



**Laboratoire d'Arithmétique,
Calcul formel et d'Optimisation**



UMR CNRS 6090

Slice-Continuous Sets in Reflexive Banach Spaces: Convex Constrained Optimization and Strict Convex Separation

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**Rapport de recherche n° 2004-03
Déposé le 1 juin 2004**

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SLICE-CONTINUOUS SETS IN REFLEXIVE BANACH SPACES: CONVEX CONSTRAINED OPTIMIZATION AND STRICT CONVEX SEPARATION

EMIL ERNST, MICHEL THÉRA, AND CONSTANTIN ZĂLINESCU

ABSTRACT. The concept of continuous set has been used in finite dimension by Gale and Klee and recently by Auslender and Coutat. Here, we introduce the notion of slice-continuous set in a general reflexive Banach space and we show that the class of such sets can be viewed as a subclass of the class of continuous sets. Further, we prove that every non constant real-valued convex and continuous function, which has a global minima, attains its infimum on every nonempty convex and closed subset of a reflexive Banach space if and only if its nonempty level sets are slice continuous. Thereafter, we provide a new separation property for closed convex sets, in terms of slice-continuity, and conclude this article by comments.

1. INTRODUCTION AND NOTATIONS

This article concerns two closely related topics: constrained convex optimization and the strict convex separation principle in a reflexive Banach space X . More precisely, we characterize

(P1) all the non-constant real-valued continuous and convex functions such that the constrained convex optimization problem:

$$\text{Find } x \in K \text{ s.t. } \Phi(x) \leq \Phi(y) \quad \forall y \in K,$$

has a solution for every nonempty closed and convex subset K of X ,

as well as

(P2) the class of those nonempty closed and convex subsets C of X which may be strictly separated by a closed hyperplane from any

1991 *Mathematics Subject Classification.* 47H05, 52A41, 39B82.

Key words and phrases. Constrained optimization, strict convex separation, slice-continuous set, well-positioned set, asymptote, continuous set.

The research of Michel Théra has been supported by NATO Collaborative Linkage Grant 978488. The contribution of C. Zălinescu was partially done during his stay at Université de Limoges, 2003.

disjoint nonempty closed convex set $D \subset X$. This means that there exists a continuous linear functional f such that

$$\sup_{x \in C} \langle f, x \rangle < \inf_{y \in D} \langle f, y \rangle.$$

Two subfamilies of closed convex sets play a crucial role in solving these problems:

- (1) the class $\mathcal{C}1$ of those closed convex sets which admit no boundary rays or asymptotes;

and

- (2) the class $\mathcal{C}2$ of those closed convex sets for which the support function (as defined in Rockafellar's book [13]) is continuous at every nonnull continuous functional.

The work of Gale and Klee [12] (see also Auslender and Coutat [6]) proves that these two classes coincide in \mathbb{R}^n , and their elements are called *continuous* sets. It is also proved that, in the finite dimensional setting, a nonempty closed convex set can be strictly separated from any other disjoint nonempty closed convex set if and only if it is continuous. Moreover, on the basis of the results from [6] and [12], it can easily be established that a non-constant, real-valued, convex function attains its infimum on every nonempty closed convex subset of \mathbb{R}^n if and only if the function attains its infimum on \mathbb{R}^n and all its level sets are continuous.

Let us also note that in several recent results including, for instance, the characterization of the closure of the linear image of convex sets ([5]), or existence theorems for generalized non-coercive equilibrium problems ([11]), finite-dimensional continuous closed convex sets play a crucial role.

In the framework of infinite dimensional reflexive Banach spaces, it is well known that

- (D1) classes $\mathcal{C}1$ and $\mathcal{C}2$ no longer coincide,

and

- (D2) neither the strict separation, nor the solvability of the constrained convex optimization problem are guaranteed by any of the above mentioned classes of closed and convex sets (for a definition and several properties of infinite-dimensional continuous sets see [9]).

Thus, the aim of this paper is to define, in the framework of general reflexive Banach spaces, a class of closed convex sets enjoying separation and solvability properties similar to those of *continuous* sets in \mathbb{R}^n .

A recently established weaker strict separation result suggests a possible way to avoid difficulties (D1) and (D2). Namely, it is proved that

a nonempty closed and convex subset of a reflexive Banach space can be strictly separated from every disjoint nonempty closed and convex set such that the two convex sets have no common recession half-line if and only if it is well-positioned. The concept of well-positioned closed convex set (introduced by Adly et al. in [1]) is a geometric notion equivalent, in the framework of reflexive Banach spaces, to the absence of lines and to weak local compactness (see [4]). The necessity of well-positionedness in this separation problem was established by Adly et al. in [2], while sufficiency goes back to Dieudonné [10].

In light of these considerations, the notion we seek should clearly capture the properties of well-positionedness and those of closed convex sets without boundary rays and asymptotes. In this respect, Proposition 1 proves that the class of closed convex slice-continuous sets (see Definition 5), that is closed convex sets for which every nonempty intersection with a closed linear manifold is continuous with respect to the closed linear manifold, coincides with the class of well-positioned closed convex sets with no boundary rays and no asymptotes.

The main results of this article, Theorems 1 and 2, prove that a non-constant real-valued convex and continuous function Φ which attains its infimum on a reflexive Banach space X , attains its infimum on every nonempty closed and convex set if and only if every of its nonempty level sets is slice-continuous. It is also proven that the same condition characterizes the class of nonempty closed and convex sets which may be strictly separated by a closed hyperplane from any disjoint nonempty closed convex set.

A direct characterization of non-constant real-valued convex and continuous functions Φ attaining their infimum on every nonempty closed and convex set is also provided (Proposition 4): the function Φ is required to be the sum between a coercive and a linear functional, and every half-line of X on which Φ is bounded from above must meet $\operatorname{argmin}_X \Phi$.

The article is organized as follows. Section 2 proves the necessity of the absence of boundary rays and asymptotes (Lemma 2) and of the well-positionedness (Lemma 5). The class of closed and continuous slice-continuous sets is defined and studied in Section 3. Sections 4 and 5 are devoted to the statement and the proof of the main results of the paper, Theorems 1 and 2. The last section contains dimension-reduction variants of the main results, and concluding remarks.

Throughout the paper, we suppose that X is a reflexive Banach space (unless otherwise stated) with continuous dual X^* . The norms in X and X^* will be denoted by $\|\cdot\|$ and $\|\cdot\|_*$, and the primal and dual closed unit balls of X and X^* by \mathbb{B}_X and \mathbb{B}_{X^*} , respectively.

The measure of the distance between two subsets S and T of X is given by $\text{gap}(S, T) = \inf_{x \in S, y \in T} \|x - y\| > 0$. As usual,

$$S^\circ = \{f \in X^* : \langle f, w \rangle \leq 0 \ \forall w \in S\}$$

is the negative polar cone of the set S of X , and S° reduces to the orthogonal $S^\perp = \{f \in X^* : \langle f, w \rangle = 0 \ \forall w \in S\}$ when S is a linear subspace of X . We will use the notations $\text{Int } S$ and $\text{Bd } S$ to denote respectively the norm-interior and the norm-boundary of a set S in X or in X^* . We recall that the *recession cone* to the closed convex set S is the closed convex cone S^∞ defined as

$$S^\infty = \{v \in X : \forall \lambda > 0, \forall x_0 \in S, x_0 + \lambda v \in S\},$$

(see [13] as a reference book). A set S is called *linearly bounded* whenever $S^\infty = \{0\}$.

If $\Phi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, $\text{Dom } \Phi$ is the set of all $x \in X$ for which $\Phi(x)$ is finite, and we say that Φ is *proper* if $\text{Dom } \Phi \neq \emptyset$. When Φ is a proper lower semi-continuous convex function, the *recession function* Φ^∞ of Φ is the proper lower semi-continuous convex function whose epigraph is the recession cone for the epigraph of Φ , i.e., $\text{epi } \Phi^\infty = (\text{epi } \Phi)^\infty$. Equivalently

$$\Phi^\infty(x) = \lim_{t \rightarrow +\infty} \frac{\Phi(x_0 + tx)}{t},$$

where x_0 is any element such that $\Phi(x_0)$ is finite. Given a closed convex subset S of X , the domain of the *support function* given by

$$\sigma_S(f) := \sup_{x \in S} \langle f, x \rangle$$

is the *barrier cone* of S :

$$\mathcal{B}(S) = \{f \in X^* : \sigma_S(f) < +\infty\} = \text{Dom } \sigma_S.$$

Finally, we use the symbol “ \rightarrow ” to denote the strong convergence and “ \rightharpoonup ” to denote the weak convergence on X .

