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# LOWER SEMICONTINUOUS REGULARIZATION FOR VECTOR-VALUED MAPPINGS

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*Dedicated to Alex Rubinov*

ABSTRACT. The concept of the lower limit for vector-valued mappings is the main focus of this work. We first introduce a new definition of adequate lower and upper level sets for vector-valued mappings and establish some of their topological and geometrical properties. Characterization of semicontinuity for vector-valued mappings is thereafter presented. Then, we define the concept of vector lower limit, proving its lower semicontinuity, and furnishing in this way a concept of lower semicontinuous regularization for mappings taking their values in a complete lattice. The results obtained in the present work subsume the standard ones when the target space is finite dimensional. In particular, we recapture the scalar case with a new flexible proof. In addition, extensions of usual operations of lower and upper limits for vector-valued mappings are explored. The main result is finally applied to obtain a continuous D.C. decomposition of continuous D.C. mappings.

## 1. INTRODUCTION

The concept of lower semicontinuity introduced for scalar functions by R. Baire is a fundamental ingredient in different fields of mathematical analysis. It has been used in different contexts and in particular by D. Hilbert and L. Tonelli in the calculus of variations. The increasing

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developments of optimization theory, especially Pareto optimization has manifested the strong need to extend this concept to vector-valued mappings. This has motivated a quite number of mathematicians to tackle this topic. The first efforts in this direction go back to Théra [21] (1978), G. Gierz et al [9] (1980), Penot and Théra [17] (1982), G. Gerrits [8] (1985), H. Holwerda [10] (1989), D.T. Luc [12] (1989), J. M. Borwein and Théra [3] (1990), Combari, Laghdir and Thibault [4] (1994). We refer also to the recent contributions of M. Akian [1] (1999) and M. Akian and I. Singer [2] (2003).

A basic fact in real analysis is that every real-valued function  $f$  not necessarily lower semi-continuous, admits a lower semicontinuous regularization, l.s.c regularization for short, defined by means of the lower limit of  $f$  :

$$(1.1) \quad \bar{f}(x) := \liminf_{y \rightarrow x} f(y).$$

A very natural and challenging question is, therefore, to determine a concept of l.s.c regularization for vector-valued mappings. It seems that, since the contribution of Théra [21], in which he provided some types of l.s.c regularizations for mappings with values in order complete Daniell spaces and lattices Daniell spaces, a little bit of attention has been focused on the topic. Therefore, there is still a need to advance this direction in the general setting of mappings with values in partially ordered spaces not necessarily Daniell.

The main scope of this paper is then to define an appropriate l.s.c regularization for mappings with values in a complete Banach lattice. Thus, after defining a suitable lower limit, we will mostly be devoted in proving its semicontinuity.

On the way to do it, inspired by the ideas of Penot and Théra [17] and motivated by Combari *et al* [4], we introduce a concept of lower and upper "level" sets. We first study these sets, show that they own nice properties, both topological and geometrical, and establish the link between them and semicontinuity. Then, we succeed in defining the concept of lower limit for a vector-valued mapping  $f$  at a point  $x$  in its domain. We will use the notation  $v\text{-}\liminf_{y \rightarrow x} f(y)$  rather than the standard one  $\liminf_{y \rightarrow x} f(y)$  in order to do make clear that we are in the framework of vector-valued mappings.

We now outline the plan of the remaining contents of the work, which we organize in nine sections. Section 2 includes the notations and most necessary definitions used later. Section 3 introduces adequate lower and upper level sets and illustrates them by some examples. The fourth section deals with characterizations of semicontinuity for vector-valued mappings. Section 5 presents a study of some of the topological and geometrical properties of the lower level set. In the following Section, 6, we reach our goal by proving that the vector lower limit we consider defines a l.s.c regularization for a given vector-valued map. We begin with Hilbert-valued maps and then consider Banach lattice-valued ones. In Section 7 we check that our contribution can be viewed as an extension of standard results. In particular, we recapture the scalar case with a more flexible proof. In Section 8, we extend the usual operations of estimation of lower and upper limits of the sum of two vector-valued mappings. Section 9 aims to apply the main result to obtain a continuous decomposition for D.C.-mappings in the vector case, which extends some previous results in the literature.

## 2. NOTATIONS AND DEFINITIONS

Throughout this paper,  $E$  and  $F$  are real-vector topological spaces. For a subset  $S$  in  $E$  or  $F$ ,  $\text{Int } S$  and  $\text{cl } S$  denote the interior and the closure of  $S$ , respectively. Let  $C \subset F$  be a closed and convex cone, which is supposed to be pointed, that is  $C \cap -C = \{0\}$ , and with nonempty interior. The cone  $C$  defines a partial order on  $F$  denoted by  $\leq_c$  and defined by

$$(2.1) \quad y_1 \leq_c y_2 \Leftrightarrow y_2 \in y_1 + C.$$

The positive polar cone  $C_+^*$  of  $F$  is defined by

$$(2.2) \quad C_+^* = \{y^* \in F^* : \langle y^*, y \rangle \geq 0 \text{ for all } y \in C\},$$

where  $F^*$  is the continuous dual of  $F$  and  $\langle \cdot, \cdot \rangle$  the corresponding duality pairing. The set  $C_0$  used in the sequel is defined by

$$C_0 := \{w \in C : \xi(w) > 0, \forall \xi \in C_+^* \setminus \{0\}\}.$$

**Remark 2.1.** Notice that we have  $\text{Int } C \subset C_0$ , we refer for instance to [15].

$F^\bullet$  will stand for  $F \cup \{+\infty\}$ , where  $+\infty$  denotes the greatest element of  $F$  with respect to the order  $\leq_c$ . We will write  $x <_c y$ , for  $x, y \in E$ , if  $y - x \in \text{Int } C$ . The order between subsets in  $F$  is defined as follows:

**Definition 2.2.** Let  $A$  and  $B$  be two subsets of  $F^\bullet$ . We use the notation  $A \leq_c B$ , if for each  $x \in A$  and each  $y \in B$ , we have  $y - x \in C$ .

It is equally worth to recall that a subset  $A$  of  $F$  is said to be *directed upwards* if for every  $a, b$  in  $A$  there exists  $c \in A$  such that  $a \leq_c c$  and  $b \leq_c c$ . By analogy, *directed downwards* subsets can be defined.

For a given subset  $A \subset F$ , there may, or may not, be  $a \in F$  with the following property :

for every  $c \in F$ ,  $a \leq_c c$  if and only if  $b \leq_c c$  for every  $b \in A$ ,

that is,  $a \leq_c c$  if, and only if,  $c$  is an upper bound for  $A$ . Obviously, if  $a$  exists it is unique; it is called the *least upper bound* of  $A$  and denoted by  $\sup_{\leq_c} A$  or simply  $\sup A$  if there is no risk of confusion on the order. In a similar way,  $\inf A$ , whenever it exists, is called the *greatest lower bound* of  $A$  and it is the element of  $F$  such that

for every  $c \in F$ ,  $c \leq_c \inf A$  if and only if  $c \leq_c b$  for every  $b \in A$ .

We shall recall that  $F$  is called a *lattice* whenever  $\sup\{a, b\}$  and  $\inf\{a, b\}$  exist for all elements  $a, b$  in  $F$ . It follows that finite subsets in a lattice have a least upper bound and a greatest lower bound. Finally, a lattice such that non-empty subsets bounded from above and directed upwards have a least bound are called *complete lattice* or else *Dedekind lattice*<sup>1</sup>. We point out here that no confusion should be done between complete lattice and totally ordered spaces.

The domain of a function,  $f : E \rightarrow F^\bullet$ , is denoted by  $\text{Dom } f$  and is defined by

$$\text{Dom } f = \{x \in E \mid f(x) <_c +\infty\},$$

and its epigraph by

$$(2.3) \quad \text{epi } f = \{(x, y) \in E \times F \mid y \in f(x) + C\}.$$

Recall now the following definitions:

**Definition 2.3.**  $f$  is said to be  $C$ -convex, if for every  $\alpha \in [0, 1]$  and  $x_1, x_2 \in E$  one has

$$(2.4) \quad \alpha f(x_1) + (1 - \alpha) f(x_2) \in f(\alpha x_1 + (1 - \alpha) x_2) + C.$$

**Definition 2.4.** A mapping  $f : E \rightarrow F$  is said to be  $C$ -D.C., if there exist two  $C$ -convex mappings  $g$  and  $h$  such that:

$$f(x) = g(x) - h(x) \quad \forall x \in E.$$

The pair  $(g, h)$  of  $C$ -convex maps will be called a  $C$ -D.C. decomposition of  $f$ .

We recall now the definitions of lower semicontinuity and sequential lower semicontinuity of a vector-valued mapping introduced respectively in [17] and [4].

<sup>1</sup>Some authors use the terminology of conditionally complete lattice as in [2].

