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# Continuous Sets and Non-Attaining Functionals in Reflexive Banach Spaces

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# CONTINUOUS SETS AND NON-ATTAINING FUNCTIONALS IN REFLEXIVE BANACH SPACES

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ABSTRACT. In this paper we prove, in the framework of reflexive Banach spaces, that a linear and continuous functional $f$ achieves its supremum on every small $\varepsilon$-uniform perturbation of a closed convex set $C$ containing no lines, if and only if $f$ belongs to the norm-interior of the barrier cone of $C$. This result is applied to prove that every closed convex subset $C$ of a reflexive Banach space $X$ which contains no lines is continuous if and only if every small $\varepsilon$-uniform perturbation of $C$ does not allow non-attaining linear and continuous functionals. Finally, we define a new class of non-coercive variational inequalities and state a corresponding open problem.

## 1. INTRODUCTION AND NOTATIONS

Throughout the paper, we suppose that $X$ is a reflexive Banach space with continuous dual $X^*$. The norms in $X$ and $X^*$ will be denoted by $\|\cdot\|$ and $\|\cdot\|_*$, and the primal and dual closed unit balls of $X$ and $X^*$ by $\mathbb{B}_X$ and $\mathbb{B}_{X^*}$, respectively. Given a closed convex subset $C$ of $X$ and $\varepsilon > 0$, we call *$\varepsilon$-uniform perturbation* of $C$ every closed convex set $C_\varepsilon$ which satisfies:

$$C_\varepsilon \subseteq C + \varepsilon\mathbb{B}_X \text{ and } C \subseteq C_\varepsilon + \varepsilon\mathbb{B}_X. \tag{1}$$

The main purpose of this note is to determine, when the closed convex subset $C$ of $X$ is given, the class of all the linear continuous functionals on $X$ which attain their supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ for a small $\varepsilon$.

The main result of this note (Theorem 1, Section 2) claims that a linear continuous functional $f$ achieves its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ ($\varepsilon$ sufficiently small) if and only if $f$ belongs to the norm-interior of the barrier cone of $C$, that is the cone of all the linear continuous functionals bounded from above on $C$. Using a recent characterization of the interior of the barrier cone ([1]) we deduce (Corollary 1) that $f$ reaches

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its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ ($\varepsilon$ sufficiently small) if and only if $C$ is well-positioned and $f$ belongs to the norm-interior of the negative polar cone of the recession cone of $C$.

Theorem 1 gives thus a necessary and sufficient condition for a linear continuous functional on $X$ to achieve its supremum not only on a given closed convex set $C$ but also on any $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ ($\varepsilon$ sufficiently small).

Throughout the paper, as customary, given a closed convex set $C$, by a non-attaining functional we mean a linear continuous functional bounded from above on $C$ which does not reach its supremum on $C$. In Theorem 1 we characterize the class of all closed convex subsets $C$ of a reflexive Banach space $X$ such that any sufficiently small $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ disallows non-attaining functionals. Under theses assumptions, we show (Proposition 1, Section 3) that all the sufficiently small $\varepsilon$-uniform perturbations of a closed convex set $C$ disallow non-attaining functionals if and only if $C$ is a continuous set, in the sense of Gale and Klee ([6]) for sets in finite dimensional spaces. The reader is refereed to [4]) for a definition and several properties of infinite-dimensional continuous sets.

Finally, by virtue of a remark of Del Piero [5], we use Proposition 1 to define a class of non-coercive variational inequalities for which a natural necessary condition for the existence of solutions is also sufficient, and this for every small $\varepsilon$-uniform perturbation of the data involved in the problem. This result should be applicable to variational problems as they often arise in finance or in engineering problems in which data are known only with a certain precision and it is desired that further refinement of the data should not cause substantial changes in the existence of a solution.

We end up this note with an open question: can we characterize the class of all semi-coercive variational inequalities for which the above mentioned necessary condition is also sufficient?