2. NECESSARY CONDITIONS

When defining the class of non-constant real-valued convex continuous functions attaining their infimum on every nonempty closed convex set, we will proceed by elimination. Lemmata 2 and 5 collect conditions disallowing the constrained optimization problem to have a solution on every nonempty closed convex set.

2.1. Asymptotes of convex sets. In this subsection we will extend the notion of asymptote well known in real analysis to the case of closed convex sets in a normed vector space (for a definition of the asymptote in a general topological vector space, see [8]).

Definition 1. *Let C be a nonempty, closed convex subset of the normed vector space X . We say that the half-line $A := y_0 + \mathbb{R}_+w$ (with $y_0, w \in X$ and $\|w\| = 1$) is an asymptote of C , and that w is an asymptotic direction of C , if $A \cap \text{Int } C = \emptyset$ and $\text{gap}(A \setminus r\mathbb{B}_X, C) = 0$ for every $r \geq 0$.*

Remark 1. In order to simplify the notations, the above definition does not distinguish, as customary, between boundary rays, that is half-lines laying within $\text{Bd } C$, and asymptotes, that is half-lines A disjoint from C fulfilling $\text{gap}(A \setminus r\mathbb{B}_X, C) = 0$ for every $r \geq 0$. In the sequel, the notion of asymptote will thus be understood in the sense of Definition 1 (a half-line called asymptote in the present article may accordingly correspond either to a boundary ray, or to an asymptote, as classically defined). As a consequence of Definition 1, one may easily remark that every $w \in C^\infty$ with $\|w\| = 1$ is an asymptotic direction when $\text{Int } C = \emptyset$.

Rather than the previous definition, we will use in the sequel the following characterization of asymptotic directions.

Lemma 1. *Let X be a Banach space and $w \in X$ with $\|w\| = 1$. The following two statements are equivalent:*

- (a) *w is an asymptotic direction of C ;*
- (b) *$w \in C^\infty$ and the half-line $B := z_0 + \mathbb{R}_+w$ is disjoint from C for some $z_0 \in X$.*

Proof of Lemma 1: (a) \Rightarrow (b). Consider $w \in X$, $\|w\| = 1$, an asymptotic direction of C , and $A := y_0 + \mathbb{R}_+w$ an asymptote of C of direction w . Let us first prove that $w \in C^\infty$. Indeed, as $\text{gap}(A \setminus r\mathbb{B}_X, C) = 0$ for every $r \geq 0$, it follows that there are sequences $(t_n)_{n \in \mathbb{N}^*} \subset \mathbb{R}_+$ and $(x_n)_{n \in \mathbb{N}^*} \subset C$ such that

$$\|y_0 + t_n w\| > n, \quad \|x_n - y_0 - t_n w\| \leq 1 \quad \forall n \in \mathbb{N}^*.$$

In particular this yields $t_n \geq n - \|y_0\|$, and therefore $t_n \rightarrow +\infty$. Then, from the relation $\|t_n^{-1}x_n - t_n^{-1}y_0 - w\| \leq t_n^{-1}$ we obtain that $t_n^{-1}x_n \rightarrow w$, and so $w \in C^\infty$.

Let us now prove that there is $z_0 \in X$ such that $B := z_0 + \mathbb{R}_+w$ is disjoint from C . When A and C are disjoint there is nothing to prove. Hence, let us consider the case $A \cap C \neq \emptyset$. Take $x_0 \in A \cap C$ and remark that $x_0 \notin \text{Int } C$,

which means that $x_0 \in \text{Bd } C$. We distinguish two cases, depending whether the interior of C is empty or not.

Case 1: $\text{Int } C \neq \emptyset$. Let $c_0 \in \text{Int } C$ and take $z_0 := 2y_0 - c_0$. Assume that there exists $t \geq 0$ such that $z_0 + tw \in C$. Then,

$$y_0 + \frac{1}{2}tw = \frac{1}{2}c_0 + \frac{1}{2}(z_0 + tw) \in \text{Int } C.$$

Since $y_0 + \frac{1}{2}tw \in A$, we get the contradiction $A \cap \text{Int } C \neq \emptyset$.

Case 2: $\text{Int } C = \emptyset$. Without loss of generality, we (may) assume that $0 \in C$. For every $n \in \mathbb{N}^*$, the set

$$C_n := w + n(C - w)$$

is closed and has an empty interior (remark that $C_1 = C$). As X is of second Baire category, it follows that the countable union of closed sets with empty interior $\bigcup_{n \in \mathbb{N}^*} C_n$ is a proper subset of X . Accordingly, there is some element $z_0 \in X$ such that $z_0 \notin C_n$ for every $n \in \mathbb{N}^*$, which means that the open half-line $D := w + \mathbb{R}_+(z_0 - w)$ does not meet the set C .

To the end of obtaining a contradiction, suppose that $B \cap C \neq \emptyset$, that is $z_0 + \nu w \in C$ for some $\nu \geq 0$. Then, on one hand, because $0 \in C$, $v := \frac{1}{1+\nu}(z_0 + \nu w) \in C$, and on the other hand $v = w + \frac{1}{1+\nu}(z_0 - w) \in D$. This contradicts the fact that C and D are disjoint, and therefore we have proved that $B \cap C = \emptyset$.

(b) \Rightarrow (a). Let $w \in C^\infty$, $\|w\| = 1$, such that $B := z_0 + \mathbb{R}_+w$ does not meet C for some $z_0 \in X$. As noticed above, w is an asymptotic direction when $\text{Int } C = \emptyset$. So, assume that $\text{Int } C \neq \emptyset$. Without loss of generality, we (may) assume that $0 \in \text{Int } C$. Set

$$S := \{\lambda \in (0, 1] : (\lambda z_0 + \mathbb{R}_+w) \cap \text{Int } C = \emptyset\}.$$

Then, $S = \{\lambda \in (0, 1] : \lambda z_0 \notin (\text{Int } C) - \mathbb{R}_+w\}$, and obviously S is a closed subset of $[0, 1]$ which does not contain 0. Let $\lambda_0 := \min S \in (0, 1]$ and take $y_0 := \lambda_0 z_0$ and $A := y_0 + \mathbb{R}_+w$. Because $\lambda_0 \in S$ we have that $A \cap \text{Int } C = \emptyset$. Consider a sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ in $(0, \lambda_0)$ converging to λ_0 . By the choice of λ_0 note that

$$(\lambda_n z_0 + \mathbb{R}_+w) \cap \text{Int } C \neq \emptyset \quad \forall n \in \mathbb{N}^*.$$

Therefore, for every $n \geq 1$ there exists $t_n \geq 0$ such that $\lambda_n z_0 + t_n w \in \text{Int } C$. Fixing some $r \geq 0$, as $w \in C^\infty$ we have that $\lambda_n z_0 + t'_n w \in C$ for $t'_n := r + t_n + \|z_0\|$. It follows that

$$\text{gap}(A \setminus r\mathbb{B}_X, C) \leq \|(\lambda_0 z_0 + t'_n w) - (\lambda_n z_0 + t'_n w)\| = (\lambda_0 - \lambda_n)\|z_0\|.$$

Since the last term goes to 0, this yields $\text{gap}(A \setminus r\mathbb{B}_X, C) = 0$, and so A is an asymptote of C . \square

Note that the implication (b) \Rightarrow (a) of the preceding lemma is proved in [7, Prop. 2.4.1] when X is finite dimensional. Moreover, we used the fact that X is complete only for (a) \Rightarrow (b) in the case $\text{Int } C = \emptyset$.

2.2. Two necessary conditions. The first condition preventing the non-constant real-valued convex and continuous function Φ from attaining its infimum on every nonempty closed and convex subset of X states that at least one of the level sets C_M of Φ with $M \geq \inf_X \Phi$ and

$$C_M = \{x \in X : \Phi(x) \leq M\},$$

has asymptotes.