**Definition 2.5.** [17] A mapping  $f : E \rightarrow F^\bullet$  is said to be *lower semicontinuous* (l.s.c) at  $\bar{x} \in E$ , if for any neighborhood  $V$  of zero and for any  $b \in F$  satisfying  $b \leq_c f(\bar{x})$ , there exists a neighborhood  $U$  of  $\bar{x}$  in  $E$  such that

$$f(U) \subset b + V + C \cup \{+\infty\}.$$

**Remark 2.6.** Following [17], if  $f(\bar{x}) \in F$  then Definition 2.5 amounts to saying that for all neighborhood  $V$  of zero (in  $F$ ), there exists a neighborhood  $U$  of  $\bar{x}$  such that

$$(2.5) \quad f(U) \subset f(\bar{x}) + V + C \cup \{+\infty\}.$$

**Definition 2.7.** [4] A mapping  $f : E \rightarrow F$  is said to be *sequentially lower semicontinuous* (s-l.s.c) at  $\bar{x} \in E$ , if for any  $b \in F$  satisfying  $b \leq_c f(\bar{x})$  and for any sequence  $(x_n)$  of  $E$  which converges to  $\bar{x}$ , there exists a sequence  $(b_n)$  (in  $F$ ) converging to  $b$  and satisfying  $b_n \leq_c f(x_n)$ , for every  $n \in \mathbb{N}$ .

**Remark 2.8.** For  $\bar{x} \in \text{Dom } f$ , Definition 2.7 can be expressed simply as follows: For each sequence  $(x_n)$  converging to  $\bar{x}$ , there exists a sequence  $(b_n)$  converging to  $f(\bar{x})$  such that  $b_n \leq_c f(x_n)$  for all  $n \in \mathbb{N}$ .

Note that it has been proved in [4] that Definitions 2.5 et 2.7 coincide whenever  $E$  and  $F$  are metrizable.

**Definition 2.9.**  $F$  is said to be *normal* if  $F$  has a basis of order-convex neighborhood of zero of the form  $V = (V + C) \cap (V - C)$ .

**Remark 2.10.** It is worth mentioning as well that:

- The sequential upper semicontinuity of  $f$  (s-u.s.c for brevity) is defined by saying that  $-f$  is s-l.s.c.
- If  $(F, C)$  is *normal*, one may check that  $f$  is sequentially continuous at  $\bar{x} \in E$  with  $f(\bar{x}) \in F$ , if and only if  $f$  is s-l.s.c and s-u.s.c at  $\bar{x}$ .
- Whenever  $E$  is metrizable and  $F = \mathbb{R}$ , the s-l.s.c continuity coincides with the classical lower semicontinuity. In this case, a function is s-l.s.c at every point of  $E$  if, and only if its epigraph is closed in  $E \times F$ .
- Note that every l.s.c vector-valued mapping has a closed epigraph (see [3]), but the converse is not true as the following counterexample furnished in [17] shows:

The mapping  $h : \mathbb{R} \rightarrow \mathbb{R}^2$  defined by :

$$h(x) = \begin{cases} (0, 0) & \text{if } x = 0 \\ \left( \frac{1}{|x|}, -1 \right) & \text{otherwise,} \end{cases}$$

is not not l.s.c at  $(0, 0)$  while its epigraph (with respect to  $C = \mathbb{R}_+^2$ ) is closed.

We end up these preliminaries by recalling that the lower part of Painlevé-Kuratowski set-convergence, of a sequence  $(A_n)$  of subsets of  $F$ , is given by

$$\liminf_n A_n = \{y \in F : y = \lim_n y_n, \text{ there exists } n_0 : \text{for all } n \geq n_0, y_n \in A_n\}.$$

The sequence  $(A_n)$  will be said lower convergent to  $A \subset F$  in the sense of Painlevé-Kuratowski if, and only if

$$A \subset \liminf_n A_n.$$

### 3. ADEQUATE LOCAL LOWER AND UPPER LEVEL SETS

In the present section, on the way to our objective, we introduce adequate notions of local lower and upper "level" sets for vector-valued mappings defined from  $E$  into  $F^\bullet$ .

**Definition 3.1.** Let  $f$  be an extended-vector-valued mapping and  $\bar{x} \in \text{Dom } f$ . Denoting by  $\vartheta(\bar{x})$  the family of neighborhoods of  $\bar{x}$ , we introduce the following "level" sets:

$$(3.1) \quad A_{\bar{x}}^f := \{y \in F \mid \forall V \in \vartheta(y), \exists U \in \vartheta(\bar{x}), f(U) \subset V + C \cup \{+\infty\}\};$$

$$(3.2) \quad B_{\bar{x}}^f := \{y \in F \mid \forall V \in \vartheta(y), \exists U \in \vartheta(\bar{x}), f(U) \subset V - C \cup \{+\infty\}\};$$

$$(3.3) \quad s - A_{\bar{x}}^f := \{y \in F \mid \forall (x_n) \rightarrow \bar{x}, \exists (b_n) \rightarrow y, b_n \leq_c f(x_n) \quad \forall n \in \mathbb{N}\};$$

$$(3.4) \quad s - B_{\bar{x}}^f := \{y \in F \mid \forall (x_n) \rightarrow \bar{x}, \exists (b_n) \rightarrow y, b_n \geq_c f(x_n) \quad \forall n \in \mathbb{N}\}.$$

**Remark 3.2.** In order to illustrate these definitions we present the following examples:

**Example 1:**

Let  $f$  be the real-valued function defined by:

$$f(x) = \begin{cases} x & \text{if } x < 0 \\ x + 1 & \text{otherwise.} \end{cases}$$

Note that  $f$  is not lower semicontinuous and observe that

$$A_x^f = \begin{cases} ]-\infty, x] & \text{if } x \leq 0 \\ ]-\infty, x + 1] & \text{otherwise.} \end{cases}$$

**Remark 3.3.** Let us point out that for any extended real-valued function  $f$ ,  $\text{cl } A_x^f = ]-\infty, \liminf_{x \rightarrow \bar{x}} f(x)]$ . In Section 7 we present the proof in details for finite dimensional-valued functions.

**Example 2:**

Let  $H$  be a separable Hilbert space and let  $(e_n)_{n \in \mathbb{N}}$  be a basis of  $H$ . We suppose that the order is defined by the closed convex cone  $H_+$  given by

$$H_+ = \{x \in H \mid \langle e_p, x \rangle \geq 0, \forall p \in \mathbb{N}\}.$$

Denoting  $\langle e_i, f \rangle$  by  $f_i$ , we consider the function  $f : \mathbb{R} \longrightarrow H$  defined by

$$\begin{cases} f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise} \end{cases} \\ f_i(x) = 0 \end{cases} \quad \text{if } i \neq 1.$$

We observe that  $f$  is lower semicontinuous everywhere except at 0, and we check that<sup>2</sup>

$$A_x^f = \prod_{p=1}^{\infty} \langle e_p, A_x^f \rangle.$$

Indeed,

$$\langle e_p, A_x^f \rangle = A_x^{\langle e_p, f \rangle}.$$

Then, for  $p > 1$ , we have

$$A_x^{\langle e_p, f \rangle} = A_x^{f_p} = ]-\infty, 0].$$

For  $p = 1$ , we see that

$$A_x^{\langle e_1, f \rangle} = A_x^{f_1} = ]-\infty, 0].$$

Hence,

$$A_x^f = \prod_{p=1}^{\infty} \langle e_p, A_x^f \rangle = \prod_{p=1}^{\infty} ]-\infty, 0] = -H_+.$$

**Example 3:**

Consider the mapping  $f = (f_1, f_2) : \mathbb{R} \longrightarrow \mathbb{R}^2$  defined by

$$\begin{cases} f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{otherwise.} \end{cases} \\ f_2(x) = x. \end{cases}$$

A straightforward calculation shows that  $\text{cl } A_x^f = ]-\infty, 0] \times ]-\infty, x]$  if  $x \leq 0$  and  $\text{cl } A_x^f = ]-\infty, 1] \times ]-\infty, x]$  otherwise.

<sup>2</sup>as it is done later in Lemma 6.2.

Remark that  $\text{cl } A_x^f$  is convex, directed upwards and in addition we have

$$A_x^f = A_x^f - \mathbb{R}_+^2.$$

In Section 5, we will prove that these properties of this lower "level" set hold in general.

In the next section, we characterize the semicontinuity of vector-valued mappings in terms of the above level sets.

#### 4. CHARACTERIZATION OF SEMICONTINUITY FOR VECTOR-MAPPINGS

We begin with the following proposition which gives the link between the level sets and the s-l.s.c, l.s.c, s-u.s.c and u.s.c.

**Proposition 4.1.** Let  $f$  be an extended vector map and  $\bar{x} \in \text{Dom } f$ . Then,

- (1)  $f$  is s-l.s.c at  $\bar{x}$  if, and only if,  $f(\bar{x}) \in s - A_{\bar{x}}^f$ ;
- (2)  $f$  is l.s.c en  $\bar{x}$  if, and only if,  $f(\bar{x}) \in A_{\bar{x}}^f$ ;
- (3)  $f$  is s-u.s.c at  $\bar{x}$  if, and only if,  $f(\bar{x}) \in s - B_{\bar{x}}^f$ ;
- (4)  $f$  is s.c.s at  $\bar{x}$  if, and only if,  $f(\bar{x}) \in B_{\bar{x}}^f$ .

*Proof.* The proof follows from the definitions. □

Now, we can determine the level sets of semicontinuous extended-vector-valued mappings. First, we establish the equivalence between  $s-A_{\bar{x}}^f$  and  $A_{\bar{x}}^f$  once  $E$  and  $F$  are metrizable.