For the convenience of the reader, we now introduce some additional notations. As usual, $j : X^* \to X$ is the *duality mapping* given by $\langle f, j(f)\rangle = \|f\|_*^2$ and $\|j(f)\| = \|f\|_*$, (see for example [8]),

$$S^\circ = \{f \in X^* : \langle f, w\rangle \leq 0 \ \forall w \in S\}$$

is the negative polar cone of the set $S$ of $X$, and $S^\circ$ reduces to the orthogonal

$$S^\perp = \{f \in X^* : \langle f, w\rangle = 0 \ \forall w \in S\}$$

when $S$ is a linear subspace of $X$. The linear subspace of $X$ parallel to the largest linear manifold contained in $C$ will be denoted by $l(C)$:

$$l(C) = C^\infty \cap (-C^\infty).$$

We will use the notations $\operatorname{Int} S$ and $\operatorname{Bd} S$ to denote respectively the norm-interior and the norm-boundary of a set $S$ in $X$ or in $X^*$. We recall that the *recession cone* (see [7]) to the closed convex set $S$ is the closed convex cone $S^\infty$ defined by

$$S^\infty = \{v \in X : \ \ \forall\, \lambda > 0,\, \forall\, x_0 \in S,\ x_0 + \lambda v \in S\}\,,$$

and that a set $S$ is called *linearly bounded* whenever $S^\infty = \{0\}$.

If $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, $\operatorname{Dom}\Phi$ is the set of all $x \in X$ for which $\Phi(x)$ is finite, and we say that $\Phi$ is *proper* if $\operatorname{Dom}\Phi \neq \emptyset$. When $\Phi$ is a proper lower semi-continuous convex function, the *recession* function $\Phi^\infty$ of $\Phi$ is the proper lower semi-continuous convex function whose epigraph is the recession cone for the epigraph of $\Phi$, i.e., $\operatorname{epi}\Phi^\infty = (\operatorname{epi}\Phi)^\infty$. Equivalently

$$\Phi^\infty(x) = \lim_{t\to+\infty} \frac{\Phi(x_0 + tx)}{t},$$

where $x_0$ is any element such that $\Phi(x_0)$ is finite. Given a closed convex subset $S$ of $X$, the domain of the *support function* given by

$$\sigma_S(f) := \sup_{x\in S}\langle f, x\rangle$$

is the *barrier cone* of $S$:

$$\mathcal{B}(S) = \{f \in X^* : \sigma_S(f) < +\infty\} = \operatorname{Dom}\sigma_S.$$

Finally, we use the symbols "$\to$" and "$\rightharpoonup$" to denote the strong convergence and the weak convergence on $X$, respectively.

## 2. THE MAIN RESULT

Let us consider a closed convex set $C$ which contains no lines and a continuous linear functional $f$ on $X$. The following two lemmata collect conditions on $C$ and $f$ allowing us to construct for every $\varepsilon > 0$ an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ on which $f$ does not achieve its supremum.

**Lemma 1.** *Let $C$ be a non-void closed convex subset of $X$ which contains no lines, and $f \in \mathcal{B}(C)$, $\|f\|_* = 1$ for which there is $w \in C^\infty$, $\|w\| = 1$, such that $\langle f, w\rangle = 0$. Then, for every $\varepsilon > 0$, there is an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ such that $f$ does not reach its supremum on $C_\varepsilon$.*

*Proof of Lemma 1*: If $f$ does not reach its supremum on $C$, then take $C_\varepsilon = C$ and Lemma 1 follows. Suppose now that $f$ attains its supremum on $C$ at $\overline{x}$, i.e.,

$$\langle f, \overline{x}\rangle \geq \langle f, x\rangle \ \forall x \in C.$$

Set

$$D=\left\{\overline{x}+\nu w+\mu j(f)\ :\ 0\leq\nu,0\leq\mu\leq\frac{\nu}{1+\nu}\varepsilon\right\},$$

and take $C_\varepsilon$ as the closed convex hull of $C$ and $D$ (denoted by $\overline{\mathrm{co}}(C,D)$) $C_\varepsilon=\overline{\mathrm{co}}(C,D)$. Remark that $D\subset A+\varepsilon\mathbb{B}_X$, where $A=\overline{x}+\mathbb{R}_+w$ is a half-line in $C$. Accordingly, $C_\varepsilon\subset C+\varepsilon\mathbb{B}_X$. As obviously $C\subset C_\varepsilon\subset C_\varepsilon+\varepsilon\mathbb{B}_X$, it follows that the closed convex set $C_\varepsilon$ is an $\varepsilon$-uniform perturbation of $C$.