Lemma 2. *Let X be a Banach space and let $\Phi : X \rightarrow \mathbb{R}$ be a continuous convex function. If one of the nonempty level sets of Φ has an asymptote, then there is a two-dimensional nonempty closed convex subset of X on which the function Φ does not attain its infimum.*

Proof of Lemma 2: Suppose that, for some $M \in \mathbb{R}$, the level set C_M of Φ has asymptotes, and let $w \in C^\infty$ be an asymptotic direction of C_M ; of course, $\|w\| = 1$. From Lemma 1 it follows that there is $z_0 \in X$ such that the half-line $B := z_0 + \mathbb{R}_+ w$ does not meet C_M . Without loss of generality, we (may) assume that $0 \in C_M$. Define

$$T = \{\lambda \in [0, 1] : (\lambda z_0 + \mathbb{R}_+ w) \cap C_M \neq \emptyset\},$$

and put $d := \sup T \in [0, 1]$. We claim that $[0, d) \subset T$. Indeed, if $0 \leq \lambda < d$, then there exists $\mu \in T$ with $\lambda < \mu \leq d$. Then $\mu z_0 + tw \in C_M$ for some $t \geq 0$. Since $0 \in C_M$, we obtain that

$$\frac{\lambda}{\mu}(\mu z_0 + tw) = \lambda z_0 + \frac{\lambda t}{\mu} w \in C_M.$$

This proves that $\lambda \in T$. If $d = 1$ define $z'_0 := 2z_0$ and

$$T' := \{\lambda \in [0, 1] : (\lambda z'_0 + \mathbb{R}_+ w) \cap C_M \neq \emptyset\}.$$

As above we obtain that T' is an interval and $d' := \sup T' \leq \frac{1}{2}$ because $\frac{1}{2} \notin T'$. So, replacing if necessary z_0 by $2z_0$, we (may) assume that $d < 1$.

Observe that z_0 and w are linearly independent (otherwise $z_0 = \mu w$ for some $\mu \in \mathbb{R}$; then we get the contradiction $\hat{E}\hat{E} z_0 = 0 + \mu w \in C \cap B$). Consider the set

$$(1) \quad K := \left\{ \tau z_0 + \nu w : d < \tau \leq 1, \nu \geq \frac{1 - \tau}{\tau - d} \right\}.$$

A straightforward calculation shows that K is a closed convex subset of X which obviously contains B . Moreover, from the definition of T and d we deduce that K and C_M are disjoint.

Accordingly,

$$\Phi(x) > M \quad \forall x \in K,$$

and in order to prove Lemma 2 let us define a sequence $(y_n)_{n \in \mathbb{N}^*} \subset K$ such that

$$\lim_{n \rightarrow \infty} \Phi(y_n) = M.$$

To this respect, because $d = \sup T$, there are sequences $(\tau_n)_{n \in \mathbb{N}^*} \subset T$ converging to d and $(\nu_n)_{n \in \mathbb{N}^*} \subset \mathbb{R}_+$ such that $\tau_n z_0 + \nu_n w \in C_M$ for every $n \in \mathbb{N}^*$. Let $\zeta_n := \max\{\nu_n, n + 1, (1 - d)^{-1}\}$; of course, $\zeta_n \rightarrow \infty$. Since $w \in C_M^\infty$ and $\zeta_n \geq \nu_n$, we have that

$$x_n := \tau_n z_0 + \zeta_n w \in C_M \quad \forall n \in \mathbb{N}^*.$$

Let

$$\lambda_n := \frac{1 - d}{1 - \tau_n} - \frac{1}{\zeta_n}, \quad n \in \mathbb{N}^*.$$

Obviously, the sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ converges to 1. Because $\tau_n \leq d < 1$ we have

$$\lambda_n < (1 - d)/(1 - \tau_n) < 1.$$

On the other hand, we have that

$$\zeta_n \geq 1/(1 - d) \geq (1 - \tau_n)/(1 - d),$$

and so $\lambda_n \geq 0$. Therefore $\lambda_n \in [0, 1)$ for every n . Consider

$$y_n := \lambda_n x_n + (1 - \lambda_n) z_0 = \tau'_n z_0 + \nu'_n w,$$

where $\tau'_n := 1 - \lambda_n(1 - \tau_n)$ and $\nu'_n := \lambda_n \zeta_n$. A simple verification shows that $\tau'_n > d$ and $\nu'_n = (1 - \tau'_n)/(\tau'_n - d)$. Therefore $y_n \in K$ for every $n \in \mathbb{N}^*$. But the relation

$$M < \Phi(y_n) \leq \lambda_n \Phi(x_n) + (1 - \lambda_n) \Phi(z_0) \leq \lambda_n M + (1 - \lambda_n) \Phi(z_0)$$

holds for every $n \in \mathbb{N}^*$, and taking into account that the sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ converges to 1, and passing to the limit we obtain that $\lim_{n \rightarrow \infty} \Phi(y_n) = M$. Therefore $\inf_{x \in K} \Phi(x) = M$, infimum which is not attained. \square

In order to state the second condition ensuring the existence of at least one nonempty closed and convex set on which the function Φ does not attain its infimum (Lemma 5), we recall the concept of *well-positioned convex set*, introduced recently by Adly et al. ([1]).

Definition 2. A nonempty subset C of a normed vector space X is well-positioned if there exist $x_0 \in X$ and $g \in X^*$ such that:

$$\langle g, x - x_0 \rangle \geq \|x - x_0\|, \quad \forall x \in C.$$

It follows directly from the definition that when C is well-positioned, the sets $x + \lambda C$ and B are well-positioned for every $x \in X$, $\lambda \in \mathbb{R}$ and $\emptyset \neq B \subset C$.

The following geometric result will be useful in the proof of the Lemma 3 and also in the next section.

Theorem [2, Theorem 4.2]. *Let X be a reflexive Banach space, and $C \subseteq X$ be a nonempty closed convex set which contains no lines. Then C is not well-positioned if and only if $C \cap L$ is unbounded and linearly bounded for some closed linear manifold L of X .*

Lemma 3. *Let C be a nonempty closed convex subset of X , and assume that C does not contain lines and is not well-positioned. Then there is a closed linear manifold K of X disjoint from C such that $\text{gap}(K, C) = 0$.*

Proof of Lemma 3: It follows from [2, Theorem 4.2] (recalled above) that there is a closed linear manifold L of X , such that $C \cap L$ is a closed convex linearly bounded and unbounded set. Accordingly (see the proof of [2, Theorem 2.2]), there is $g \in X^*$ such that

$$(2) \quad \sup_{x \in C \cap L} \langle g, x \rangle = 1 \text{ and } \langle g, x \rangle < 1 \quad \forall x \in C \cap L.$$

The closed linear manifold K defined by

$$K := \{x \in L : \langle g, x \rangle = 1\}$$

will be a good candidate for doing the job (remark that K is nonempty as g is not constant, thus surjective, on L). Relation (2) implies, on one hand, that $K \cap C = \emptyset$, and on the other hand that there is a sequence $(x_n)_{n \in \mathbb{N}^*} \subset C \cap L$ with $1 > \langle g, x_n \rangle$ for every $n \in \mathbb{N}^*$ and such that $\lim_{n \rightarrow \infty} \langle g, x_n \rangle = 1$. If $V(L)$ denote the linear subspace of X parallel to L , as g is not constant on L , we can find $v_0 \in V(L)$ such that $\langle g, v_0 \rangle = 1$.

Accordingly, for every $n \in \mathbb{N}^*$, the element $y_n := x_n + (1 - \langle g, x_n \rangle)v_0$ belongs to L ; moreover, $\langle g, y_n \rangle = 1$, and so y_n is an element of K . Hence

$$\text{gap}(K, C) \leq \|y_n - x_n\| = (1 - \langle g, x_n \rangle)\|v_0\| \quad \forall n \in \mathbb{N}^*;$$

as $\lim_{n \rightarrow \infty} \langle g, x_n \rangle = 1$, it follows that $\text{gap}(K, C) = 0$. \square

Lemma 4. *Let C be a proper closed and convex subset of a normed linear space X , and $w \in X$ such that $\|w\| = 1$. If the line $x_0 + \mathbb{R}w$ lies within C for some $x_0 \in C$, then w is an asymptotic direction of C .*

Proof of Lemma 4: As both w and $-w$ belong to C_M^∞ , for every non-null linear function $h \in \mathcal{B}(C)$ (such a h exists since C is a proper subset of X)

we simultaneously have $\langle h, w \rangle \leq 0$ and $\langle h, -w \rangle \leq 0$, that is $\langle h, w \rangle = 0$. Obviously, for every $z_0 \in X$ such that

$$\langle h, z_0 \rangle > \sup_{x \in C_M} \langle h, x \rangle,$$

the half-line $B := z_0 + \mathbb{R}_+ w$ and C are disjoint. Thus (see Lemma 1) w is an asymptotic direction of C . \square

We now state the second condition ensuring the existence of at least one nonempty closed and convex set on which the function Φ does not attain its infimum.

Lemma 5. *Let Φ be a non-constant real-valued convex and continuous function, and suppose that one of its nonempty level sets is not well-positioned. Then, there is either a closed linear manifold, or a two-dimensional closed convex subset of X on which the function Φ does not attain its infimum.*

Proof of Lemma 5: Let C_M be a non well-positioned nonempty level set of Φ . When the closed and convex set C_M contains at least a line, we apply Lemma 4 to deduce that the set C_M admits at least an asymptote, and then Lemma 2 to prove that there is a two-dimensional closed convex set on which Φ does not attain its infimum.