**Proposition 4.2.** Assume that  $E$  and  $F$  are metrizable. Let  $f : E \rightarrow F^\bullet$ ,  $\bar{x} \in \text{Dom } f$ . Then, we have

$$s - A_{\bar{x}}^f = A_{\bar{x}}^f \quad \text{and} \quad s - B_{\bar{x}}^f = B_{\bar{x}}^f.$$

*Proof.* Let  $y \in s - A_{\bar{x}}^f$  and consider  $y \notin A_{\bar{x}}^f$ . There exists a neighborhood  $V$  of zero (in  $F$ ) such that for every neighborhood  $U$  of  $\bar{x}$ ,  $f(U)$  is not included in  $y + V + C \cup \{+\infty\}$ . For  $n \in \mathbb{N}^*$ , take  $U_n = B(\bar{x}, \frac{1}{n})$  and select  $x_n \in B(\bar{x}, \frac{1}{n})$  such that

$$(4.1) \quad f(x_n) - y \notin V + C \cup \{+\infty\}.$$

As  $(x_n)$  converges to  $\bar{x}$  and  $y \in s - A_{\bar{x}}^f$ , there exists a sequence  $(b_n)$  in  $F$  such that

$$(4.2) \quad \lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

Thus, for  $n$  large enough i.e,  $n \geq N$ , we get

$$(4.3) \quad b_n - y \in V, \quad \forall n \geq N.$$

Using (4.3) and the fact that

$$b_n - y \leq_c f(x_n) - y, \quad \forall n \geq N,$$

we obtain

$$f(x_n) - y \in V + C \cup \{+\infty\}, \quad \forall n \geq N,$$

a contradiction with (4.1). Then  $y \in A_{\bar{x}}^f$ .

Conversely, let  $y \in A_{\bar{x}}^f$ . Then, for each neighborhood  $V$  of origin, there exists a neighborhood  $U$  of  $\bar{x}$  such that

$$f(U) \subset y + V + C \cup \{+\infty\}.$$

For each integer  $k \geq 1$ , take  $V_k = B(0, \frac{1}{k})$  and select a neighborhood  $U_k$  of  $\bar{x}$  verifying

$$(4.4) \quad f(U_k) \subset y + B\left(0, \frac{1}{k}\right) + C \cup \{+\infty\}.$$

Let  $(x_n)$  be a sequence converging to  $\bar{x}$ . For every  $k \geq 1$ , there exists an integer  $N_k$  such that

$$(4.5) \quad \forall n \geq N_k, \quad x_n \in U_k.$$

Consider the following sequence of integers  $S(k)$ :

$$\begin{aligned} S(1) &= N_1 \\ S(k+1) &= \max[S(k) + 1, N_{k+1}]. \end{aligned}$$

(Remark that  $S(k)$  is strictly increasing). For some  $k \in \mathbb{N}^*$ , for  $n \in \{S(k), \dots, S(k+1) - 1\}$ , taking into account relations (4.4) and (4.5), we can pick an element  $v_{n,k} \in B(0, \frac{1}{k})$  such that

$$(4.6) \quad y + v_{n,k} \leq_c f(x_n).$$

Now, for each integer  $n \neq 0$  such that  $n < S(1)$ , pick an element  $b_n \in F$  such that  $b_n \leq_c f(x_n)$ . The sequence  $(y_n)$  given by

$$\begin{aligned} y_n &= b_n - y \quad \text{if } n < S(1) \\ y_n &= v_{n,k} \quad \text{si } n \in [S(k), S(k+1)] \end{aligned}$$

converges to zero, the sequence  $b_n = y_n + y$  converges to  $y$ , and according to relation (4.6) we have

$$b_n \leq_c f(x_n) \quad \forall n \geq 1.$$

Consequently,  $y \in s - A_{\bar{x}}^f$ .

In a similar way, we show that if  $E$  and  $F$  are metrizable, then :

$$s - B_{\bar{x}}^f = B_{\bar{x}}^f,$$

the proof is complete.  $\square$

**Remark 4.3.** In what follows, we assume that  $E$  and  $F$  are metrizable and adopt the notation  $A_{\bar{x}}^f$  for a lower level set.

Next, we prove the following elementary property.

**Proposition 4.4.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Then,

- (1)  $A_{\bar{x}}^f = A_{\bar{x}}^f - C$ ;
- (2)  $B_{\bar{x}}^f = B_{\bar{x}}^f + C$ .

*Proof.* 1) Let  $y \in A_{\bar{x}}^f$ ,  $c \in C$ ,  $(x_n)$  be a sequence in  $E$  converging to  $\bar{x}$ . There exists a sequence  $(b_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

Set  $a_n = b_n - c$  and remark that

$$(4.7) \quad \lim_{n \rightarrow +\infty} a_n = y - c.$$

As  $f(x_n) - b_n \in C \cup \{+\infty\}$  and  $c \in C$  we have

$$f(x_n) - b_n + c \in C \cup \{+\infty\}$$

for every  $n \in \mathbb{N}$ . Consequently

$$(4.8) \quad a_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

Using (4.7) and (4.8), we deduce that  $y - c \in A_{\bar{x}}^f$  for every  $c \in C$  and  $y \in A_{\bar{x}}^f$ , whence

$$A_{\bar{x}}^f - C \subset A_{\bar{x}}^f.$$

The other inclusion is true because  $0 \in C$ .

2) The second equality can be established by inverting the order.  $\square$

**Theorem 4.5.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Then,

- (1)  $f$  is l.s.c at  $\bar{x} \iff A_{\bar{x}}^f = f(\bar{x}) - C$ ;
- (2)  $f$  is u.s.c at  $\bar{x} \iff B_{\bar{x}}^f = f(\bar{x}) + C$ .

*Proof.* ( $\Rightarrow$ )

Suppose that  $f$  is l.s.c at  $\bar{x}$  and  $y \in A_{\bar{x}}^f$ . We can easily see that (as detailed in Proposition 5.1),

$$y - f(\bar{x}) \in -C.$$

Therefore

$$(4.9) \quad A_{\bar{x}}^f \subset f(\bar{x}) - C.$$

Since  $f$  is l.s.c at  $\bar{x}$ ,  $f(\bar{x}) \in A_{\bar{x}}^f$ , by Proposition 4.4, we have

$$(4.10) \quad f(\bar{x}) - C \subset A_{\bar{x}}^f.$$

We hence deduce, via (4.9) and (4.10), that

$$f(\bar{x}) - C = A_{\bar{x}}^f.$$

( $\Leftarrow$ ) Assume that  $A_{\bar{x}}^f = f(\bar{x}) - C$ . As  $0 \in C$ ,  $f(\bar{x}) \in A_{\bar{x}}^f$ . Following Proposition 4.1,  $f$  is lower semicontinuous at  $\bar{x}$ .

2) The second equivalence can be established similarly.  $\square$

**Corollary 4.6.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Assume that  $C$  is pointed and  $(F, C)$  is normal. Then, the assertions below are equivalent.

- i)  $f$  is continuous at  $\bar{x}$ ;
- ii)  $A_{\bar{x}}^f \cap B_{\bar{x}}^f \neq \emptyset$ .

*Proof.* ii) $\Rightarrow$ i).

Let  $y \in A_{\bar{x}}^f \cap B_{\bar{x}}^f$ . Then, for each sequence  $(x_n)$  that converges to  $\bar{x}$ , there exist two sequences  $(a_n)$  and  $(b_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} a_n = y \quad \text{and} \quad a_n \leq_c f(x_n), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad f(x_n) \leq_c b_n, \quad \forall n \in \mathbb{N}.$$

In particular, for the stationary sequence  $x_n = \bar{x}$ , for every  $n \in \mathbb{N}$ , there exists two sequences  $(a'_n)$  and  $(b'_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} a'_n = y \quad \text{and} \quad a'_n \leq_c f(\bar{x}), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow +\infty} b'_n = y \quad \text{and} \quad f(\bar{x}) \leq_c b'_n, \quad \forall n \in \mathbb{N}.$$

Since  $C$  is closed,

$$f(\bar{x}) - y \in C \cap (-C).$$

Since  $C$  is pointed, then  $f(\bar{x}) - y \in A_{\bar{x}}^f \cap B_{\bar{x}}^f$ . Following Remark 4.1,  $f$  is l.s.c and u.s.c at  $\bar{x}$ . The normality of  $(F, C)$  ensures the continuity of  $f$  at  $\bar{x}$ .  $\square$

## 5. PROPERTIES OF LOWER AND UPPER LEVEL SETS

This section is devoted to the study of the behavior of the lower and upper level sets. We show that they have remarkable properties, both topological and geometrical. We focus only on the lower level set, because the properties of the local upper level set can be deduced straightforwardly by similarity.

At first, we provide a bound for  $A_{\bar{x}}^f$  and  $B_{\bar{x}}^f$ .

**Proposition 5.1.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . The following assertion hold:

$$A_{\bar{x}}^f \leq_c \{f(\bar{x})\} \leq_c B_{\bar{x}}^f.$$

*Proof.* Let  $y \in A_{\bar{x}}^f$  and consider the sequence  $(x_n)$  given by  $x_n = \bar{x}$  for every  $n \in \mathbb{N}$ . There exists a sequence  $(b_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq_c f(\bar{x}), \quad \forall n \in \mathbb{N}.$$

Since  $C$  is closed,  $y \leq_c f(\bar{x})$ . Hence,  $A_{\bar{x}}^f \leq_c \{f(\bar{x})\}$ .