On $D$, the supremum of $f$ is $\langle f,\overline{x}\rangle+\varepsilon$, while on $C$ the supremum of $f$ is $\langle f,\overline{x}\rangle$. Hence, the supremum of $f$ on $C_\varepsilon$ is $\langle f,\overline{x}\rangle+\varepsilon$. Let us show that $f$ does not reach this value on $C_\varepsilon$.

Suppose by contradiction that there is $\tilde{x}\in C_\varepsilon$ such that

$$\langle f,\tilde{x}\rangle=\langle f,\overline{x}\rangle+\varepsilon.$$

As $\tilde{x}\in\overline{\mathrm{co}}(C,D)$, select a sequence $(a_n)_{n\in\mathbb{N}^*}\subset\mathrm{co}(C,D)$ norm-converging to $\tilde{x}$. As obviously $D$ is closed and convex, for every $a_n\in\mathrm{co}(C,D)$ we may pick $d_n\in D$ (that is $d_n=\overline{x}+\nu_n w+\mu_n j(f)$ for some $0\leq\nu_n$ and $0\leq\mu_n\leq\frac{\nu_n}{1-\nu_n}\varepsilon$), $c_n\in C$ and $\lambda_n\in[0,1]$ such that

$$a_n=\lambda_n c_n+(1-\lambda_n)d_n.$$

As $\langle f,w\rangle=0$ we deduce that

$$\langle f,a_n\rangle-\langle f,\overline{x}\rangle=\lambda_n\left(\langle f,c_n\rangle-\langle f,\overline{x}\rangle\right)+(1-\lambda_n)\mu_n.$$

Since $\lim_{n\to+\infty}a_n=\tilde{x}$, then

$$\lim_{n\to+\infty}\left(\mu_n+\lambda_n(\langle f,c_n\rangle-\langle f,\overline{x}\rangle)-\mu_n)\right)=\varepsilon.$$

As $\mu_n\leq\varepsilon$ and $\lambda_n(\langle f,c_n\rangle-\langle f,\overline{x}\rangle-\mu_n)\leq 0$, the previous relation implies that

$$\lim_{n\to+\infty}\left(\lambda_n(\langle f,c_n\rangle-\langle f,\overline{x}\rangle-\mu_n)\right)=0\text{ and }\lim_{n\to+\infty}\mu_n=\varepsilon. \tag{2}$$

Using the fact that $\langle f,c_n\rangle-\langle f,\overline{x}\rangle-\mu_n\leq-\mu_n$, we derive

$$\limsup_{n\to\infty}\left(\langle f,c_n\rangle-\langle f,\overline{x}\rangle-\mu_n\right)\leq-\varepsilon. \tag{3}$$

Combining relations (3) and (2) it follows that $(\lambda_n)\to 0$, while since $(\mu_n)\to\varepsilon$ we deduce that $(\nu_n)\to\infty$. Let us observe that

$$\lambda_n c_n=-(1-\lambda_n)\nu_n w+a_n-(1-\lambda_n)(\overline{x}+\mu_n j(f)) \tag{4}$$

and that $(1-\lambda_n)\nu_n>0$, for $n$ large enough ($1-\lambda_n\to 1$ and $\nu_n\to\infty$). Hence, dividing by $(1-\lambda_n)\nu_n$ we obtain

$$\frac{\lambda_n}{(1-\lambda_n)\nu_n}c_n=-w+\frac{a_n-(1-\lambda_n)(\overline{x}+\mu_n j(f))}{(1-\lambda_n)\nu_n}. \tag{5}$$

Being convergent, the sequence $(a_n)$ is bounded, so the previous relation implies that

$$\frac{\lambda_n}{(1-\lambda_n)\nu_n} c_n \to -w.$$

As $t_n := \dfrac{\lambda_n}{(1-\lambda_n)\nu_n} \to 0$, $c_n \in C$ and $t_n c_n \to -w$ as $n \to +\infty$, it follows that $-w \in C^\infty$, that is $0 \neq w \in l(C)$, contradicting the fact that the closed convex set $C$ contains no lines. As a result, $f$ does not achieve its supremum on the $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$. □

In order to state the second condition ensuring the existence of at least one $\varepsilon$-uniform perturbation of $C$ on which $f$ does not attain its supremum, let us recall the concept of *well-positioned convex sets*, introduced by Adly *et al.* in a recent paper ([1]).