In the case when the level set C_M does not contain lines, the closed linear manifold on which the function Φ does not attain its infimum is the closed linear manifold K obtained by setting C_M for C in the proof of Lemma 3. Indeed, in this case $K \cap C_M = \emptyset$, so

$$\Phi(x) > M \quad \forall x \in K,$$

and the conclusion of Lemma 5 will follow, in the same way to the proof of Lemma 2, by defining a sequence $(y_n)_{n \in \mathbb{N}^*} \subset K$ such that $\lim_{n \rightarrow \infty} \Phi(y_n) = M$.

As g is not constant on L , there is $y_0 \in L$ such that $\langle g, y_0 \rangle = 2$. Set $y_n := \lambda_n x_n + (1 - \lambda_n) y_0$, where x_n was defined in Lemma 3 and $\lambda_n := (2 - \langle g, x_n \rangle)^{-1}$; the choice of x_n shows that $0 < \lambda_n < 1$ for every $n \in \mathbb{N}^*$, and that the sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ tends to 1. As a convex combination of x_n and y_0 , both in L , the element y_n belongs to L for $n \in \mathbb{N}^*$. Because $\langle g, y_n \rangle = 1$, we have that $(y_n)_{n \in \mathbb{N}^*} \subset K$. Taking into account that $x_n \in C_M$, the convexity of Φ yields

$$M < \Phi(y_n) \leq \lambda_n \Phi(x_n) + (1 - \lambda_n) \Phi(y_0) \leq \lambda_n M + (1 - \lambda_n) \Phi(y_0)$$

for every $n \in \mathbb{N}^*$. Since the sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ converges to 1 we obtain that $\lim_{n \rightarrow \infty} \Phi(y_n) = M$. The proof of Lemma 5 is thus complete. \square

3. DEFINITION AND PROPERTIES OF SLICE-CONTINUOUS SETS

By virtue of Lemmata 2 and 5 it follows that if the function Φ attains its infimum on every nonempty closed convex set, then all its level sets are well-positioned (Lemma 5) and have no asymptotes (Lemma 2).

The object of this section is to explore the relations between the well-positioned closed convex sets with no asymptotes and the continuity of their support function. Let us extend (in Definition 3) the Definition 1 from lines to closed linear manifolds, and clarify (in Definition 4) the notion of continuity with respect to a closed linear manifold.

Definition 3. *Let C be a nonempty, closed and convex subset of a normed vector space X . We say that the closed linear manifold L of X is an asymptotic linear manifold of C if $\text{gap}(L \setminus r\mathbb{B}_X, C) = 0$ for every $r \geq 0$, and $L \cap \text{Int } C = \emptyset$.*

Note that an asymptotic linear manifold of finite dimension must necessarily contain at least one asymptote (in the sense of Definition 1), but that an infinite dimensional closed linear manifold of X may be an asymptotic linear manifold even in the absence of any asymptote. Remark also that a line is an asymptotic (one-dimensional) linear manifold for a closed and convex set if and only if one of its half-lines is an asymptote.

Definition 4. *Let L be a closed linear manifold of a normed linear space X and let $V(L)$ denote the closed subspace of X parallel to L . We say that a nonempty closed convex subset C of X is continuous with respect to L if $C - x_0$ is a continuous subset of the normed vector space $V(L)$ for some $x_0 \in L$.*

We can now define the central notion of this study. It corresponds to the class of those closed convex subsets of X which satisfies neither the hypothesis of Lemma 2, nor those of Lemma 5.

Definition 5. *We say that a nonempty closed convex subset C of a normed vector space X is slice-continuous if $C \cap L$ is continuous with respect to L for every closed linear manifold L which meets C .*

Proposition 1. *Let C be a nonempty proper closed convex subset of X . The following statements are equivalent:*

- (a) C is slice-continuous;
- (b) C is continuous and has no asymptotes;
- (c) C has no asymptotic linear manifolds;
- (d) C is well-positioned and admits no asymptotes;

(e) for every closed linear manifold L which meets C , the barrier cone of $C \cap L$ is the union between $V(L)^\perp$ and a nonempty norm-open set.

Remark 2. According to (b), and similarly to the class of continuous closed convex sets in \mathbb{R}^n , we deduce that the class of slice-continuous closed convex sets is characterized by both the absence of boundary lines and asymptotes, and the continuity (except maybe at 0), of their support functional. As a consequence, it follows that in \mathbb{R}^n the classes of slice-continuous and of continuous closed convex sets coincide.

Remark 3. The equivalence between (a) and (c) together with Definition 5 highlights the difference between finite-dimensional and infinite-dimensional reflexive Banach spaces. Indeed, from (c) it follows that slice-continuous sets must disallow not only (half-line) asymptotes, as in \mathbb{R}^n , but also asymptotic linear manifolds of any dimension. The mere definition of slice-continuous sets implies that not only the support function of C must be requested to be continuous (except at 0), but also that the support functions of all nonempty linear slices $C \cap L$ must be continuous except on $V(L)^\perp$.

Every unbounded linearly bounded closed and convex set is not well-positioned, and thus, although it admits no asymptotes, cannot be a slice-continuous set. Taking the closed convex set C defined by

$$C := \left\{ x = (x_1, x_2, \dots) \in \ell_2 : x_1 \geq 0, \sum_{i=2}^{\infty} |x_i| \leq \sqrt{x_1} \right\},$$

we have that

$$C^\infty = \mathbb{R}_+ e_1, \quad \mathcal{B}(C) = \{0\} \cup \{g \in \ell_2 : g_1 < 0\}$$

and $\text{Int } C \neq \emptyset$, where $e_1 := (1, 0, \dots) \in \ell_2$ (and similarly e_n). It follows that σ_C is continuous on $\ell_2 \setminus \{0\}$ and e_1 is an asymptotic direction of C , which means that C is a continuous set with asymptotes.

We may thus conclude that, in infinite dimensional reflexive Banach spaces, the class of slice-continuous sets is a proper subset of both the classes of closed convex sets without asymptotes, and of continuous closed convex sets.

Proof of Proposition 1: As when X is one-dimensional, every proper closed convex set obviously fulfills all the statements of Proposition 1, we assume that $\dim X \geq 2$ (we need this assumption in order to construct two-dimensional linear manifolds of X).

Step 1 [(d) \Leftrightarrow (e)]: (d) \Rightarrow (e). Consider C to be a well-positioned set without asymptotes, and L a closed linear manifold such that the intersection $C \cap L$ is nonempty. We prove that

$$(3) \quad \mathcal{B}(C \cap L) = V(L)^\perp \cup \text{Int } \mathcal{B}(C \cap L).$$

As the inclusion “ \supseteq ” is obvious, let us prove the converse one. Assume that there is some f in $\mathcal{B}(C \cap L)$ which is not in $V(L)^\perp \cup \text{Int } \mathcal{B}(C \cap L)$. Because $C \cap L$ is well-positioned, the norm-interior of the convex set $\mathcal{B}(C \cap L)$ is nonempty. Hence, there exists some $w \in X^{**}$ of norm 1 such that

$$\langle f, w \rangle \geq \langle h, w \rangle \quad \forall h \in \mathcal{B}(C \cap L).$$

Because X is reflexive we (may) consider that $w \in X$. The set $\mathcal{B}(C \cap L)$ is a cone, thus

$$\langle f, w \rangle \geq 0 \geq \langle h, w \rangle \quad \forall h \in \mathcal{B}(C \cap L).$$

Accordingly,

$$w \in [\mathcal{B}(C \cap L)]^\circ = (C \cap L)^\infty = C^\infty \cap V(L).$$

Finally, remark that the half-line $B := z_0 + \mathbb{R}_+ w$ is disjoint from $C \cap L$ for every $z_0 \in L$ such that

$$\langle f, z_0 \rangle > \sup_{x \in C \cap L} \langle f, x \rangle$$

(such an element z_0 exists because $f \in \mathcal{B}(C \cap L)$ and $f \notin V(L)^\perp$), and thus, because $B \subset L$, B is disjoint from C . From Lemma 1 it follows that w is an asymptotic direction for C , contradicting thus assumption (d). Therefore, relation (3) holds. As $\text{Int } \mathcal{B}(C \cap L)$ is nonempty, the conclusion follows.