In the same manner, we show that  $\{f(\bar{x})\} \leq_c B_{\bar{x}}^f$  by inverting the order.  $\square$

The useful property of directness is also verified, precisely we have the following:

**Proposition 5.2.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Suppose that  $F$  is a Banach lattice. Then,  $A_{\bar{x}}^f$  is directed upwards and  $B_{\bar{x}}^f$  is directed downwards.

*Proof.* Take  $y_1, y_2 \in A_{\bar{x}}^f$  and  $(x_n)$  a sequence in  $E$  converging to  $\bar{x}$ . Then there exist two sequences  $(b_n)$  and  $(b'_n)$  in  $F$  such that

$$(5.1) \quad \lim_{n \rightarrow +\infty} b_n = y_1 \quad \text{and} \quad b_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}$$

and

$$(5.2) \quad \lim_{n \rightarrow +\infty} b'_n = y_2 \quad \text{and} \quad b'_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

As the map :

$$\begin{aligned} \text{sup} : F \times F &\rightarrow F \\ (x, y) &\mapsto \text{sup}(x, y) \end{aligned}$$

is uniformly continuous (see [20] Proposition 5.2 page 83), taking into consideration (5.1) and (5.2), we obtain

$$\lim_{n \rightarrow +\infty} \text{sup}(b_n, b'_n) = \text{sup}(y_1, y_2) \quad \text{and} \quad \text{sup}(b_n, b'_n) \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

It follows that  $\text{sup}(y_1, y_2) \in A_{\bar{x}}^f$ .

□

**Proposition 5.3.** For each  $\xi \in Y^*$ ,  $\xi(\sup A_{\bar{x}}^f) = \sup \xi(A_{\bar{x}}^f)$ .

*Proof.* Let  $\xi \in Y^*$ .  $A_{\bar{x}}^f$  being directed upwards, thanks to Proposition 4.1 of [2], it suffices to check that  $\xi$  is continuous with respect to the Scott topology. Since  $\mathbb{R}$  is continuous (in sense of [2] see example 1.1 in [2] for further details) and  $\xi$  is lower semicontinuous continuous (because continuous), it follows from [Theorem 4.2, [2]] that  $\xi$  is Scott-continuous. The proof is then complete. □

**Proposition 5.4.** Let  $f : E \rightarrow F^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Then  $A_{\bar{x}}^f$  and  $B_{\bar{x}}^f$  are convex.

*Proof.* Take  $y_1, y_2 \in A_{\bar{x}}^f$ ,  $\lambda \in [0, 1]$  and  $(x_n)$  a sequence in  $E$  that converges to  $\bar{x}$ , then there exist two sequences  $(b'_n)$  and  $(b''_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} b'_n = y_1 \quad \text{and} \quad b'_n \leq_c f(x_n), \quad \forall n \in \mathbb{N},$$

and

$$\lim_{n \rightarrow +\infty} b''_n = y_2 \quad \text{and} \quad b''_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}.$$

Take  $b_n = \lambda b'_n + (1 - \lambda) b''_n$ . On one hand we have

$$(5.3) \quad \lim_{n \rightarrow +\infty} b_n = \lambda y_1 + (1 - \lambda) y_2.$$

On the other hand,

$$\begin{aligned} f(x_n) - b_n &= f(x_n) - \lambda b'_n - (1 - \lambda) b''_n \\ &= \lambda (f(x_n) - b'_n) + (1 - \lambda) (f(x_n) - b''_n). \end{aligned}$$

As  $f(x_n) - b'_n \in C \cup \{+\infty\}$  and  $f(x_n) - b''_n \in C \cup \{+\infty\}$ ,

$$(5.4) \quad b_n \leq_c f(x_n), \quad \forall n \in \mathbb{N}$$

we deduce from (5.3) and (5.4) that

$$\lambda y_1 + (1 - \lambda) y_2 \in A_{\bar{x}}^f.$$

Similarly, we can prove that  $B_{\bar{x}}^f$  is convex. □

## 6. THE OBJECTIVE OF THE PAPER

The study of the l.s.c regularization of vector-valued mappings has been initiated by Théra for maps with values in complete (lattice) Daniell spaces. Our ambition here is to define a l.s.c of a vector-valued mapping  $f : E \rightarrow F^\bullet$ , when  $F$  is a complete Banach lattice ordered by a closed convex cone  $C$  with nonempty interior.

Assuming that for every  $\bar{x} \in \text{Dom } f$ ,  $A_{\bar{x}}^f \neq \emptyset$ , let us agree to introduce the following mapping:

$$I_f(\bar{x}) := \sup A_{\bar{x}}^f.$$

**Remark 6.1.**  $F$  being complete lattice,  $A_{\bar{x}}^f$  is upper bounded, it follows that  $I_f$  is well defined i.e.,  $\sup A_{\bar{x}}^f$  exists.

We first begin with the case where  $F = H$  is a separable complete Hilbert lattice space. Let  $(e_n)_{n \in \mathbb{N}}$  be a basis of  $H$ . The order on  $H$  is given by the closed convex cone given by

$$H_+ = \{x \in H \mid \langle e_p, x \rangle \geq 0, \forall p \in \mathbb{N}\}.$$

Clearly, the polar cone of  $H_+$  is equal to  $H_+$ , i.e.,  $H_+^* = H_+$ .

In order to conclude the semicontinuity of  $I_f$  in this first case with the use of semicontinuity of the usual scalar lower limit, we need two ingredients.

**Lemma 6.2.** *Let  $f : E \rightarrow H^\bullet$  and  $\bar{x} \in \text{Dom } f$ . Assume that  $A_{\bar{x}}^f \neq \emptyset$ . Then, for each  $p \in \mathbb{N}$ , one has*

$$\langle e_p, A_{\bar{x}}^f \rangle = A_{\bar{x}}^{\langle e_p, f \rangle},$$

where  $\langle e_p, A_{\bar{x}}^f \rangle := \left\{ \langle e_p, y \rangle \mid y \in A_{\bar{x}}^f \right\}$ .

*Proof.* Let  $p \in \mathbb{N}$ ,  $y \in A_{\bar{x}}^f$  and  $(x_n)$  be a sequence in  $E$  converging to  $\bar{x}$ , then there is a sequence  $(b_n)$  in  $H$  such that

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq f(x_n), \quad \forall n \in \mathbb{N},$$

which implies that

$$\lim_{n \rightarrow +\infty} \langle e_p, b_n \rangle = \langle e_p, y \rangle \quad \text{and} \quad \langle e_p, b_n \rangle \leq \langle e_p, f(x_n) \rangle, \quad \forall n \in \mathbb{N}$$

and therefore  $\langle e_p, y \rangle \in A_{\bar{x}}^{\langle e_p, f \rangle}$ . It follows that

$$(6.1) \quad \langle e_p, A_{\bar{x}}^f \rangle \subset A_{\bar{x}}^{\langle e_p, f \rangle} \quad \text{for each } p \in \mathbb{N}.$$

At this stage, let us show that

$$A_{\bar{x}}^{\langle e_p, f \rangle} \subset \langle e_p, A_{\bar{x}}^f \rangle$$

for each  $p \in \mathbb{N}$ .

So, let  $(x_n)$  be a sequence in  $E$  converging to  $\bar{x}$ ,  $p \in \mathbb{N}$  and  $y \in A_{\bar{x}}^{\langle e_p, f \rangle}$ . Then consider  $(y_n)$  in  $\mathbb{R}$  such that

$$\lim_{n \rightarrow +\infty} y_n = y \quad \text{and} \quad y_n \leq \langle e_p, f(x_n) \rangle, \quad \forall n \in \mathbb{N}.$$

Let  $z_0 \in A_{\bar{x}}^f$  be fixed and take  $z := z_0 + (y - z_0^p) e_p$ , where  $z_0^p = \langle z_0, e_p \rangle$ . Let us show that  $z \in A_{\bar{x}}^f$ .

As  $z_0 \in A_{\bar{x}}^f$ , there exists  $(z_n)$  such that

$$(6.2) \quad \lim_{n \rightarrow +\infty} z_n = z_0 \quad \text{and} \quad z_n \leq f(x_n), \quad \forall n \in \mathbb{N}.$$

Set  $z'_n = z_n + (y_n - z_n^p) e_p$  with  $z_n^p = \langle z_n, e_p \rangle$ . Since  $y \in A_{\bar{x}}^{\langle e_p, f \rangle}$ , according to (6.2) we have

$$\lim_{n \rightarrow +\infty} z'_n = z \quad \text{and} \quad z'_n \leq f(x_n), \quad \forall n \in \mathbb{N}$$

with  $\langle e_p, z \rangle = y$ . We deduce that

$$(6.3) \quad A_{\bar{x}}^{\langle e_p, f \rangle} \subset \langle e_p, A_{\bar{x}}^f \rangle.$$

Using (6.1) and (6.3) we obtain  $A_{\bar{x}}^{\langle e_p, f \rangle} = \langle e_p, A_{\bar{x}}^f \rangle$  for each  $p \in \mathbb{N}$ .  $\square$

**Lemma 6.3.** [17] *Let  $Y = \prod_i Y_i$  be the space product of a family of ordered vector spaces  $Y_i$ . Then, a map  $f : E \rightarrow Y$  is l.s.c, if and only if, its projections  $f_i = p_i \circ f$  are l.s.c.*

Here,  $p_i$  denotes the projection from  $Y$  into  $Y_i$ .