**Definition 1.** *A nonempty subset $C$ of the normed vector space $X$ is* well-positioned *if there exist $x_0 \in X$ and $g \in X^*$ such that:*

$$\langle g, x - x_0 \rangle \geq \|x - x_0\|, \quad \forall\, x \in C.$$

It follows directly from the definition that when $C$ is well-positioned, the sets $x + \lambda C$ and $B$ are well-positioned for every $x \in X$, $\lambda \in \mathbb{R}$ and $\emptyset \neq B \subset C$.

**Lemma 2.** *Let $C$ be a nonempty closed convex subset of $X$ containing no lines and which is not well-positioned, and $f \in \mathcal{B}(C)$, $\|f\|_* = 1$. Then, for every $\varepsilon > 0$, there is an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ on which $f$ does not attain its supremum.*

*Proof of Lemma 2*: Notice first that when $f$ does not reach its supremum on $C$ it is sufficient to set $C_\varepsilon = C$, and when there is $w \in C^\infty$, $\|w\| = 1$, such that $\langle f, w \rangle = 0$, we may apply Lemma 1. Consider now the remaining case, that is when

$$\langle f, w \rangle < 0 \quad \forall w \in C^\infty,\ \|w\| = 1, \tag{6}$$

and $f$ attains its supremum on $C$ at $\overline{x} \in C$. In order to achieve the proof of Lemma 2 we shall construct an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ on which $f$ does not attain its supremum.

Take $\overline{y} = \overline{x} + \varepsilon j(f)$ and consider

$$B = \left\{ x \in \overline{\text{co}}(\overline{y}, C) : \langle f, x \rangle = \langle f, \overline{x} \rangle \right\}. \tag{7}$$

Obviously, $B$ is a closed convex set. As $\overline{\text{co}}(\overline{y}, C)^\infty = C^\infty$, it follows that $B^\infty \subset C^\infty$; taking into account relation (6) we deduce that

$$\langle f, w \rangle < 0 \quad \forall w \in B^\infty,\ \|w\| = 1. \tag{8}$$

On the other hand, relation (7) implies that

$$\langle f, w\rangle = 0 \quad \forall w \in B^\infty. \tag{9}$$

Combining relations (8) and (9), it follows that $B^\infty = \{0\}$. Accordingly, $B$ is a linearly bounded closed convex set.

Let us now prove that $B$ is unbounded. Indeed, by contradiction, let us suppose that $B \subseteq \rho\mathbb{B}_X$ for some $\rho > 0$. Let $x \in C$; observe that the convex combination

$$z = \frac{\langle f, \overline{x}\rangle - \langle f, x\rangle}{\varepsilon + \langle f, \overline{x}\rangle - \langle f, x\rangle}\overline{y} + \frac{\varepsilon}{\varepsilon + \langle f, \overline{x}\rangle - \langle f, x\rangle}x \tag{10}$$

of $\overline{y}$ and $x$ belongs to $C_\varepsilon$, and, as $\langle f, z\rangle = \langle f, \overline{x}\rangle$, we deduce that $z \in B$. Accordingly, $\|z\| \leq \rho$; in addition, as $\overline{x} \in B$, we have $\|\overline{x}\| \leq \rho$, and therefore $\|z - \overline{x}\| \leq 2\rho$.