(e) \Rightarrow (d) Let C be a nonempty closed and convex set fulfilling (e). Without loss of generality we (may) suppose that $0 \in C$. Put $L = X$ in (e), to deduce that $\mathcal{B}(C) \setminus \{0\}$ is a nonempty norm-open set. Accordingly, the norm-interior of the barrier cone of C is nonempty, and thus (see Proposition 2.1 from [1]) C is well-positioned.

It remains to prove that C does not admit asymptotes. In order to obtain a contradiction, suppose that $w \in X$ with $\|w\| = 1$ an asymptotic direction of C . From Lemma 1 it follows that $w \in C^\infty$ and the half-line $B := z_0 + \mathbb{R}_+ w$ and C are disjoint, for some $z_0 \in X$. Similarly to the proof of Lemma 2, we have that z_0 and w are linearly independent. Consider the two-dimensional linear manifold (in fact linear space)

$$L := \{\tau z_0 + \nu w : \tau, \nu \in \mathbb{R}\};$$

hence $V(L) = L$. Note that necessarily,

$$(4) \quad C \cap L \subset \{\tau z_0 + \nu w : \tau, \nu \in \mathbb{R}, \tau \leq 1\}.$$

Indeed, if the inclusion (4) fails, then $x := \tau z_0 + \nu w \in C$ for some $\tau, \nu \in \mathbb{R}$, $\tau > 1$. It follows that $x' := \tau z_0 + \nu' w \in C$, where $\nu' := \max(\nu, 0)$. Because $0 \in C$, we get the contradiction $\tau^{-1}x' = z_0 + \tau^{-1}\nu'w \in C \cap B$.

Because $z_0 \notin \mathbb{R}w$, there is $h \in X^*$ such that $\langle h, w \rangle = 0$ and $\langle h, z_0 \rangle = 1$. Taking into account (4), we have that $\langle h, x \rangle \leq 1$ for every $x \in C \cap L$. Hence $w \in (C \cap L)^\circ = (\mathcal{B}(C \cap L))^\circ$ and $h \in \mathcal{B}(C \cap L) \setminus L^\perp$. According to our hypothesis we obtain that $h \in \text{Int } \mathcal{B}(C \cap L)$. It follows that $h + r\mathbb{B}_{X^*} \subset \mathcal{B}(C \cap L)$ for some $r > 0$. Because $w \in (\mathcal{B}(C \cap L))^\circ$, we have that

$$r\langle g, w \rangle = \langle h + rg, w \rangle \leq 0 \quad \forall g \in \mathbb{B}_{X^*},$$

whence the contradiction $r \leq 0$. The proof of the equivalence (d) \Leftrightarrow (e) is thus complete.

Step 2 [(e) \Leftrightarrow (a)]: (e) \Rightarrow (a). Let L be a closed linear manifold such that $C \cap L \neq \emptyset$, and $x_0 \in L$. By our hypothesis, we have that $\mathcal{B}(C \cap L) = V(L)^\perp \cup \Omega$ with Ω a nonempty open set. It follows that $\Omega \subset \text{Int } \mathcal{B}(C \cap L)$, and so $\text{Int } \mathcal{B}(C \cap L) \neq \emptyset$ and $\mathcal{B}(C \cap L) = V(L)^\perp \cup \text{Int } \mathcal{B}(C \cap L)$, that is (3) holds.

The restriction operator $\text{Res} : X^* \rightarrow V(L)^*$ defined for every $f \in X^*$ as

$$\langle \text{Res}(f), x \rangle = \langle f, x \rangle \quad \forall x \in V(L),$$

is a linear, continuous and surjective operator between two Banach spaces. Accordingly,

$$\text{Int}(\text{Res}^{-1}(A)) = \text{Res}^{-1}(\text{Int}(A)),$$

for every subset A of $V(L)^*$.

If $\mathcal{B}^L(C \cap L - x_0)$ is the barrier cone of $(C \cap L - x_0)$ viewed as a subset of the reflexive Banach space $V(L)$, we observe that

$$\mathcal{B}(C \cap L) = \text{Res}^{-1}(\mathcal{B}^L(C \cap L - x_0))$$

and we deduce that

$$\text{Int } \mathcal{B}(C \cap L) = \text{Res}^{-1}(\text{Int } \mathcal{B}^L(C \cap L - x_0)).$$

Thus, relation (3) implies that

$$(5) \quad \mathcal{B}^L(C \cap L - x_0) = \{0\} \cup \text{Int } \mathcal{B}^L(C \cap L - x_0).$$

Observe that in a Banach space E , every lower semicontinuous convex function $f : E \rightarrow \mathbb{R} \cup \{+\infty\}$, is (norm) continuous at $x \in E$ if and only if x is in the set $(E \setminus \text{Dom } f) \cup \text{Int}(\text{Dom } f)$. Applying this remark to the support function $\sigma_{C \cap L - x_0}^L : V(L)^* \rightarrow \mathbb{R} \cup \{+\infty\}$ of the closed convex subset $(C \cap L - x_0)$ of $V(L)$, we deduce that $\sigma_{C \cap L - x_0}^L$ is (norm) continuous

on $U := (V(L)^* \setminus \mathcal{B}^L(C \cap L - x_0)) \cup \text{Int } \mathcal{B}^L(C \cap L - x_0)$. But, by (5), we have that

$$U = (V(L)^* \setminus (\{0\} \cup \text{Int } \mathcal{B}^L(C \cap L - x_0))) \cup \text{Int } \mathcal{B}^L(C \cap L - x_0)$$

and thus $U \supset V(L)^* \setminus \{0\}$. Therefore, $\sigma_{C \cap L - x_0}^L$ is continuous on $V(L)^* \setminus \{0\}$.

(a) \Rightarrow (e). C being a proper (closed and convex) barrier cone $\mathcal{B}(C)$ is nonempty. Moreover, $\mathcal{B}(C) \setminus \{0\} = \sigma_C^{-1}(\mathbb{R}) \setminus \{0\}$, and applying assumption (a) to $L = X$, we deduce that $\mathcal{B}(C) \setminus \{0\}$ is an open set, and therefore $\text{Int } \mathcal{B}(C)$ is nonempty. Let L be a closed linear manifold such that $C \cap L \neq \emptyset$. Clearly the inclusion $\text{Int } \mathcal{B}(C) \subset \text{Int } \mathcal{B}(C \cap L)$, yields $\text{Int } \mathcal{B}(C \cap L) \neq \emptyset$. Accordingly, in order to prove (e), it is sufficient to show that (3) holds. As the inclusion \supset in (3) is obvious, let us prove the converse one. For this consider $h \in \mathcal{B}(C \cap L)$. The case $h \in V(L)^\perp$ being obvious, let suppose that $h \notin V(L)^\perp$. Then, by our hypothesis, $\sigma_{C \cap L - x_0}^L$ is continuous at $\text{Res}(h)$, and so $\text{Res}(h) \in \text{Int } \mathcal{B}^L(C \cap L - x_0)$. As

$$\text{Int } \mathcal{B}(C \cap L) = \text{Res}^{-1}(\text{Int } \mathcal{B}^L(C \cap L - x_0)),$$

it follows that (3) holds, and the proof of the equivalence (a) \Leftrightarrow (e) from Proposition 1 is complete.

Step 3 [(c) \Leftrightarrow (d)]: (d) \Rightarrow (c). Let C be a well-positioned closed convex set without asymptotes, and suppose that there is an asymptotic linear manifold L of C . Thus there are sequences $(x_n)_{n \in \mathbb{N}^*} \subset L$ and $(y_n)_{n \in \mathbb{N}^*} \subset C$ such that $\|x_n\| \rightarrow \infty$ and $\|x_n - y_n\| \rightarrow 0$. Accordingly, $\|y_n\| \rightarrow \infty$ and $\|y_n\|/\|x_n\| \rightarrow 1$.

As C is well-positioned, there is $x_0 \in X$ and $f \in X^*$ such that $\langle f, x - x_0 \rangle \geq \|x - x_0\|$ for every $x \in C$. Thus

$$\left\langle f, \frac{y_n}{\|y_n\|} - \frac{x_0}{\|y_n\|} \right\rangle \geq \left\| \frac{y_n}{\|y_n\|} - \frac{x_0}{\|y_n\|} \right\| \geq 1 - \frac{\|x_0\|}{\|y_n\|},$$

whence $\langle f, w \rangle \geq 1$ for every weak cluster point w of the sequence $(y_n/\|y_n\|)_{n \in \mathbb{N}^*}$. As X is reflexive, by virtue of the previous inequality we can choose $w \neq 0$. On one hand, as $\|y_n\| \rightarrow \infty$, we deduce that $w \in C^\infty$. On the other hand, since $\|x_n - y_n\| \rightarrow 0$, then $w \in V(L)$, as a weak cluster point of the sequence $(x_n/\|x_n\|)_{n \in \mathbb{N}^*}$. Let $z_0 \in L$ and set $A := z_0 + \mathbb{R}_+ w$. If A and C are disjoint, Lemma 1 implies that $w/\|w\|$ is an asymptotic direction for C ; when A meets C , then A is obviously an asymptote of C . In both cases, we obtain a contradiction with condition (d) which states that C has no asymptotes.