**Theorem 6.4.** *Let  $f : E \rightarrow H^\bullet$  and let  $\bar{x} \in \text{Dom } f$ . Then,  $I_f$  is l.s.c at  $\bar{x}$ .*

*Proof.* According to Proposition 5.3, for all  $p \geq 0$  we have

$$\langle e_p, I_f(\bar{x}) \rangle = \langle e_p, \sup A_{\bar{x}}^f \rangle = \sup \langle e_p, A_{\bar{x}}^f \rangle.$$

Then, using Lemma 6.2 and Remark 3.3, we deduce that

$$\langle e_p, I_f(\bar{x}) \rangle = \sup A_{\bar{x}}^{f_p} = \liminf_{x \rightarrow \bar{x}} f_p(x),$$

where  $f_p = \langle e_p, f \rangle$ . Clearly,  $\langle e_p, I_f \rangle$  is l.s.c for each  $p$ , thus  $f$  is l.s.c.  $\square$

Next, we present the more general case where  $F$  is a complete Banach lattice in which we do not make recourse to semicontinuity of the scalar lower limit.

**Theorem 6.5.** *Suppose  $F$  is a complete Banach lattice. Then,  $I_f$  is lower semicontinuous at every  $x$  in  $\text{Dom } f$ , and therefore  $I_f$  defines a l.s.c regularization of  $f$ .*

We prove first some technical Lemmata that will be useful for proving the main result.

**Lemma 6.6.** *For every convex cone  $C$  in  $F$ , we have*

$$(6.4) \quad (F \setminus C) - C = F \setminus C.$$

*Proof.* The proof is standard and based on the convexity of  $C$ .  $\square$

**Lemma 6.7.** *We have  $I_f(\bar{x}) \in cl A_{\bar{x}}^f$ .*

*Proof.* It suffices to prove for each  $\mu \in \text{Int } C$  that

$$I_f(\bar{x}) - \mu \in cl A_{\bar{x}}^f.$$

To this end, we shall prove that for every  $y \in F$  such that

$$I_f(\bar{x}) - y \in \text{Int } C$$

then  $y \in cl A_{\bar{x}}^f$ .

Assume on the contrary that there exists  $y$  such that

$$I_f(\bar{x}) - y \in \text{Int } C \text{ and } y \notin cl A_{\bar{x}}^f.$$

As  $A_{\bar{x}}^f = A_{\bar{x}}^f - C$  and  $C$  is closed, it follows that

$$cl(A_{\bar{x}}^f) - C \subset cl(A_{\bar{x}}^f - C) = cl A_{\bar{x}}^f,$$

therefore  $0 \notin y - cl(A_{\bar{x}}^f) + C$ . We see that  $y - cl(A_{\bar{x}}^f) + C \neq \emptyset$  and  $y - A_{\bar{x}}^f + C = y - A_{\bar{x}}^f$  is convex (because  $A_{\bar{x}}^f$  is convex)

According to the Hahn-Banach Separation Theorem, select some  $y^* \in F^*$  such that  $y^* \neq 0$  and

$$(6.5) \quad \langle y^*, y - z + r \rangle > 0,$$

for each  $z \in cl A_{\bar{x}}^f$  and  $r \in C$ .

Let us show that  $y^* \in C_+^* \setminus \{0\}$ . Let  $r \in C$  and let  $z \in cl A_{\bar{x}}^f$ . Thanks to (6.5), we deduce

$$\langle y^*, y - z \rangle > -\langle y^*, nr \rangle$$

for each  $n \in \mathbb{N}$ , and therefore

$$\frac{1}{n} \langle y^*, y - z \rangle > -\langle y^*, r \rangle.$$

Letting  $n$  go to  $+\infty$ , we obtain  $\langle y^*, r \rangle \geq 0$  for  $r \in C$ . Thus,

$$y^* \in C^* \setminus \{0\}.$$

Now, let  $r \in C$ . We have  $\frac{1}{n}r \in C$ , consequently

$$\left\langle y^*, y - z + \frac{1}{n}r \right\rangle \geq 0,$$

by passing to the limit whenever  $n$  goes to  $+\infty$ , we obtain

$$\langle y^*, y \rangle \geq \langle y^*, z \rangle \text{ for each } z \in A_{\bar{x}}^f.$$

Therefore,

$$\langle y^*, y \rangle \geq \sup_{z \in A_{\bar{x}}^f} \langle y^*, z \rangle.$$

Since  $A_{\bar{x}}^f$  is directed upwards, it follows from Proposition 5.3 that

$$\langle y^*, I_f(\bar{x}) \rangle = \sup_{y \in A_{\bar{x}}^f} \langle y^*, y \rangle.$$

Accordingly,

$$(6.6) \quad \langle y^*, y \rangle \geq \left\langle y^*, \sup_{z \in A_{\bar{x}}^f} z \right\rangle.$$

Keeping in mind that  $I_f(\bar{x}) - y \in \text{Int } C$  and  $y^* \in C_+^* \setminus \{0\}$ , thanks to Remark 2.1 we conclude that

$$\left\langle y^*, \sup A_{\bar{x}}^f - y \right\rangle > 0,$$

and consequently

$$(6.7) \quad \langle y^*, I_f(\bar{x}) \rangle > \langle y^*, y \rangle.$$

Relations (6.6) and (6.7) provide the contradiction.

Then, for each  $y \in F$  such  $I_f(\bar{x}) - y \in \text{Int } C$ , we have

$$y \in \text{cl } A_{\bar{x}}^f.$$

In particular, taking  $y = I_f(\bar{x}) - \mu$  ( $\mu \in \text{Int } C$ ), We derive that

$$I_f(\bar{x}) - y = \mu \in \text{Int } C,$$

we therefore deduce that

$$I_f(\bar{x}) - \mu \in \text{cl } A_{\bar{x}}^f.$$

$\mu$  being arbitrary in  $\text{Int } C$ , we end up at

$$I_f(\bar{x}) \in \text{cl } A_{\bar{x}}^f.$$

□

As a consequence of the previous Lemma we derive the following technical Lemma.

**Lemma 6.8.** *For each  $\bar{y} \in A_{\bar{x}}^f$  such that  $\bar{y} <_c I_f(\bar{x})$ , there exists a sequence  $(\beta_k)_k$  in  $\text{cl } A_{\bar{x}}^f$  such that*

$$(6.8) \quad \beta_k \rightarrow I_f(\bar{x}) \text{ as } k \text{ goes to } +\infty \text{ and } \bar{y} <_c \beta_k, \text{ for all } k.$$

*Proof.* For  $k > 0$ , take

$$(6.9) \quad \beta_k = I_f(\bar{x}) - \frac{1}{k}(I_f(\bar{x}) - \bar{y}).$$

Observe that

$$\beta_k - \bar{y} = \frac{k-1}{k}(I_f(\bar{x}) - \bar{y}) \in \text{Int } C$$

and conclude that

$$\bar{y} <_c \beta_k, \forall k.$$

On the other hand, Lemma 6.7 implies

$$I_f(\bar{x}) \in \text{cl } A_{\bar{x}}^f.$$

Then, (6.9) leads to

$$\beta_k \in \text{cl}(A_{\bar{x}}^f) - C \subset \text{cl}(A_{\bar{x}}^f - C) = \text{cl } A_{\bar{x}}^f.$$

Clearly,  $(\beta_k)$  converges to  $I_f(\bar{x})$ . The proof of the Lemma is therefore complete.  $\square$

Now, for  $\bar{x} \in \text{Dom } f$ , we introduce the following sets

$$(6.10) \quad E_f(\bar{x}) := \{y \in A_{\bar{x}}^f \mid y <_c I_f(\bar{x})\}$$

and

$$(6.11) \quad H_f(\bar{x}) := \{y \in \text{cl } A_{\bar{x}}^f \mid y \leq_c I_f(\bar{x})\}$$

and establish the following:

**Lemma 6.9.** *We have  $\text{cl } E_f(\bar{x}) = H_f(\bar{x})$ .*

*Proof.* Let  $y \in \text{cl}(E_f(\bar{x}))$ . There exists a sequence  $(y_k)$  converging to  $y$  such that  $y_k \in E_f(\bar{x})$ , for all  $k$ . Clearly,  $y \in \text{cl } A_{\bar{x}}^f$  and

$$(6.12) \quad I_f(\bar{x}) - y_k \in \text{Int } C \subset C \quad \forall k.$$

Passing to the limit in (6.12) whenever  $k$  goes to  $+\infty$ , we obtain that

$$I_f(\bar{x}) - y \in C,$$

or equivalently  $y \in H_f(\bar{x})$ .