Standard calculations yield

$$z - \overline{x} = \frac{\varepsilon\left((x - \overline{y}) + \langle f, \overline{y} - x\rangle\, j(f)\right)}{\langle f, \overline{y} - x\rangle},$$

from which we obtain

$$\frac{\|(x - \overline{y}) + \langle f, \overline{y} - x\rangle\, j(f)\|}{\langle f, \overline{y} - x\rangle} \leq \frac{2\rho}{\varepsilon}.$$

Hence

$$\|x - \overline{y}\| \leq \left\langle -\left(1 + \frac{2\rho}{\varepsilon}\right) f, x - \overline{y}\right\rangle \quad \forall x \in C. \tag{11}$$

As relation (11) contradicts the fact that the set $C$ is not well-positioned we obtain the unboundedness of $B$. Accordingly, the set $B - \overline{x}$ is an unbounded linearly bounded closed convex set, and thus (see [2, Theorem 2.2]), there is a linear continuous functional $g$ such that

$$\inf_{y \in B} \langle g, y - \overline{x}\rangle < \langle g, x - \overline{x}\rangle \leq 1 \quad \forall\, x \in B. \tag{12}$$

Now, take

$$\begin{aligned} C_\varepsilon \quad &= \Big\{ x \in \overline{\mathrm{co}}(\overline{y}, C) : \\ &\langle g, x + \langle f, \overline{x} - x\rangle\, j(f) - \overline{x}\rangle + \frac{3}{\varepsilon}\langle f, x\rangle \leq 2 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle \Big\}. \end{aligned} \tag{13}$$

Obviously, $C_\varepsilon$ is a closed convex set which belongs to $\overline{\mathrm{co}}(\overline{y}, C)$, whence $C_\varepsilon \subseteq C + \varepsilon\mathbb{B}_X$. Moreover, as (see relation (10))

$$x + \langle f, \overline{x} - x\rangle\, j(f) - \overline{x} = \left(1 + \frac{\langle f, \overline{x} - x\rangle}{\varepsilon}\right)(z - \overline{x}),$$

we deduce from relation (12) that

$$\langle g, x + \langle f, \overline{x} - x\rangle\, j(f) - \overline{x}\rangle \leq 1 + \frac{\langle f, \overline{x} - x\rangle}{\varepsilon}. \tag{14}$$

Taking into account that $\langle f, x\rangle \leq \langle f, \overline{x}\rangle$ for every $x \in C$ we deduce from (14) that

$$\langle g, x + \langle f, \overline{x} - x\rangle\, j(f) - \overline{x}\rangle + \frac{3}{\varepsilon}\langle f, x\rangle \leq 1 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle \quad \forall x \in C,$$

and therefore by virtue of Definition (13) of $C_\varepsilon$ we have $C \subset C_\varepsilon$. Accordingly, $C_\varepsilon$ is a $\varepsilon$-uniform perturbation of $C$ in the sense of (1).

Fix $z \in B$; for $\lambda \in [0,1]$ put $z(\lambda) = \lambda z + (1-\lambda)\overline{y}$ and define

$$h(\lambda) = \langle g, z(\lambda) + \langle f, \overline{x} - z(\lambda)\rangle\, j(f) - \overline{x}\rangle + \frac{3}{\varepsilon}\langle f, z(\lambda)\rangle.$$

We have

$$h(0) = 3 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle > 2 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle,$$

while

$$h(1) = \langle g, z - \overline{x}\rangle + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle \leq 1 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle < 2 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle.$$

As the map $h : [0,1] \to \mathbb{R}$ is continuous, there is $\overline{\lambda} \in (0,1)$ such that

$$h(\overline{\lambda}) = 2 + \frac{3}{\varepsilon}\langle f, \overline{x}\rangle. \tag{15}$$

Obviously $z(\overline{\lambda}) \in \overline{\text{co}}(\overline{y}, C)$ and thus $z(\overline{\lambda}) \in C_\varepsilon$. Relation (15) yields

$$\langle f, z(\overline{\lambda})\rangle = \langle f, \overline{x}\rangle + \varepsilon\left(1 - \frac{3}{3 - \langle g, z - \overline{x}\rangle}\right),$$

and thus

$$\sup_{x \in C_\varepsilon}\langle f, x\rangle \geq \langle f, \overline{x}\rangle + \varepsilon\left(1 - \frac{3}{3 - \inf_{z \in B}\langle g, z - \overline{x}\rangle}\right). \tag{16}$$

