})$ i.e.,

$$\text{cl } \mathcal{L}_f(x) = \bigcap_{\xi \in F^*, \alpha \in \mathbb{R}} \Delta_\xi(\alpha),$$

where

$$\Delta_\xi(\alpha) = \{y \in F \mid \langle \xi, \mathcal{L}_f(x) \rangle \leq \alpha \Rightarrow \langle \xi, y \rangle \leq \alpha\},$$

where $\alpha \in \mathbb{R}$ and $\xi \in F^*$. Consider an half space $\Delta_\xi(\alpha)$. Making use of Proposition 4.8, we deduce that

$$\langle \xi, J_f(x) \rangle = \sup \langle \xi, \mathcal{L}_f(x) \rangle \leq \alpha.$$

Thus, $J_f(x) \in \text{cl } \mathcal{L}_f(x)$. Therefore, $J_f(x)$ is an ideal efficient point of $\text{cl } \mathcal{L}_f(x)$ and is unique thanks to the pointedness of the cone C . $\square$

5. THE MAIN RESULTS

5.1. Characterization of lower semicontinuity of vector-valued mappings. The aim of this section is to present a new characterization of lower semicontinuity for vector-valued mappings by means of their hypographical profile mappings, which will allow us to obtain a l.s.c regularization.

We begin with the following

Lemma 5.1. Let $f : E \rightarrow F^\bullet$ and $\bar{x} \in \text{Dom } f$. We have

$$\mathcal{L}_f(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x).$$

Proof. Take $e \in \text{Int } C$. It suffices to prove for every $\varepsilon > 0$ that

$$\mathcal{L}_f(\bar{x}) - \varepsilon e \subset \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x).$$

Suppose on the contrary that there is some $y_\varepsilon = y - \varepsilon e \notin \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x)$ with $y \in \mathcal{L}_f(\bar{x})$. Then, there is $\eta > 0$ and a sequence (x_k) converging to $\bar{x}$ such that

$$B(y_\varepsilon, \eta) \cap \mathcal{L}_f(x_k) = \emptyset.$$

Using the fact that $y_\varepsilon \notin \mathcal{L}_f(x_k)$, we can select a sequence $(z_p^k)_p$ converging to x_k such that for every p ,

$$f(z_p^k) - y_\varepsilon \notin C.$$

For each k , we pick $z_{p(k)}^k$ in $B(x_k, \frac{1}{k})$. Therefore, the sequence $(z_{p(k)}^k)_k$ converges to $\bar{x}$ and $f(z_{p(k)}^k) - y_\varepsilon \notin C$. On the other hand, since $y \in \mathcal{L}_f(\bar{x})$, then we can find a sequence (b_k) converging to y and $b_k \leq_C f(z_{p(k)}^k)$. For k large enough, $y_\varepsilon = y - \varepsilon e \leq_C b_k \leq_C f(z_{p(k)}^k)$, a contradiction. $\square$

Theorem 5.2. Assume that for every $x \in \text{Dom } f$, $\mathcal{L}_f(x)$ is nonempty and admits a supremum. The following assertions are equivalent:

- (a₁) f is lower semicontinuous at a point x in its domain;
- (a₂) $\mathcal{G} : E \rightrightarrows F$, defined for $x \in \text{Dom } f$ by $\mathcal{G}(x) = \mathcal{L}_f(x) \cup \mathcal{H}_f^s(x)$, is lower semicontinuous at $\bar{x}$;
- (a₃) $\sup \mathcal{L}_f(x) = f(x)$;
- (a₄) $\mathcal{H}_f : E \rightrightarrows F$ is lower semicontinuous at x .
- (a₅) $\mathcal{L}_f(x) = \mathcal{H}_f(x) = \mathcal{G}(x)$.

Remark 5.3. Let f be an extended-real-valued function and let $x \in \text{Dom } f$. If f is not l.s.c at x , then $\text{cl } \mathcal{L}_f(x) =]-\infty, \liminf_{z \rightarrow x} f(z)]$, $\mathcal{H}_f(x) =]-\infty, f(x)]$, and $\mathcal{G}(x) =]-\infty, f(x)[$.