Conversely, let  $\bar{y} \in H_f(\bar{x})$ . Then,  $\bar{y} \in \text{cl } A_{\bar{x}}^f$  and  $I_f(\bar{x}) - \bar{y} \in C$ . Therefore, there exists a sequence  $(y_k)$  in  $A_{\bar{x}}^f$  such that  $y_k \rightarrow \bar{y}$ . Consider now a sequence  $(\nu_k)$  in  $\text{Int } C$  such that  $\nu_k \rightarrow 0$ . Take  $\bar{y}_k := y_k - \nu_k$  and observe that  $(\bar{y}_k)_k$  also converges to  $\bar{y}$ . On the other hand, Proposition 4.4 implies that

$$(6.13) \quad \bar{y}_k \in A_{\bar{x}}^f - \text{Int } C \subset A_{\bar{x}}^f - C = A_{\bar{x}}^f.$$

Remark also that

$$\begin{aligned} I_f(\bar{x}) - \bar{y}_k &= (I_f(\bar{x}) - y_k) + \nu_k \\ &\in C + \text{Int } C = \text{Int } C. \end{aligned}$$

Thus,  $\bar{y} \in \text{cl } E_f(\bar{x})$ . The proof of the Lemma is therefore established.  $\square$

**Remark 6.10.** Remark that  $A_{\bar{x}}^f \subset \text{cl } E_f(\bar{x}) = H_f(\bar{x}) = \text{cl } A_{\bar{x}}^f$ .

The following result will play a key role to derive the lower semicontinuity of the lower limit.

**Lemma 6.11.** *For every sequence  $(x_k)_k$  converging to  $\bar{x}$ , the sequence  $(A_{x_k}^f)_k$  is lower convergent to  $A_{\bar{x}}^f$  in the sense of Painlevé-Kuratowski.*

*Proof.* Let us first observe that, thanks to Remark 6.10, it suffices to prove that

$$(6.14) \quad E_f(\bar{x}) \subset \liminf_k A_{x_k}^f.$$

Let  $\bar{y} \in E_f(\bar{x})$ . We shall prove that

$$\bar{y} \in \liminf_k A_{x_k}^f.$$

Obviously,  $\bar{y} <_c I_f(\bar{x})$ . Let  $(\beta_k) \in \text{cl } A_{\bar{x}}^f$  be the sequence provided by Lemma 6.8 such that  $\beta_k \rightarrow I_f(\bar{x})$  and

$$(6.15) \quad \bar{y} <_c \beta_k, \forall k > 0.$$

Consider now  $y_k \rightarrow \bar{y}$  with  $y_k <_c \bar{y}$  for all  $k \geq 0$ . We claim that

$$(6.16) \quad \exists \delta_0 > 0 : \forall k > 0, y_k \leq_c f(x) \quad \forall x \in B(\bar{x}, \delta_0).$$

Otherwise, for each  $\delta > 0$ , there exists  $k > 0$  and  $x_\delta \in B(\bar{x}, \delta)$  such that

$$f(x_\delta) - y_k \notin C.$$

Thus,

$$\begin{aligned} f(x_\delta) - \bar{y} &\in y_k - \bar{y} + F \setminus C \\ &\in -C + F \setminus C = F \setminus C. \end{aligned}$$

Therefore, we can exhibit a sequence  $(w_n)$  converging to  $\bar{x}$  such that

$$(6.17) \quad f(w_n) - \bar{y} \in F \setminus C, \quad \forall n.$$

Now, consider some  $k_0 > 0$ . Since  $\beta_{k_0} \in \text{cl } A_{\bar{x}}^f$ , there exists a sequence  $(\gamma_m) = (\gamma_m(k_0))_m$  such that

$$\gamma_m \rightarrow \beta_{k_0}$$

with  $\gamma_m \in A_{\bar{x}}^f$ . Thus, from the definition of the lower level set, there exists a sequence  $(\rho_n)_n := (\rho_n(k_0, m))_n$  converging to  $\gamma_m$  and satisfying for each  $n$

$$(6.18) \quad f(w_n) - \rho_n \in C, \text{ or else } \rho_n - f(w_n) \in -C.$$

Hence, by (6.17), (6.18) and Lemma 6.6, we obtain

$$(6.19) \quad \rho_n - \bar{y} \in (F \setminus C) - C = F \setminus C.$$

Therefore

$$(6.20) \quad \rho_n - \bar{y} \notin \text{Int } C.$$

Passing to limit in (6.20) whenever  $n$  goes to  $+\infty$  we obtain

$$(6.21) \quad \gamma_m - \bar{y} \notin \text{Int } C,$$

and thus, by passing to limit in (6.21) whenever  $m$  goes to  $+\infty$ , we obtain

$$\beta_{k_0} - \bar{y} \notin \text{Int } C$$

contradicting (6.15). Then (6.16) is true.

Now, using (6.16), let us prove that  $y_k \in A_{x_k}^f$ . To this end, let  $(x_k^n)_n \rightarrow x_k$ . For some  $n_0 := n_0(k)$ , we have  $x_k^n \in B(x_k, \frac{\delta_0}{2}) \forall n \geq n_0$ . Let  $(z_k^n)_n$  be a sequence converging to  $y_k$  such that for each  $k$

$$z_k^n \leq_c y_k.$$

Take  $y_k^n = z_k^n$  for  $n \geq n_0$  and  $y_k^n = f(x_k^n)$  for  $n < n_0$ .

We can suppose that  $x_k \in B(\bar{x}, \frac{\delta_0}{2})$  and we note that  $x_k^n \in B(\bar{x}, \delta_0)$ . Hence, thanks to (6.16), we see that

$$y_k^n \leq_c f(x_k^n) \forall n,$$

Consequently,  $y_k \in A_{x_k}^f$  for all  $k$ . Thus,  $\bar{y} \in \liminf_k A_{x_k}^f$ . Hence,

$$(6.22) \quad E_f(\bar{x}) \subset \liminf_k A_{x_k}^f.$$

Since  $\liminf_k A_{x_k}^f$  is closed, it follows that

$$(6.23) \quad \text{cl}(E_f(\bar{x})) \subset \liminf_k A_{x_k}^f.$$

Then, Remark 6.10 implies that

$$(6.24) \quad A_{\bar{x}}^f \subset \liminf_k A_{x_k}^f.$$

The proof is therefore complete.  $\square$

We are now ready to conclude the main result of the paper.

*Proof.* Let  $\bar{x} \in \text{Dom } f$  and let  $(x_n)$  be a sequence converging to  $\bar{x}$ . Thanks to Lemma 6.7, there exists a sequence  $(y_k)$  converging to  $I_f(\bar{x})$  such that  $y_k \in A_{\bar{x}}^f$ . From Lemma 6.11, it follows that  $y_k \in \liminf_n A_{x_n}^f$ . Select  $y_k^n \in A_{x_n}^f$  such that  $(y_k^n)_n$  converges to  $y_k$  as  $n$  goes to  $+\infty$ . Following the beginning of the proof of Lemma 6.11 we can assume that  $(y_k^n)$  converges uniformly in  $k$  to  $y_k$ . Indeed, we can consider  $y_k^n = y_k - \nu_n$  with  $\nu_n \in \text{Int } C$  such that

$$\lim_{n \rightarrow +\infty} \nu_n = 0.$$

Clearly,

$$y_k^n \leq_c \sup A_{x_n}^f = I_f(x_n).$$

This being true for all  $k$ , let  $k(n)$  be a map such that  $k(n) \rightarrow +\infty$ . Take  $b_n = y_{k(n)}^n$ ; this sequence converges to  $I_f(\bar{x})$  (thanks to the uniform convergence of  $y_k^n$  to  $y_k$ ) and satisfies

$$b_n \leq_c I_f(x_n).$$

The proof is complete.  $\square$

**Remark 6.12.** From now on, we will use the following notation

$$v - \liminf_{x \rightarrow \bar{x}} f(x) := I_f(\bar{x}) = \sup A_{\bar{x}}^f$$

**Corollary 6.13.** The assertions below are equivalent

- $f$  is lower semicontinuous at  $\bar{x} \in \text{Dom } f$ ;
- $f(\bar{x}) \leq_c v - \liminf_{x \rightarrow \bar{x}} f(x)$ .

*Proof.* By Proposition 4.1,  $f$  is l.s.c at  $\bar{x}$  if, and only if  $f(\bar{x}) \in A_{\bar{x}}^f$ . Then, if  $f$  is l.s.c at  $\bar{x}$ , as  $v - \liminf_{x \rightarrow \bar{x}} f(x)$  is the least upper bound of  $A_{\bar{x}}^f$ , it follows that

$$(6.25) \quad f(\bar{x}) \leq_c v - \liminf_{x \rightarrow \bar{x}} f(x).$$

Conversely, suppose that (6.25) holds. We know that  $f(\bar{x})$  is an upper bound of  $A_{\bar{x}}^f$ , a fortiori

$$v - \liminf_{x \rightarrow \bar{x}} f(x) \leq_c f(\bar{x}).$$

Hence, as the cone  $C$  is pointed, we derive

$$v - \liminf_{x \rightarrow \bar{x}} f(x) = f(\bar{x}).$$

Therefore, thanks to the characterization of semicontinuity of [4] and our main result, we deduce that  $f$  is necessarily l.s.c at  $\bar{x}$ . Indeed, having in mind

$$I_f(z) := v - \liminf_{x \rightarrow z} f(x) \leq_c f(z), \forall z \in E,$$

for any sequence  $(x_n)$  such that  $x_n \rightarrow \bar{x}$ , by Theorem 6.5, there exists a sequence  $(b_n)$  converging to  $I_f(\bar{x}) = f(\bar{x})$  such that

$$b_n \leq_c I_f(x_n) \leq_c f(x_n) \quad \forall n.$$

The proof is established. □

**Remark 6.14.** Similarly, we can define the upper semicontinuous regularization of  $f$  by

$$v - \limsup_{x \rightarrow \bar{x}} f(x) := \inf B_{\bar{x}}^f.$$

## 7. COMPATIBILITY WITH THE STANDARD CASES

Now, we derive from the main result a more flexible proof for the well know result that says : for every real-valued function, the lower limit is lower semicontinuous. We consider here, more generally, finite dimensional-valued mappings.