(c) \Rightarrow (d). Let C be a closed and convex subset of X which has no asymptotic linear manifolds. Obviously, C has no asymptotes, as the support line of any asymptote is an asymptotic (one-dimensional) linear manifold of C . In order to prove that C is well-positioned, let us first remark that C contains no lines. Indeed, from Lemma 4 it follows that any closed and convex set containing a line admits asymptotes, and thus asymptotic linear manifolds. Finally, for the closed convex set C , which contains no lines and is not well-positioned, we may apply Lemma 3 to deduce the existence of a closed linear manifold K of X disjoint from C such that $\text{gap}(K, C) = 0$. As $K \cap r\mathbb{B}_X$ is a bounded closed and convex set disjoint from C and X is reflexive, it follows that $\text{gap}(C \cap r\mathbb{B}_X, K) > 0$ for every $r \geq 0$, and thus $\text{gap}(C \setminus r\mathbb{B}_X, K) = 0$ for every $r \geq 0$.

In conclusion, if C contains a line, it has an asymptote, and thus an asymptotic (one-dimensional) linear manifold, while if C does not contain lines but it is not well-positioned, then C admits the asymptotic (infinite-dimensional) linear manifold K . Accordingly, any closed convex set without asymptotic linear manifolds is well-positioned.

Step 4 [(b) \Leftrightarrow (d)]: (d) \Rightarrow (b). Let C be a closed convex well-positioned set with no asymptotes ; in order to prove that C fulfills (b) it is sufficient to remark that, from the implication (a) \Rightarrow (d) it follows that the support functional of C is continuous on $X^* \setminus \{0\}$.

(b) \Rightarrow (d). Assume that C fulfills (b). Then its support functional is continuous on $X^* \setminus \{0\}$ and finite on $\mathcal{B}(C)$, and this yields that $\mathcal{B}(C) \setminus \{0\}$ is an open subset of X^* . As C is a nonempty proper closed and convex subset of X , its barrier cone $\mathcal{B}(C)$ contains at least one non-null element; hence $\text{Int } \mathcal{B}(C) \neq \emptyset$. By virtue of Proposition 2.1 from [1] it follows that C is well-positioned; from (b) we also observe that C has no asymptotes, so C satisfies (d). \square

An important step in proving the main result of this section is the following topological property of slice-continuous closed convex sets.

Proposition 2. *Any two disjoint closed and convex nonempty subsets from a reflexive Banach space may be strictly separated by a closed hyperplane provided that at least one of them is slice-continuous.*

Proof of Proposition 2: Let C_1 and C_2 be two disjoint closed and convex nonempty subsets from the reflexive Banach space X , and suppose that C_1 is slice-continuous. Let us first prove that $C_1^\infty \cap C_2^\infty = \{0\}$.

On the contrary, let us suppose that there is $w \in C_1^\infty \cap C_2^\infty$ such that $\|w\| = 1$. Pick $y_0 \in C_2$; as $w \in C_2^\infty$, it follows that $A := y_0 + \mathbb{R}_+ w \subseteq C_2$.

Since C_1 and C_2 are disjoint, we deduce that the closed half-line A does not meet C_1 , while $w \in C_1^\infty$. Accordingly, w is an asymptotic direction of C_1 . Taking into account that C_1 is a slice-continuous set, this contradicts the equivalence (a) \Leftrightarrow (b) from Proposition 1. Hence, $C_1^\infty \cap C_2^\infty = \{0\}$.

Proposition 2 follows now as a consequence of the equivalence (a) \Leftrightarrow (d) in Proposition 1 and of Theorem 5.1 in [2], which states that any two nonempty closed and convex disjoint sets with no common recession half-line may be strictly separated provided that one of them is well-positioned. \square

4. THE MAIN RESULT

We can now characterize all the non-constant, real-valued, convex and continuous functions attaining their infimum on every closed convex subset of X . This characterization is given in terms of level sets.

Theorem 1. *Let $\Phi : X \rightarrow \mathbb{R}$ be a non-constant, convex and continuous function which attains its infimum on a reflexive Banach space X . Then the following statements are equivalent:*

- (a) Φ attains its infimum on each nonempty closed convex subset of X ;
- (b) every nonempty level set of Φ is a slice-continuous set.

Proof of Theorem 1: The implication (a) \Rightarrow (b) follows immediately from Lemmata 2 and 5, and from the equivalence (a) \Leftrightarrow (d) from Proposition 1.

(b) \Rightarrow (a) Assume that every nonempty level set of Φ is a slice-continuous set. Consider $K \subset X$ a nonempty closed convex set. Without loss of generality, we (may) assume that $0 \in K$. Set $m := \inf_{x \in K} \Phi$. If $m = \Phi(0)$ there is nothing to prove. So, let $m < \Phi(0)$ and assume that $\operatorname{argmin}_K \Phi = \emptyset$, that is $K \cap C_m = \emptyset$. Because $m \geq \inf \Phi$ and Φ attains its infimum on X , the set C_m is nonempty. As K and C_m are two closed, convex and disjoint sets, and as C_m is a slice-continuous set, Proposition 2 implies that there are $h \in X^*$, $\|h\|_* = 1$, and $t \in \mathbb{R}$, such that

$$(6) \quad \langle h, x \rangle \leq t \leq \langle h, y \rangle \quad \forall x \in C_m, \forall y \in K.$$

Consequently, h is a non-null element of the barrier cone of the slice-continuous set C_m . Using Definition 5 for $L = X$, we obtain

$$\mathcal{B}(C_m) = \{0\} \cup \operatorname{Int} \mathcal{B}(C_m),$$

and thus $h \in \operatorname{Int} \mathcal{B}(C_m)$. Since $(C_m^\infty)^\circ = \operatorname{cl} \mathcal{B}(C_m)$ (by the Bipolar Theorem) and $\operatorname{Int} \mathcal{B}(C_m) \neq \emptyset$, by a classical result on convex sets, we obtain that

$$\operatorname{Int} \mathcal{B}(C_m) = \operatorname{Int}(C_m^\infty)^\circ.$$

Because $C_m^\infty = C_{\Phi(0)}^\infty$, $h \in \text{Int}(C_{\Phi(0)}^\infty)^\circ$; using a classical result (see for instance Lemma 2.1 from [1]), there are $R, \gamma \in \mathbb{R}$ with $\gamma > 0$ such that

$$\langle h, x \rangle \leq R - \gamma \|x\|, \quad \forall x \in C_{\Phi(0)}.$$

From this relation and (6) we obtain that

$$t \leq R - \gamma \|x\|, \quad \forall x \in C_{\Phi(x_0)} \cap K.$$

Accordingly, $C_{\Phi(0)} \cap K$ is bounded; being also closed and convex, $C_{\Phi(0)} \cap K$ is weakly compact. Therefore there exists some $\bar{x} \in C_{\Phi(0)} \cap K$ such that

$$\Phi(\bar{x}) = \inf\{\Phi(x) : x \in C_{\Phi(0)} \cap K\} = \inf_X \Phi = m,$$

a contradiction. The proof of Theorem 1 is therefore complete. \square

4.1. Characterization of slice-continuous functions. The next proposition shows that the condition requesting that all the level sets of a function are slice-continuous sets may be relaxed to only two of the level sets.

Proposition 3. *Let X be a normed vector space and $\Psi : X \rightarrow \overline{\mathbb{R}}$ be a proper lower semicontinuous convex function and $r_0 > \inf_X \Psi$. Then*

(a) $(C_r)^\infty = \{u \in X : \Psi_\infty(u) \leq 0\}$ for every $r \in \mathbb{R}$ with $C_r \neq \emptyset$, where $C_r := \{x \in X : \Psi(x) \leq r\}$;

(b) $\mathcal{B}(C_r) = \mathcal{B}(C_{r_0})$ for every $r \in \mathbb{R}$ with $r > \inf \Psi$;

(c) If C_{r_0} is well-positioned, then C_r is well-positioned for every $r \in \mathbb{R}$ with $C_r \neq \emptyset$;

(d) If $w \in X$ is an asymptotic direction of C_{r_0} , then w is an asymptotic direction of C_r for every $r \in \mathbb{R}$ with $C_r \neq \emptyset$;

(e) If X is reflexive and C_{r_0} is a slice-continuous set, then C_r is a slice-continuous set for every $r > \inf \Psi$.