We first introduce the sets

$$(5.1) \quad E_f(\bar{x}) := \{y \in \mathcal{L}_f(\bar{x}) \mid y <_C \sup \mathcal{L}_f(\bar{x})\},$$

and

$$(5.2) \quad Q_f(\bar{x}) := \{y \in \text{cl } \mathcal{L}_f(\bar{x}) \mid y \leq_C \sup \mathcal{L}_f(\bar{x})\},$$

and we establish the following technical Lemma:

Lemma 5.4. We assert that $\text{cl}(E_f(\bar{x})) = Q_f(\bar{x}) = \text{cl } \mathcal{L}_f(\bar{x})$.

Proof. Clearly, $Q_f(\bar{x}) = \text{cl}(\mathcal{L}_f(\bar{x}))$. Let us prove the first equality. Let $y \in \text{cl}(E_f(\bar{x}))$ and select a sequence $(y_k)_k$ in $E_f(\bar{x})$ converging to y . Therefore, for all $k > 0$, we have $y_k \in \mathcal{L}_f(\bar{x})$ and

$$y_k <_C \sup \mathcal{L}_f(\bar{x}).$$

Thus, $y \in \text{cl}(\mathcal{L}_f(\bar{x}))$ and $y \leq_C \sup \mathcal{L}_f(\bar{x})$, which means that $y \in Q_f(\bar{x})$ and therefore $\text{cl}(E_f(\bar{x})) \subseteq H_f(\bar{x})$.

Now let $\bar{y} \in H_f(\bar{x})$. We have $\bar{y} \in \text{cl} \mathcal{L}_f(\bar{x})$ and

$$\bar{y} \leq_C \sup \mathcal{L}_f(\bar{x}).$$

Hence, there exists a sequence $(y_k)_k$ in $\mathcal{L}_f(\bar{x})$ converging to $\bar{y}$.

Consider now a sequence $(\nu_k)_k > 0$ of real numbers converging to 0 and take, for all $k > 0$, $\bar{y}_k := y_k - \nu_k$. As $\mathcal{L}_f(\bar{x})$ is free disposal we have $\bar{y}_k \in \mathcal{L}_f(\bar{x})$. Obviously,

$$\bar{y}_k <_C \sup \mathcal{L}_f(\bar{x}).$$

Accordingly,

$$\bar{y} \in \text{cl} E_f(\bar{x}).$$

The proof of the Lemma is therefore complete. $\square$

Proof. of Theorem 5.2: we first establish the part " $(a_1) \Leftrightarrow (a_2)$ ". Let $\bar{\beta} \in \mathcal{G}_f(\bar{x})$.

First case: if $\bar{\beta} \in \mathcal{L}_f(\bar{x})$, according to Lemma 5.4, it suffices to consider $\bar{\beta} \in E_f(\bar{x})$, that is $\bar{\beta} <_C \sup \mathcal{L}_f(\bar{x})$.

We have, $\bar{\beta} <_C \sup \mathcal{L}_f(\bar{x})$. Thanks to Lemma 5.1, there exists a sequence (β_k) converging to β such that $\beta_k \in \mathcal{L}_f(x_k)$ for all $k \geq k_0$. Thus,

$$\bar{\beta} \in \liminf_k \mathcal{L}_f(x_k),$$

and thus

$$\bar{\beta} \in \liminf_k \mathcal{G}_f(x_k).$$

Second case: if $\beta \in \mathcal{H}_f^s(\bar{x})$, clearly, $\beta <_C f(\bar{x})$. f being supposed to be lower semicontinuous at $\bar{x}$, it follows that

$$f(\bar{x}) \leq_C \sup \mathcal{L}_f(\bar{x}),$$

and consequently $\beta <_C \sup \mathcal{L}_f(\bar{x})$. Let us first prove that $\beta \in \text{cl}[\mathcal{L}_f(\bar{x})]$. Indeed, take $\alpha \in \mathbb{R}$ and $\xi \in F^*$ such that $\langle \xi, \mathcal{L}_f(\bar{x}) \rangle \leq \alpha$. As