On the other hand, for every $x \in C_\varepsilon$

$$\langle f, x\rangle \leq \langle f, \overline{x}\rangle + \varepsilon\left(1 - \frac{3}{3 - \langle g, z - \overline{x}\rangle}\right). \tag{17}$$

Hence, relations (12), (15) and (17) infer that for every $x \in C_\varepsilon$ we have

$$\langle f, x\rangle < \langle f, \overline{x}\rangle + \varepsilon\left(1 - \frac{3}{3 - \inf_{z \in B}\langle g, z - \overline{x}\rangle}\right) = \sup_{y \in C_\varepsilon}\langle f, y\rangle.$$

Accordingly, the linear continuous functional $f$ does not reach its supremum on the $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$, the proof of Lemma 2 is thus complete. ☐

The main result of this note characterizes all the linear continuous functionals which achieve their supremum on every sufficiently small $\varepsilon$-uniform perturbation of a given closed and convex set.

**Theorem 1.** *Let $C$ be a non-void closed convex subset of $X$ and $f$ a non-null linear continuous functional. Then, there is $\varepsilon > 0$ such that $f$ reaches its supremum on every closed convex subset $C_\varepsilon$ of $X$ fulfilling relation (1) if and only if $f$ belongs to the norm-interior of the barrier cone of $C$.*

*Proof of Theorem 1*: Consider $f \in \operatorname{Int} \mathcal{B}(C)$. From Corollary 2.1 of [1] select $R_f$ and $\gamma_f \in \mathbb{R}$ such that

$$\langle f, x\rangle \leq R_f - \gamma_f \|x\| \quad \forall x \in C. \tag{18}$$

Using relations (1) and (18) we deduce that

$$\langle f, x\rangle \leq \varepsilon \|f\|_* + R_f - \gamma_f \|x\| \quad \forall x \in C_\varepsilon. \tag{19}$$

Remark that $f \in \mathcal{B}(C_\varepsilon)$ and consider $(x_n)_{n\in\mathbb{N}^*} \subset C_\varepsilon$ a maximizing sequence of $f$, i.e., a sequence satisfying

$$\langle f, x_n\rangle \to \sup_{y\in C_\varepsilon} \langle f, y\rangle\,. \tag{20}$$

Accordingly, for $n$ large enough we have

$$\langle f, x_n\rangle \geq \sup_{y\in C_\varepsilon} \langle f, y\rangle - 1,$$

and from (19) it follows that

$$\|x_n\| \leq \frac{1}{\gamma_n}\left(1 + \varepsilon\|f\|_* + R_f - \sup_{y\in C_\varepsilon}\langle f, y\rangle\right).$$

The sequence $(x_n)_{n\in\mathbb{N}^*} \subset C_\varepsilon$ is therefore bounded, and, as $X$ is reflexive and $C$ is a closed and convex (thus weakly closed) set, the sequence $(x_n)$ has a weak cluster point $w \in C_\varepsilon$. From relation (20) we derive that

$$\langle f, w\rangle = \lim_{n\to\infty} \langle f, x_n\rangle = \sup_{y\in C_\varepsilon}\langle f, y\rangle\,,$$

which means that $f$ attains its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$.

In order to prove that a continuous linear functional $f$ which achieves its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of a closed convex set $C$ belongs to the norm-interior of the barrier cone of $C$, let us first remark that every such functional must be bounded from above on $C$.

If we suppose that $C$ is a not well-positioned, from Lemma 2 we deduce that, for every $\varepsilon > 0$, there is an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ (in the sense of (1)) on which $f$ does not attain its supremum, a contradiction. If we suppose that there is $w \in C^\infty$ such that $\langle f, w\rangle = 0$, Lemma 1 proves that there is an $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ on which $f$ does not attain its supremum, once again a contradiction.