Proof of Proposition 3: The assertion (a) is well-known. Before studying the other assertions, let us recall that for $\Psi(x_0) < r_1 < r_2 < \infty$ one has

$$(7) \quad C_{r_1} \subset C_{r_2} \subset \frac{r_2 - \Psi(x_0)}{r_1 - \Psi(x_0)} C_{r_1} - \frac{r_2 - r_1}{r_1 - \Psi(x_0)} x_0.$$

Taking into account the preceding relation and Proposition 1, only assertion (d) needs an explanation. Let w be an asymptotic direction of C_{r_0} and $r \in \mathbb{R}$ with $C_r \neq \emptyset$. It follows that $\|w\| = 1$ and, by (the note after) Lemma 1, $w \in C_{r_0}^\infty$. By (a), $w \in C_r^\infty$. If $\text{Int } C_{r_0} = \emptyset$, from (7) we have that $\text{Int } C_r = \emptyset$, and so w is an asymptotic direction of C_r . Assume that $\text{Int } C_{r_0} \neq \emptyset$. From (7) we obtain that $\text{Int } C_r \neq \emptyset$, except, possibly, when $r = \inf \Psi$; in this latter case, as above, we have that w is an asymptotic

direction of C_r . So, let $\text{Int } C_r \neq \emptyset$. Because w is an asymptotic direction of C_{r_0} , again by Lemma 1, there exists $z_0 \in X$ such that

$$(z_0 + \mathbb{R}_+ w) \cap \text{Int } C_{r_0} = \emptyset.$$

Take $x_0 \in X$ such that $\Psi(x_0) < r_0$. If $r > r_0$, set

$$y_r := \frac{r - \Psi(x_0)}{r_0 - \Psi(x_0)} z_0 - \frac{r - r_0}{r_0 - \Psi(x_0)} x_0.$$

Taking into account that $z_0 \notin (\text{Int } C_{r_0}) - \mathbb{R}_+ w$, we have that

$$y_r \notin \frac{r - \Psi(x_0)}{r_0 - \Psi(x_0)} (\text{Int } C_{r_0}) - \frac{r - r_0}{r_0 - \Psi(x_0)} x_0 - \mathbb{R}_+ w.$$

From (7) we obtain that $y_r \notin (\text{Int } C_r) - \mathbb{R}_+ w$, or, equivalently,

$$(y_r + \mathbb{R}_+ w) \cap \text{Int } C_r = \emptyset.$$

If $r \leq r_0$, set $y_r := z_0$. Hence $(y_r + \mathbb{R}_+ w) \cap \text{Int } C_r = \emptyset$. By (the note after) Lemma 1, we obtain that w is an asymptotic direction of C_r . \square

Note that the assertions (b)–(e) are not valid if $r_0 = \inf \Psi$. Indeed, let

$$\Psi : \mathbb{R}^2 \rightarrow \overline{\mathbb{R}}, \quad \Psi(x, y) = \begin{cases} \infty & \text{if } y < 0, \\ 0 & \text{if } x = y = 0, \\ x^2/y & \text{if } y > 0. \end{cases}$$

Then $\inf_{\mathbb{R}^2} \Psi = 0$ and

$$C_r = \{(x, y) \in \mathbb{R}^2 : y \geq 0, x^2 \leq ry\}, \quad \forall r \geq 0.$$

Hence $\mathcal{B}(C_0) = \mathbb{R} \times (-\infty, 0]$ and $\mathcal{B}(C_r) = \{(0, 0)\} \cup (\mathbb{R} \times (-\infty, 0))$. Moreover, $(0, 1)$ is an asymptotic direction for C_0 , but $(0, 1)$ is not an asymptotic direction for C_r with $r > 0$.

In finite dimensions (c) and (e) are true even if $r_0 = \inf \Psi$ because a nonempty closed convex set $C \subset X$ (with $\dim X < \infty$) is well-positioned if and only if C^∞ is pointed (that is C does not contain lines). For a counter-example of (c) and (e) in the case $\dim X = \infty$ and $r_0 = \inf \Psi$, take $\Psi : \ell_2 \rightarrow \mathbb{R}$, $\Psi(x) := \sum_{n \geq 1} n^{-1} |x_n|$ for $x = (x_1, \dots, x_n, \dots) \in \ell_2$. We remark that $\inf \Psi = 0$, $C_0 = \{0\}$ is slice-continuous (being well-positioned and having no asymptotes), but C_1 is unbounded ($ne_n \in C_1$) and linearly bounded; hence C_1 is not well-positioned (and therefore not slice-continuous).

Remark 4. On the basis of the previous proposition, we establish that the level sets of a non-constant real-valued convex and continuous function Φ which attains its infimum are slice-continuous if and only if $\text{argmin}_X \Phi$ is slice-continuous and C_r is well-positioned for some $r > \inf_X \Phi$.

The following proposition provides a direct characterization of all non-constant, real-valued convex and continuous slice-continuous functions attaining their infimum on X , where by a *slice-continuous function* we mean a function for which all the nonempty level sets are slice-continuous.

Proposition 4. *Let Φ be a non-constant, real-valued convex and continuous function which attains its infimum on X . The level sets of Φ are slice-continuous sets if and only if the two following conditions hold simultaneously:*

- (i) $\Phi - f$ is coercive for some $f \in X^*$;
- (ii) every half-line B of X on which Φ is bounded from above meets $\operatorname{argmin}_X \Phi$.

Proof of Proposition 4: It is well known (see for instance Lemma 5.1 in [1] followed by Proposition 3.1 from [3]) that condition (i) is equivalent to the well-positionedness of the epigraph of Φ in $X \times \mathbb{R}$. In order to prove that all the level sets of Φ are well-positioned if and only if its epigraph is also well-positioned, let us recall the following analytic characterization of well-positionedness (Proposition 2.1 in [3]).

Proposition [3, Proposition 2.1]. *Let X be a reflexive Banach space, and $C \subseteq X$ be a closed convex set which contains no lines. Then C is not well-positioned if and only if it does not contain sequences $(x_n)_{n \in \mathbb{N}^*}$ such that $\|x_n\| \rightarrow \infty$ and $(x_n/\|x_n\|)$ weakly converges to 0.*

As to every sequence $(x_n)_{n \in \mathbb{N}^*} \subset C_M$ such that $\|x_n\| \rightarrow \infty$ and $x_n/\|x_n\| \rightharpoonup 0$ corresponds the sequence $((x_n, M))_{n \in \mathbb{N}^*}$ in $\operatorname{epi} \Phi$ which satisfies $\|(x_n, M)\|_{X \times \mathbb{R}} \rightarrow \infty$ and

$$(x_n, M)/\|(x_n, M)\|_{X \times \mathbb{R}} \rightarrow 0$$

in $X \times \mathbb{R}$, it is clear that all the level sets of Φ are well-positioned whenever the epigraph of Φ is well-positioned.

In order to prove the converse implication, suppose that all the level sets of Φ are well-positioned, but its epigraph is not.

The function Φ attains its infimum on X , and thus is bounded from below, and none of its level sets contains lines. Accordingly, the epigraph of Φ does not contain lines. From the above Proposition we deduce that there is a sequence $((x_n, \tau_n))_{n \in \mathbb{N}^*} \subset \operatorname{epi} \Phi$ such that

$$\|(x_n, \tau_n)\|_{X \times \mathbb{R}} \rightarrow \infty, \quad (x_n, \tau_n)/\|(x_n, \tau_n)\|_{X \times \mathbb{R}} \rightarrow 0.$$

Accordingly, $\|x_n\| \rightarrow \infty$, $\tau_n/\|x_n\| \rightarrow 0$ and $(x_n/\|x_n\|) \rightharpoonup 0$ in X . Moreover, since Φ attains its infimum on X and its level sets are well-positioned, we have that $\tau_n \rightarrow \infty$.