$$J_f(\bar{x}) - \beta \in \text{Int } C$$

and since $\mathcal{L}_f(\bar{x})$ is free disposal, it follows that

$$\beta = J_f(\bar{x}) - (J_f(\bar{x}) - \beta) \in \mathcal{L}_f(\bar{x}),$$

and consequently

$$\langle \xi, \beta \rangle \leq \alpha.$$

Equivalently, $\beta \in \text{cl} \mathcal{L}_f(\bar{x})$. According to the above first case,

$$(5.3) \quad \text{cl}[\mathcal{L}_f(\bar{x})] \subseteq \text{cl}[\liminf_k \mathcal{L}_f(x_k)]$$

$$(5.4) \quad = \liminf_k \mathcal{L}_f(x_k) \subseteq \liminf_k \mathcal{G}_f(x_k).$$

For the proof of the part " $(a_2) \Rightarrow (a_1)$ ", let (x_k) be a converging sequence to $\bar{x}$. Observing that

$$f(\bar{x}) \in \text{cl}[\mathcal{H}_f^s(\bar{x})],$$

it follows that,

$$f(\bar{x}) \in \text{cl}[\mathcal{G}_f(\bar{x})] \subseteq \liminf_k \mathcal{G}_f(x_k).$$

Otherwise, there would exist a sequence (β_k) in $\mathcal{G}_f(x_k)$ converging to $f(\bar{x})$. As $\beta_k \in \mathcal{G}_f(x_k)$ then either $\beta_k \in \mathcal{L}_f(x_k)$, or $\beta_k \in \mathcal{H}_f^s(x_k)$. This leads to $\beta_k \leq_C f(x_k)$ or $\beta_k <_C f(x_k)$ and in any case, $\beta_k \leq_C f(x_k)$ for all k . Therefore, by the characterization of [3], f is lower semicontinuous at $\bar{x}$, establishing (a_1) .

We shall prove now that $(a_1) \Rightarrow (a_3) \Rightarrow (a_4) \Rightarrow (a_1)$ and then $(a_4) \Leftrightarrow (a_5)$.

" $(a_1) \Rightarrow (a_3)$ ". f being lower semicontinuous at $\bar{x}$, it follows that $f(\bar{x}) \in \mathcal{L}_f(\bar{x})$. Since $f(\bar{x})$ is an upper bound of $\mathcal{L}_f(\bar{x})$, it follows that $f(\bar{x}) = \sup \mathcal{L}_f(\bar{x})$, (a_3) is then satisfied.

Now assume (a_3) holds. Clearly, we always have $\mathcal{L}_f(\bar{x}) \subset \mathcal{H}_f(\bar{x})$. On the other hand, as $\text{cl} \mathcal{L}_f(\bar{x})$ is a convex closed, it can be rewritten as the intersection of all closed half spaces containing $\mathcal{L}_f(\bar{x})$. Therefore (a_3) implies that $\mathcal{H}_f(\bar{x}) \subset \text{cl} \mathcal{L}_f(\bar{x})$, concluding that $\mathcal{H}_f(\bar{x}) = \text{cl} \mathcal{L}_f(\bar{x})$. Therefore, using Lemma 5.1, it follows that

$$\mathcal{H}_f(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{H}_f(x),$$

establishing (a_4) .

Suppose that (a_4) is true and consider a sequence (x_k) converging to $\bar{x}$. By assumption we have,

$$f(\bar{x}) \in \liminf_{x \rightarrow \bar{x}} \mathcal{H}_f(x).$$

Thus, there exists a sequence (y_k) such that $y_k \in \mathcal{H}_f(x_k)$ and (y_k) converges to $f(\bar{x})$, which means that f is lower semicontinuous at $\bar{x}$.

" $(a_4) \Rightarrow (a_5)$ ". Using (a_4) and the definition of $\mathcal{L}_f(\bar{x})$ we have $\mathcal{H}_f(\bar{x}) \subset \mathcal{L}_f(\bar{x})$. The opposite inclusion is always true and then (a_5) holds.