Consider  $F = \mathbb{R}^p$ ,  $f = (f_1, f_2, \dots, f_p)$  and  $\bar{x} \in \text{Dom } f$ . Note that the order in this case goes back to the usual order of  $\mathbb{R}$ , generated by the cone  $C := \mathbb{R}_+^p$  and will be simply denoted by  $\leq$ .

We claim that

$$\text{cl } A_{\bar{x}}^f = \prod_{i=1}^p ] - \infty, \liminf_{x \rightarrow \bar{x}} f_i(x) ],$$

and

$$\text{cl } B_{\bar{x}}^f = \prod_{i=1}^p [\limsup_{x \rightarrow \bar{x}} f_i(x), +\infty[.$$

**In fact**, consider first the case  $p = 1$ . Let  $f : E \rightarrow \mathbb{R}$ ,  $\bar{x} \in \text{Dom } f$ ,  $y \in A_{\bar{x}}^f$  and  $(x_n)$  be a sequence converging to  $\bar{x}$ . There exists a sequence  $(b_n)$  such that

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq f(x_n), \quad \forall n \in \mathbb{N}.$$

This yields  $y \leq \liminf_{x \rightarrow \bar{x}} f(x)$  and thus

$$y \in \left] -\infty, \liminf_{x \rightarrow \bar{x}} f(x) \right],$$

whence,  $\text{cl } A_{\bar{x}}^f \subset ] - \infty, \liminf_{x \rightarrow \bar{x}} f(x)]$ .

Thanks to Proposition 4.4,  $A_{\bar{x}}^f$  is an interval of  $\mathbb{R}$  containing  $-\infty$ , then to prove the second inclusion, it suffices to show that

$$\liminf_{n \rightarrow +\infty} f(x_n) \in \text{cl } A_{\bar{x}}^f.$$

To this end, let  $\varepsilon > 0$  and  $(x_n)$  be a sequence in  $E$  converging to  $\bar{x}$ ; there exists  $N \in \mathbb{N}$  such that

$$\liminf_{n \rightarrow +\infty} f(x_n) - \varepsilon \leq f(x_n) \quad \text{for each } n \geq N.$$

Set

$$b_n := \begin{cases} -\infty & \text{if } n \leq N \\ \liminf_{n \rightarrow +\infty} f(x_n) - \varepsilon & \text{if } n > N. \end{cases}$$

We have  $\lim_{n \rightarrow +\infty} b_n = \liminf_{n \rightarrow +\infty} f(x_n) - \varepsilon$  and  $b_n \leq f(x_n)$  for every  $n \in \mathbb{N}$ .

This leads to

$$\liminf_{n \rightarrow +\infty} f(x_n) - \varepsilon \in A_{\bar{x}}^f$$

for all  $\varepsilon > 0$ , accordingly

$$] - \infty, \liminf_{x \rightarrow \bar{x}} f(x)] \subset \text{cl } A_{\bar{x}}^f.$$

Thus,

$$] - \infty, \liminf_{x \rightarrow \bar{x}} f(x)] = \text{cl } A_{\bar{x}}^f.$$

Let us show this inequality for  $p > 1$ . Let  $f$  be a mapping defined from  $E$  into  $\mathbb{R}^p$  where  $f = (f_1, f_2, \dots, f_p)$ . We observe that

$$\begin{aligned} A_{\bar{x}}^f &= \{y \in \mathbb{R}^p \mid \forall (x_n) \rightarrow \bar{x}, \exists (b_n) \rightarrow y \mid b_n \leq f(x_n) \quad \forall n \in \mathbb{N}\} \\ &= \left\{ (y_1, \dots, y_p) \in \mathbb{R}^p \mid \forall (x_n) \rightarrow \bar{x}, \exists (b_n^i) \xrightarrow[n \rightarrow +\infty]{} y_i \mid b_n^i \leq f_i(x_n) \quad \forall n \in \mathbb{N} \right\}, \end{aligned}$$

for  $i = 1, 2, \dots, p$ . This yields

$$\begin{aligned} A_{\bar{x}}^f &= \left\{ (y_1, \dots, y_p) \in \mathbb{R}^p \mid y_i \in A_{\bar{x}}^{f_i} \text{ for } i = 1, 2, \dots, p \right\} \\ &= \prod_{i=1}^p A_{\bar{x}}^{f_i}. \end{aligned}$$

This product being finite, it follows that

$$\begin{aligned} \text{cl } A_{\bar{x}}^f &= \text{cl } \prod_{i=1}^p A_{\bar{x}}^{f_i} \\ &= \prod_{i=1}^p ] - \infty, \liminf_{x \rightarrow \bar{x}} f_i(x)]. \end{aligned}$$

establishing the proof.  $\square$

Now, we recapture the classical result for finite dimensional valued-functions:

**Theorem 7.1.** *The lower limit for finite dimensional valued-functions is lower semicontinuous.*

*Proof.* Take  $F = \mathbb{R}^p$ . It suffices to take  $C = \mathbb{R}_+^p$  and replace  $\leq_c$  by the usual order  $\leq$  in the proof of the main result.  $\square$

**Remark 7.2.** Notice here that our proof of this classical result is independent of the closure of the epigraph.

## 8. EXTENSION OF USUAL OPERATIONS

To extend the usual operations on the lower and upper limits to the vector case, we first establish the relation between the lower level of the sum of two maps and the sum of their lower level sets.

**Proposition 8.1.** Let  $f$  and  $h$  be two vector-valued mappings from  $E$  into  $F^\bullet$  and  $\bar{x} \in \text{Dom } f \cap \text{Dom } h$ . Then the following inclusions hold:

- 1)  $A_{\bar{x}}^f + A_{\bar{x}}^h \subset A_{\bar{x}}^{f+h}$ ;
- 2)  $B_{\bar{x}}^f + B_{\bar{x}}^h \subset B_{\bar{x}}^{f+h}$ .
- 3) If moreover,  $f$  or  $h$  is continuous at  $\bar{x}$ , the inclusions in 1) and 2) become equalities.

*Proof.* 1) Let  $y \in A_{\bar{x}}^f + A_{\bar{x}}^h$ , there exist  $y_1 \in A_{\bar{x}}^f$  and  $y_2 \in A_{\bar{x}}^h$  such that  $y = y_1 + y_2$ . Let  $(x_n)$  be a sequence converging to  $\bar{x}$ . As  $y_1 \in A_{\bar{x}}^f$  and  $y_2 \in A_{\bar{x}}^h$ , there exist two sequences  $(b'_n)$  and  $(b''_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} b'_n = y_1 \quad \text{and} \quad b'_n \leq f(x_n), \quad \forall n \in \mathbb{N}$$

and

$$\lim_{n \rightarrow +\infty} b''_n = y_2 \quad \text{and} \quad b''_n \leq h(x_n), \quad \forall n \in \mathbb{N}.$$

Set  $b_n = b'_n + b''_n$ , we have  $\lim_{n \rightarrow +\infty} b_n = y_1 + y_2 = y$ , and

$$b_n = b'_n + b''_n \leq f(x_n) + h(x_n), \quad \forall n \in \mathbb{N}.$$

It follows that  $y \in A_{\bar{x}}^{f+h}$ .

2) The second inclusion can be established as in 1).

3) Suppose now that  $f$  is continuous at  $\bar{x}$  and consider a sequence  $(x_n)$  converging to  $\bar{x}$  and  $y \in A_{\bar{x}}^{f+h}$ . There exists a sequence  $(b_n)$  in  $F$  such that

$$\lim_{n \rightarrow +\infty} b_n = y \quad \text{and} \quad b_n \leq f(x_n) + h(x_n), \quad \forall n \in \mathbb{N}.$$

This yields

$$(8.1) \quad b_n - f(x_n) \leq h(x_n), \quad \forall n \in \mathbb{N}.$$

Let us take

$$a_n = \begin{cases} b_n - f(x_n) & \text{if } x_n \in \text{Dom } f \\ h(x_n) & \text{otherwise.} \end{cases}$$

$f$  being continuous at  $\bar{x}$ , it follows for  $n$  large enough, that  $x_n \in \text{Dom } f$ , and therefore

$$(8.2) \quad \lim_{n \rightarrow +\infty} a_n = y - f(\bar{x}).$$

Hence, we deduce from (8.1) and (8.2) that  $y - f(\bar{x}) \in A_{\bar{x}}^h$ . Thus,  $y \in f(\bar{x}) + A_{\bar{x}}^h$ .