Pick $y_0 \in X$ and $M > \max\{0, \Phi(y_0)\}$. As $\tau_n \rightarrow \infty$, it follows that $\tau_n > M$ for every $n \geq n_0$ and some $n_0 \in \mathbb{N}^*$. Take

$$\lambda_n := (M - \Phi(y_0)) / (\tau_n - \Phi(y_0)).$$

It is obvious that $\lambda_n \in (0, 1)$ for $n \geq n_0$ and $\lambda_n \rightarrow 0$; moreover, because $\tau_n / \|x_n\| \rightarrow 0$ and $\tau_n \rightarrow \infty$, we have that $\lambda_n \|x_n\| \rightarrow \infty$. Since $(y_0, \Phi(y_0)), (x_n, \tau_n) \in \text{epi } \Phi$, we have that

$$(z_n, M) = (1 - \lambda_n)(y_0, \Phi(y_0)) + \lambda_n(x_n, \tau_n) \in \text{epi } \Phi,$$

and so $z_n \in C_M$ for $n \geq n_0$, where $z_n := (1 - \lambda_n)y_0 + \lambda_n x_n$; hence

$$\lambda_n \|x_n\| - \|y_0\| \leq \|z_n\| \leq \lambda_n \|x_n\| + \|y_0\| \quad \forall n \geq n_0.$$

It follows that $\|z_n\| \rightarrow \infty$ and

$$\frac{z_n}{\|z_n\|} = \frac{1 - \lambda_n}{\|z_n\|} y_0 + \frac{\lambda_n \|x_n\|}{\|z_n\|} \frac{x_n}{\|x_n\|} \rightarrow 0.$$

Using [3, Proposition 2.1] recalled above, we deduce that C_M is not well-positioned. This contradiction proves that $\text{epi } \Phi$ is well-positioned.

Finally, remark that condition (ii) is equivalent to the absence of asymptotes for every level set. Indeed, from Proposition 3(a), it follows that w is a recession direction for a nonempty level set C_M of Φ if and only if Φ is bounded from above on every half-line of the form $B = z_0 + \mathbb{R}_+ w$, with $z_0 \in X$. In this case, condition (ii) prescribes $B \cap \text{argmin}_X \Phi \neq \emptyset$, which, by virtue of Lemma 1, is equivalent with the absence of asymptotes for every level set of Φ . \square

5. STRICT SEPARATION OF CONVEX SETS

In this section, we use Theorem 1 to characterize all the nonempty closed and convex subsets C of a reflexive Banach space X which can be strictly separated from every disjoint nonempty closed and convex set D , *i.e.*,

$$\exists f \in X^* \text{ s.t. } \sup_{x \in C} \langle f, x \rangle < \inf_{y \in D} \langle f, y \rangle.$$

Indeed, it is well known that two nonempty closed and convex subsets C and D of X can be strictly separated if and only if $\text{gap}(C, D) > 0$. For every nonempty closed and convex subset C of the reflexive Banach space X , set $\Phi_C : X \rightarrow \mathbb{R}$ for the real-valued function defined by

$$\Phi_C(x) = \inf_{y \in C} \|y - x\|, \quad \forall x \in X.$$

It is straightforward to prove that Φ_C is convex and continuous, and that its level sets satisfy

$$C_m = \begin{cases} \emptyset & \text{if } m < 0, \\ C + m\mathbb{B}_X & \text{if } m \geq 0. \end{cases}$$

Let us also remark that $\text{gap}(C, D) = \inf_{x \in D} \Phi_C(x)$.

The following result allows us to use Theorem 1 in deciding whether the nonempty closed and convex set C may be strictly separated from every disjoint nonempty closed and convex set D .

Lemma 6. *Let C be a nonempty closed and convex subset of a reflexive Banach space X . The two following assertions are equivalent:*

- (a) *For every nonempty closed and convex subset D of X such that $C \cap D = \emptyset$ one has that $\text{gap}(C, D) > 0$;*
- (b) *The function Φ_C attains its infimum on every nonempty closed and convex subset of X .*

Proof of Lemma 6: The implication (b) \Rightarrow (a) is easy. Indeed, let D be a nonempty closed and convex subset of X such that $C \cap D = \emptyset$. By hypothesis, there exists $\bar{y} \in D$ such that $\Phi_C(\bar{y}) = \inf_{y \in D} \Phi_C(y)$. Hence $\text{gap}(C, D) = \Phi_C(\bar{y}) > 0$ because $\bar{y} \notin C$ and C is closed.

(a) \Rightarrow (b) Assume that (a) holds, and select a nonempty closed convex set $D \subset X$. Let

$$m := \inf_{y \in D} \Phi_C(y) = \text{gap}(C, D).$$

If $C \cap D \neq \emptyset$, then $m = \Phi_C(\bar{y})$ with $\bar{y} \in C \cap D$. Assume now that $C \cap D = \emptyset$. From our hypothesis we obtain that $m = \text{gap}(C, D) > 0$. Consider $B := D + m\mathbb{B}_X$; of course, B is a nonempty convex set. Because X is reflexive, B is (weakly) closed as the sum of a weakly closed and a weakly compact set. By Lemma 5.2 in [14] we have that $\text{gap}(B, C) = \max\{\text{gap}(D, C) - m, 0\} = 0$. It follows that $B \cap C \neq \emptyset$ (otherwise, by (a), $\text{gap}(B, C) > 0$). Therefore, there exist $y \in D$ and $u \in m\mathbb{B}_X$ such that $x := y + u \in C$. It follows that $\Phi_C(y) \leq \|x - y\| = \|u\| \leq m$, which proves that Φ_C attains its infimum on D at y . \square

Proposition 1, Theorem 1 and Lemma 6 allow us to prove the main result of this section.

Theorem 2. *Let C be a nonempty closed and convex proper subset of a reflexive Banach space X . The two following assertions are equivalent:*

- (a) *C can be strictly separated from every disjoint nonempty closed and convex subset of X ;*
- (b) *C is a slice-continuous set.*

Proof of Theorem 2: Lemma 6 implies that (a) holds if and only if the function Φ_C attains its infimum on every nonempty closed and convex subset of X . From Theorem 1 we infer that Φ_C attains its infimum on every nonempty closed and convex subset of X if and only if $C + r\mathbb{B}_X$ is a slice-continuous set for every $r \geq 0$.

The proof of Theorem 2 will be achieved if we show that for each slice-continuous set C and each $r \geq 0$, the sets $C + r\mathbb{B}_X$ are slice-continuous. Assume that C is a slice-continuous set. It is obvious that $\mathcal{B}(C + r\mathbb{B}_X) = \mathcal{B}(C)$ for every $r \geq 0$. By Proposition 2.1 from [1] we know that C is well-positioned if and only if $\text{Int } \mathcal{B}(C) \neq \emptyset$, and so $C + r\mathbb{B}_X$ is well-positioned for every $r \geq 0$. Assume that $C + r\mathbb{B}_X$ has asymptotes for some $r \geq 0$. Then, by Lemma 1, there exist $w \in (C + r\mathbb{B}_X)^\infty = C^\infty$ with $\|w\| = 1$ and $z_0 \in X$ such that $B := z_0 + \mathbb{R}_+w$ is disjoint from $C + r\mathbb{B}_X$. It follows that B and C are disjoint, and so, using again Lemma 1, C has asymptotes, a contradiction. \square

6. DIMENSION REDUCTION STATEMENTS OF THE MAIN RESULTS

Let Φ be a real-valued convex and continuous function which attains its infimum on the reflexive Banach space X , such that $\text{argmin}_K \Phi = \emptyset$ for some nonempty closed and convex set K . According to Theorem 1, at least one of its nonempty level sets, say C_M , is not a slice-continuous set. Taking into account Proposition 1, the set C_M fulfills either the conditions of Lemma 2, or those of Lemma 5. Thus, there is either a closed linear manifold, or a two-dimensional closed and convex set on which Φ does not attain its infimum.

We have thus established the following dimension reduction version of the main result, Theorem 1.

Corollary 1. *Let $\Phi : X \rightarrow \mathbb{R}$ be a convex and continuous function. Then the two following statements are equivalent:*

- (a) Φ attains its infimum on each nonempty closed convex subset of X ;
- (b) Φ attains its infimum on every closed linear manifold and every two-dimensional nonempty closed convex subset of X .

Using Lemma 6, the previous Corollary implies the following dimension reduction version of Theorem 2.

Corollary 2. *Let C be a nonempty closed and convex proper subset of X . Then the two following statements are equivalent:*

- (a) C can be strictly separated from every disjoint nonempty closed and convex subset of X ;

(b) C can be strictly separated from every disjoint closed linear manifold and every disjoint nonempty two-dimensional closed and convex subset of X .

Finally, let us remark that the continuous sublinear functions form a small subset of the class of non-constant real-valued convex and continuous functions attaining their infimum on every nonempty closed convex subset of X . Likewise, the closed convex cones form a very small subclass of the nonempty closed and convex proper subsets of X with the strict separation property. Indeed, using Theorems 1 and 2, it is straightforward to prove that, in every reflexive Banach space of dimension greater than or equal to two, the only continuous sublinear functions and the only closed and convex cone with the above mentioned property are the positive homogeneous coercive functions and the singleton $\{0\}$, respectively.

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