$(a_5) \Rightarrow (a_4)$ Thanks to Lemma 5.3 and (a_5) we get easily

$$\mathcal{H}_f(\bar{x}) = \mathcal{L}_f(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{H}_f(x).$$

The proof is then finished. $\square$

5.2. A regularization. We now consider a vector-valued mapping $f : E \rightarrow F^\bullet$. The existence of an ideal efficient point (resp. supremum) of $\text{cl}[\mathcal{L}_f(\bar{x})]$ is crucial to define a regularization of f , which will be proven now to be lower semicontinuous. Precisely, we claim the following:

Theorem 5.5. *Assume F that for every $x \in \text{Dom } f$, $\mathcal{L}_f(x)$ is nonempty and admits a supremum $\sup \mathcal{L}_f(x) = J_f(x)$. Then*

- a) J_f is lower semicontinuous at every point of its domain;
- b) f is necessarily locally C -lower bounded;
- c) f is l.s.c at $x \in \text{Dom } f$ if, and only if $f(x) = J_f(x)$;
- d) J_f is the greatest lower semicontinuous map minorizing f , i.e., if $g : E \rightarrow F^\bullet$ is a lower semicontinuous map such that $g \leq_C f$, then necessarily $g \leq_C J_f$.

Proof. We start by proving a) : Put $g(x) = J_f(x)$ for each $x \in \text{Dom } f$ and take $\bar{x} \in \text{Dom } f$. Observe first that for every $x \in \text{Dom } f$, $\mathcal{H}_g(x) = \text{cl } \mathcal{L}_f(x)$ (Clearly, $\mathcal{L}_f(x) \subset \mathcal{H}_g(x)$, the other inclusion is obtained by regarding $\mathcal{L}_f(x)$ as the intersection of all closed half spaces containing $\mathcal{L}_f(x)$.)

According to Lemma 5.3, for all (x_k) converging to $\bar{x}$,

$$\text{cl } \mathcal{L}_f(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{L}_f(x_k).$$

Then,

$$\mathcal{H}_g(\bar{x}) \subset \liminf_{x \rightarrow \bar{x}} \mathcal{H}_g(x_k).$$

It follows from Theorem 5.2 that g is lower semicontinuous.

For b), according to a), J_f is lower semicontinuous. Thus, as $\text{Int } C \neq \emptyset$, from [Proposition 2.12, [13]] we deduce that J_f is locally C -lower bounded. Then, f is also locally C -lower bounded. c) follows from Theorem 5.2.

Now, let g be a l.s.c minorant of f . Observe, for $x \in \text{Dom } f$, that $\mathcal{L}_g(x) \subset \mathcal{L}_f(x)$. Accordingly, $J_g(x) \leq_C J_f(x)$. By c) we have $J_g(x) = g(x)$. The proof is therefore complete. $\square$

Remark 5.6. Notice that we have already remarked that a sufficient condition for $\mathcal{L}_f(\bar{x})$ to be nonempty is to assume that f is locally C -lower bounded, now Theorem 5.5 shows that the condition is necessary

Now, we present a special case where the regularization preserves the convexity: Let H be a separable Hilbert space and let $(e_n)_{n \in \mathbb{N}}$ be a basis of H . We suppose that the order is defined by the closed convex cone H_+ given by

$$H_+ = \{x \in H \mid \langle e_p, x \rangle \geq 0, \forall p \in \mathbb{N}\}.$$

Let $f : E \rightarrow F^\bullet$ be a H_+ -convex map. We assert that J_f is H_+ -convex. Indeed, observe first that

$$\text{epi} \langle e_p, J_f \rangle = \text{clepi} \langle e_p, f \rangle, \quad \forall p \in \mathbb{N}.$$

Since f is H_+ -convex, for each $p \in \mathbb{N}$, $\langle e_p, f \rangle$ is convex. This yields $\langle e_p, J_f \rangle$ is convex for each $p \in \mathbb{N}$, i.e., J_f is H_+ -convex.