As  $f$  is continuous at  $\bar{x}$ ,  $f$  is s-s.c.i at  $\bar{x}$ . It follows from Proposition 4.1 that  $f(\bar{x}) \in A_{\bar{x}}^f$ , then

$$y \in A_{\bar{x}}^f + A_{\bar{x}}^h.$$

The second inclusion can be concluded analogously. □

**Theorem 8.2.** *Let  $f$  and  $h$  be two vector-valued mappings from  $E$  into  $F^\bullet$  and  $\bar{x} \in \text{Dom } f \cap \text{Dom } h$ . The following assertions hold.*

- 1)  $v - \liminf_{y \rightarrow \bar{x}} f(y) + v - \liminf_{y \rightarrow \bar{x}} h(y) \leq_c v - \liminf_{y \rightarrow \bar{x}} (f + h)(y)$ ;
- 2)  $v - \limsup_{y \rightarrow \bar{x}} (f + h)(y) \leq_c v - \limsup_{y \rightarrow \bar{x}} f(y) + v - \limsup_{y \rightarrow \bar{x}} h(y)$ .
- 3) *If moreover  $f$  or  $h$  is continuous at  $\bar{x}$ , the inequalities in 1) and 2) become equalities.*

*Proof.* **1)** By virtue of Proposition 8.1, we have  $A_{\bar{x}}^f + A_{\bar{x}}^h \subset A_{\bar{x}}^{f+h}$ . Let  $z \in A_{\bar{x}}^f$  fixed; then  $z + A_{\bar{x}}^h \subset A_{\bar{x}}^{f+h}$ , and therefore

$$\sup(z + A_{\bar{x}}^h) \leq \sup A_{\bar{x}}^{f+h}.$$

Hence,

$$z + v - \liminf_{y \rightarrow \bar{x}} h(y) \leq v - \liminf_{y \rightarrow \bar{x}} (f + h)(y),$$

for all  $z \in A_{\bar{x}}^f$ . Then

$$\sup_{z \in A_{\bar{x}}^f} (z + v - \liminf_{y \rightarrow \bar{x}} h(y)) \leq v - \liminf_{y \rightarrow \bar{x}} (f + h)(y),$$

which leads to

$$\sup_{z \in A_{\bar{x}}^f} z + v - \liminf_{y \rightarrow \bar{x}} h(y) \leq v - \liminf_{y \rightarrow \bar{x}} (f + h)(y).$$

Accordingly,

$$v - \liminf_{y \rightarrow \bar{x}} f(y) + v - \liminf_{y \rightarrow \bar{x}} h(y) \leq v - \liminf_{y \rightarrow \bar{x}} (f + h)(y).$$

- 2) The second inequality can be done similarly.  
 3) Assume that  $f$  is continuous at  $\bar{x}$ , using Proposition 8.1, we obtain

$$A_{\bar{x}}^f + A_{\bar{x}}^h = A_{\bar{x}}^{f+h}.$$

Thanks to Theorem 4.5, we deduce that

$$f(\bar{x}) - C + A_{\bar{x}}^h = A_{\bar{x}}^{f+h}.$$

Proposition 4.4 implies that

$$A_{\bar{x}}^h - C = A_{\bar{x}}^h,$$

which allows to say that

$$f(\bar{x}) + A_{\bar{x}}^h = A_{\bar{x}}^{f+h},$$

whence

$$\begin{aligned} \sup \left( A_{\bar{x}}^{f+h} \right) &= \sup \left( f(\bar{x}) + A_{\bar{x}}^h \right) \\ &= f(\bar{x}) + \sup \left( A_{\bar{x}}^h \right). \end{aligned}$$

Then,

$$f(x) + v - \liminf_{y \rightarrow \bar{x}} h(y) = v - \liminf_{y \rightarrow \bar{x}} (f + h)(y).$$

Since  $f$  is continuous, Theorem 4.11 implies that  $A_{\bar{x}}^f = f(\bar{x}) - C$ , and therefore  $\sup A_{\bar{x}}^f = f(x)$ . We conclude that

$$v - \liminf_{y \rightarrow x} f(y) + v - \liminf_{y \rightarrow x} h(y) = v - \liminf_{y \rightarrow x} (f + h)(y).$$

□

## 9. APPLICATION TO VECTOR-VALUED D.C. MAPPINGS

In this section,  $H$  is as in Section 6. We shall apply our main result to show that every vector-valued D.C. mapping finite and continuous defined on a Banach space with values in  $H$  admits a continuous D.C. decomposition.

We first prove that each finite and continuous vector-valued D.C. mapping admits a lower semicontinuous D.C. decomposition.

Let  $\Omega$  be a convex open of  $E$ . Recalling, for a mapping from  $\Omega$  into  $H$ , the notation  $I_{\varphi}(x) := \sup (A_x^{\varphi})$ , we state the following:

**Proposition 9.1.** Let  $f : \Omega \rightarrow H$  be finite and continuous D.C. vector-valued mapping on  $\Omega$ . If  $(g, h)$  is a D.C. decomposition of  $f$  on  $\Omega$ , then  $(I_g, I_h)$  is a D.C. decomposition of  $f$  on  $\Omega$ .

*Proof.* Let  $g$  be an  $H_+$ -convex. At first, we claim that  $I_g$  is  $H_+$ -convex. In fact, for  $\bar{x} \in \text{Dom } g$ , as  $A_{\bar{x}}^g$  is a directed upwards, it follows from Proposition 5.3 that

$$\langle e_p, I_g(\bar{x}) \rangle = \langle e_p, \sup A_{\bar{x}}^g \rangle = \sup \langle e_p, A_{\bar{x}}^g \rangle \quad \forall p \in \mathbb{N}.$$

According to Lemma 6.2, we have

$$\sup \langle e_p, A_{\bar{x}}^g \rangle = \sup A_{\bar{x}}^{\langle e_p, g \rangle} = \liminf_{x \rightarrow \bar{x}} \langle e_p, g(x) \rangle, \quad \forall p \in \mathbb{N}.$$

Therefore,

$$\text{epi}(\langle e_p, I_g \rangle) = \overline{\text{epi} \langle e_p, g \rangle}, \quad \forall p \in \mathbb{N}.$$

Since  $g$  is  $H_+$ -convex, for each  $p \in \mathbb{N}$ ,  $\langle e_p, g \rangle$  is convex. This yields  $\langle e_p, I_g \rangle$  is convex for each  $p \in \mathbb{N}$ , i.e.,  $I_g$  is  $H_+$ -convex.

Now, let  $(g, h)$  be a D.C. decomposition of  $f$ . Let us show that  $(I_g, I_h)$  is a D.C. decomposition of  $f$  on  $\Omega$ . For this, let  $\bar{x} \in \Omega$ , we have

$$A_{\bar{x}}^{f+h} = A_{\bar{x}}^g.$$

As  $f$  is continuous at  $\bar{x}$ , by Proposition 8.1, we obtain

$$A_{\bar{x}}^f + A_{\bar{x}}^h = A_{\bar{x}}^g.$$

On the other hand, Theorem 4.5 leads to

$$f(\bar{x}) - C + A_{\bar{x}}^h = A_{\bar{x}}^g,$$

and from Proposition 4.4 we have

$$f(\bar{x}) + A_{\bar{x}}^h = A_{\bar{x}}^g.$$

Therefore

$$\begin{aligned} \sup(A_{\bar{x}}^g) &= \sup(f(\bar{x}) + A_{\bar{x}}^h) \\ &= f(\bar{x}) + \sup(A_{\bar{x}}^h). \end{aligned}$$

Hence,  $f(\bar{x}) + I_h(\bar{x}) = I_g(\bar{x})$  for every  $\bar{x} \in \Omega$ . The proof is complete.  $\square$

In the scalar case it has been shown [6] that every real D.C. continuous function admits a continuous D.C. decomposition. Here, we provide a generalization for continuous vector-valued maps from a Banach space into a separable Hilbert space.

**Theorem 9.2.** *Every finite, continuous and  $H_+$ -D.C. mapping on a convex and open subset  $\Omega$  of  $E$  into a separable Hilbert space ordered by  $H_+$  admits a continuous  $H_+$ -D.C. decomposition on  $\Omega$ .*

*Proof.* Let  $(g, h)$  be an  $H_+$ -D.C. decomposition of  $f$ . Fix a point  $x \in \Omega$ . According to Proposition 9.1,  $(I_g, I_h)$  is a lower semicontinuous  $H_+$ -D.C. decomposition on  $f$  on  $\Omega$ . On the other hand,  $f$  is continuous at  $x$ . Then,

$$x \in \text{Int}(\text{Dom } f) = \text{Int}(\text{Dom } I_g \cap \text{Dom } I_h) \subset [\text{Dom } I_g \cap \text{Int}(\text{Dom } I_h)].$$

The map  $I_h$  being  $H_+$ -convex, lower semicontinuous and proper on  $\Omega$  and

$$x \in \Omega \cap \text{Int}(\text{Dom } h),$$

it follows, for each  $p$ , that  $I_{h_p} : H \rightarrow \mathbb{R}$  is convex, lower semicontinuous and proper. Hence, as  $\mathbb{R}$  is normal, by Theorem 10.1 below we can conclude that  $I_{h_p}$  is continuous at  $x$ , for each  $p$  and then so is  $I_h$ . As  $g(y) = f(y) + h(y)$  for each  $y \in \Omega$ , it results that  $I_g$  is continuous at  $x$ . The proof is complete.  $\square$

## 10. APPENDICE.

**Theorem 10.1** (Théra, Ph.D Thesis, 1981). *Let  $f : E \rightarrow F^\bullet$  be a vector mapping. Suppose that  $F$  is normal and  $f$  is  $C$ -convex,  $s$ -l.s.c and proper. If  $\text{Int}(\text{Dom } f)$  is nonempty, then  $f$  is continuous on  $\text{Int}(\text{Dom } f)$ .*

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