Accordingly, if the continuous linear functional $f$ achieves its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of a closed convex set $C$, then $C$ is necessarily a well-positioned closed and convex set, and the following relation holds

$$\langle f, w\rangle < 0 \quad \forall w \in C^\infty, w \neq 0. \tag{21}$$

In order to achieve the proof, let us prove that $f$ belongs to the norm-interior of the barrier cone of $C$. By contradiction we suppose that $f \in \operatorname{Bd} C$. Since $C$ is well-positioned, the norm-interior of the convex set $\mathcal{B}(C)$ is nonempty. Hence, there exists some $w \in X^{**}$ of norm 1 such that

$$\langle f, w\rangle \geq \langle h, w\rangle \quad \forall\, h \in \mathcal{B}(C).$$

Because $X$ is reflexive we (may) consider that $w \in X$. The set $\mathcal{B}(C)$ is a cone, thus

$$\langle f, w\rangle \geq 0 \geq \langle h, w\rangle \quad \forall\, h \in \mathcal{B}(C). \tag{22}$$

Accordingly,

$$w \in [\mathcal{B}(C \cap L)]^\circ = (C \cap L)^\infty = C^\infty \cap V(L),$$

and from relation (22) it follows that $\langle f, w\rangle = 0$. Lemma 1 implies that there is at least one $\varepsilon$-uniform perturbation of $C$ on which $f$ does not achieve its supremum, contradiction which completely achieves the proof of Theorem 1. □

By virtue of Proposition 2.1 in [1], we deduce the following consequence of Theorem 1.

**Corollary 1.** *Let $C$ be a nonempty closed convex subset of $X$ and $f$ a non-null continuous linear functional. The following two statements are equivalent:*

*(a) $f$ achieves its supremum on every $\varepsilon$-uniform perturbation of $C$;*

*(b) $C$ is well-positioned and $f$ belongs to the norm-interior of the negative polar cone of the recession cone of $C$,*

$$\langle f, w\rangle < 0 \quad \forall w \in C^\infty,\ w \neq 0.$$

## 3. CONTINUOUS CLOSED CONVEX SETS AND NECESSARY CONDITIONS FOR NON-COERCIVE VARIATIONAL INEQUALITIES

The last section of this note is concerned with a new characterization of continuous closed and convex sets, as defined by Gale and Klee [6] (see also [4])):

**Definition 2.** *The closed convex set $C$ of $X$ is called continuous if its support functional $\sigma_C : X^* \longrightarrow \mathbb{R}$ is continuous on $X^* \setminus \{0\}$.*

Observe that in a Banach space $X$, every lower semicontinuous convex function $h : X \longrightarrow \mathbb{R} \cup \{+\infty\}$, is norm-continuous at $x \in X$ if and only if $x$ is in the set $(X \setminus \mathrm{Dom}\, h) \cup \mathrm{Int}(\mathrm{Dom}\, h)$. Applying this remark to the support function $\sigma_C : X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ of the closed convex subset $C$ of $X$, we deduce that $C$ is continuous if and only if

$$\mathcal{B}(C) = \{0\} \cup \mathrm{Int}(\mathcal{B}(C)).$$

The previous remark and Theorem 1 lead to the following result.

**Proposition 1.** *Let $C$ be a nonempty closed convex subset of $X$. Then every linear continuous functional bounded from above on $C$ achieves its supremum on every $\varepsilon$-uniform perturbation $C_\varepsilon$ of $C$ if and only if $C$ is continuous.*

We recall that an operator is called *semi-coercive* if there exist some positive constant $\kappa > 0$ and some closed subspace $U$ of $X$ such that if $\mathrm{dist}_U(x)$ denotes the distance from $x$ to $U$, we have

$$\langle Av - Au, v - u\rangle \geq \kappa\, (\mathrm{dist}_U(v-u))^2 \quad \forall u, v \in X$$
$$A(x+u) = A(x) \quad \forall x \in X \text{ and } u \in U, \text{ and } A(X) \subseteq U^{\perp}.$$

The class of semi-coercive operators contains for instance the projection operator onto a closed subspace of a Hilbert space.