5.3. A quasi-lower semicontinuous regularization. F is order complete

Since the regularizations that preserve the convexity are of interest, we will present in the following paragraph a regularization of this kind which is not unfortunately lower semicontinuous but quasi-lower semicontinuous. Consider therefore, for $f : E \rightarrow F^\bullet$, the map defined for $x \in E$ by

$$(5.5) \quad f_*(x) = \inf \{y \in F^\bullet \mid (x, y) \in \text{cl } \text{gph}(f)\}$$

$$(5.6) \quad = \inf \{y \in F^\bullet \mid (x, y) \in \text{clepi}(f)\}.$$

Proposition 5.7. Assume that F is order complete. Let $f : E \rightarrow F^\bullet$ be a C -convex map. Assume that $\text{Int } C \neq \emptyset$. Then f_* is quasi-lower semicontinuous. In addition, f_* is the greatest quasi-lower semicontinuous regularization minorizing f i.e, if a map g is quasi-lower semicontinuous and $g \leq_C f$, then $g \leq_C f_*$.

Proof. Observe that $\text{epi}(f_*) = \text{clepi}(f)$. Then, as f is C -convex, so $\text{epi}(f)$ is convex. Thus, $\text{epi}(f_*)$ is necessarily convex, which implies that f_* is C -convex. To complete the proof, it suffices to observe that if g is quasi-lower semicontinuous map minorizing f , then $g_* = g$ and then $g \leq_C f_*$. $\square$

Remark 5.8. Notice that f_* can also be defined as follows :

$$f_*(x) = \text{Inf} \{ \mu \in F^\bullet \mid x \in \text{cl} \{y \mid f(y) \leq_C \mu\} \}.$$

Remark 5.9. Suppose F is order complete and semilattice. Remark that $J_f \leq_C f_* \leq_C f$. If in addition F is Daniell, then $J_f = \bar{f}$.

6. COMMENT

A dual treatment would allow to define and obtain upper semicontinuous regularizations. Notice also that all regularizations, as well as different concepts of semicontinuity, presented here agree with those of extended-real-valued functions.

REFERENCES

1. Akian, M. and Singer, I. (2003), Topologies on lattice ordered groups, separations from closed downwards and conjugations of type Lau, *Optimization*, Vol 52, 6, 629-673.
2. M. Ait Mansour, A. Metrane and M. Théra: Lower semicontinuous regularization for vector valued maps, submitted.
3. Combari, C., Laghdir, M. et Thibault, L. : Sous-différentiel de fonctions convexes composées, *Ann. Sci. Math. Québec* 18, No 2, (1994) 119-148.
4. H. W. Corley, An Existence Result for Maximization with respect to cones, *Journal of Optimization Theory and Applications*, Vol. 31, pp. 227-281, 1980.
5. Luc, D. T. : Theory of vector optimization. Springer-verlag, Berlin Heidelberg New-York London Paris Tokyo (1989) .
6. Luc, D. T.: An existence theorem in vector optimization, *Math. Oper. Res.* **14** (1989), 693-699.
7. Ellaia, R. and Hiriart-Urruty, J.B. : The conjugate of difference of convex functions. *JOTA*, Vol 49 N° 3 (1986) .
8. Hiriart-Urruty, J.B. : From convex optimization to Nonconvex optimization. *Nonsmooth Optimization and related topics*. Editors F. H. Clarke, V. F. Demyanov, F. Gnanessi, Plenum Press, (1989), 219-239.
9. Penot, J-P: *L'optimization à la Pareto: Deux ou Trois choses que je sais d'elle*, Structures économiques et économétrie, Lyon, 18-19, Mai 1978.
10. Penot, J-P. et Théra, M. : Semi-continuous mappings in general topology, *Arch-Math.*, Vol.38, (1982) , 158-166.
11. Schaefer, H. H. (1974), *Banach lattices and positive operators*, Springer-Verlag, Berlin Heidelberg New-York.
12. Rockafellar, R. T. Wets R. J-B. : *Variational Analysis*. Springer-verlag, Berlin Heidelberg New-York, (1998).
13. Théra, M. Etude des Fonctions Convexes Vectorielles Semi-Continues, Doctorat de Troisième Cycle, N 76, Université de Pau et Des Pays de L'Adour, 1978.

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