Let $K$ be a closed convex subset of $X$, $f$ be an element in $X^*$, $A$ be a semi-coercive operator from $X$ to $X^*$, $\Phi : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous convex function that we assume to be bounded from below, and suppose that $K \cap \mathrm{Dom}\Phi \neq \emptyset$. We call semi-coercive variational inequality the problem of

$$\text{finding } u \in K \cap \mathrm{Dom}\Phi \text{ such that}$$
$$\langle Au - f, v - u\rangle \;+\; \Phi(v) - \Phi(u) \geq 0, \;\; \forall v \in K. \tag{23}$$

Proposition 1 has a direct application in the theory of semi-coercive variational inequalities. Indeed, when the variational inequality is governed by an operator which is bounded, semi-coercive and pseudo-monotone (in the sense of Brézis [3], page 132), it has been noticed that if a solution to (23) exists, then the energy functional

$$\mathcal{F}(x) = \kappa\,(\mathrm{dist}_U(x))^2 + I_K(x) + \Phi(x) - \langle f, x\rangle \quad \forall x \in X,$$

where $I_K$ denotes the indicator functional of $K$, is bounded from below on $X$, and that if the energy functional is coercive on $X$, then (23) has a solution (see for instance the proof of Proposition 3.1 in [1]).

Remark that $\mathcal{F}$ is bounded from below if and only if the linear continuous functional

$$(f,-1) : (X\times\mathbb{R})^* \to \mathbb{R}, \langle (f,-1),(x,a)\rangle = \langle f,x\rangle - a, \quad \forall (x,a)\in X\times\mathbb{R}$$

is bounded from above in $X \times \mathbb{R}$ on the epigraph of $\Psi$ defined by

$$\Psi(x) = \kappa\,(\mathrm{dist}_U(x))^2 + I_K(x) + \Phi(x)$$

and that $\mathcal{F}$ is coercive if and only if $(f,-1)$ belongs to the norm-interior of the barrier cone of the same epigraph.

Thus, Proposition 1 and Proposition 3.1 from [1] imply that, if the epigraph of $\Psi$ is a continuous subset of $X \times \mathbb{R}$, then the boundedness from below of the energy $\mathcal{F}$ is a necessary and sufficient condition for the existence of a solution for the variational inequality (23) for every $\varepsilon$-uniform perturbation of the data involved in the problem. As every continuous set is well-positioned, we use Theorem 4.1 from [1] to deduce that, whenever the epigraph of $\Psi$ is a continuous subset of $X \times \mathbb{R}$, then the energy functional $\mathcal{F}$ is bounded from below if and only if

$$\langle f,u\rangle < \Phi^\infty(u), \quad \forall u \in K^\infty \cap U, \;\; u \neq 0.$$

The following result summarizes the previous reasoning.

**Proposition 2.** *If the epigraph of $\Psi$ is a continuous subset of $X \times \mathbb{R}$, then relation*

$$\langle f,u\rangle < \Phi^\infty(u), \quad \forall u \in K^\infty \cap U, \;\; u \neq 0$$

*(equivalent to the boundedness from below of the energy $\mathcal{F}$) is a necessary and sufficient condition for the existence of a solution to the variational inequality (23). Moreover, the existence of a solution is achieved also for every instance involving a bounded and semi-coercive operator $A_\varepsilon$, a linear functional $f_\varepsilon$, a proper lower semi-continuous convex function $\Phi_\varepsilon$ that is bounded from below, and a closed convex set $K_\varepsilon$ such that*

$K_\varepsilon \cap \operatorname{Dom}\Phi_\varepsilon \neq \emptyset$, *and*

$$\begin{gathered}\|A(x)-A_\varepsilon(x)\|_* < \varepsilon, \quad \forall x \in X\\ \|f-f_\varepsilon\|_* < \varepsilon,\\ K \subset K_\varepsilon + \varepsilon\mathbb{B}_X \text{ and } K_\varepsilon \subset K + \varepsilon\mathbb{B}_X,\\ \Phi(x)-\varepsilon \leq \Phi_\varepsilon(x) \leq \Phi(x)+\varepsilon, \quad \forall x \in X.\end{gathered}$$

Finally, let us remark that Proposition 2 does not provide a complete characterization of semi-coercive variational inequalities for which the necessary condition involving the boundedness from below of the energy $\mathcal{F}$ is also sufficient for the existence of solutions, characterization which at our knowledge, remains an open problem.

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