Bounce law at the corners of convex billiards

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Abstract

Let \( H \) be a finite dimensional space and let \( C \) be a convex subset of \( H \). Given any elastic shock solution \( x(.) \) of the differential inclusion

\[
\ddot{x}(t) + N_C(x(t)) \ni 0, \quad t > 0,
\]

the bounce of the trajectory at a regular point of the boundary of \( C \) follows the Descartes law. The aim of the paper is to exhibit the bounce law at the corners of the boundary. For that purpose, we define a sequence \((C_\varepsilon)\) of regular sets tending to \( C \) as \( \varepsilon \to 0 \), then we consider the approximate differential inclusion \( \ddot{x}_\varepsilon(t) + N_{C_\varepsilon}(x_\varepsilon(t)) \ni 0 \), and finally we pass to the limit when \( \varepsilon \to 0 \). For approximate sets defined by \( C_\varepsilon = C + \varepsilon B \) (where \( B \) is the unit euclidean ball of \( H \)), we recover the bounce law associated with the Moreau-Yosida regularization.

Key words: Convex billiards, Set regularization, Variational approximation, Evolution differential inclusions, Shock solutions, Descartes law

1 Introduction

Let \( H \) be a finite dimensional space and \( C \) be a convex subset of \( H \). In this paper, we are interested in the bounce of a material point moving inside the convex billiard \( C \). For that purpose, we modelize the motion by the following differential inclusion

\[
(x) \quad \ddot{x}(t) + N_C(x(t)) \ni 0, \quad t > 0,
\]

where \( N_C(x(t)) \) denotes the outward normal cone to \( C \) at \( x(t) \). The modelling of shocks in mechanics by using a second order differential inclusion has been initiated by Moreau (see for example [8–11]). In our situation, we focus on elastic shock

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solutions of (E): the trajectories of such solutions are straight lines outside the boundary of $C$ and they do not dissipate energy at the shocks, i.e. the incident and reflected velocity vectors have the same norm. When the shock occurs at a regular point of the boundary, one can easily show that every elastic shock solution satisfies the classical Descartes law: the incident and reflected directions are symmetric with respect to the (unique) normal direction. On the opposite, when the shock occurs at a corner of the boundary, the problem is much more involved and the aim of this paper is to find reasonable answers.

A first approach consists in approximating the indicator function $\delta_C$ of $C$ by a sequence of smooth convex functions $\Phi_n : H \to \mathbb{R} \cup \{+\infty\}$ whose domain is an open subset of $H$. Then one has to study the following ordinary differential equations

$$\ddot{x}_n(t) + \nabla \Phi_n(x_n(t)) = 0,$$

(1.1)

and finally to pass to the limit when $n \to +\infty$. Attouch, Cabot, Redont [2] have used such an approach with a general epi-convergent scheme and they have proved that any trajectory $x(.)$ obtained as a limit of approximate trajectories $x_n(.)$ of (1.1) is an elastic shock solution of (E). In particular, the shock law at a regular point of the boundary of $C$ is prescribed by the Descartes law. However, when the trajectory $x(.)$ meets a corner of the boundary, then the shock law actually depends on the approximation scheme (see [2]). In the particular case of the Moreau-Yosida approximation, Shatzman [14] has given an explicit formula for the shock law at the corners. We propose in this paper an alternative approach, still based on approximation. Indeed, given any sequence $(C_n)$ of regular sets converging toward $C$ in the sense of Painlevé-Kuratowski, we consider the following differential inclusion

$$(E_\varepsilon) \quad \dot{x}_\varepsilon(t) + N_{C_\varepsilon}(x_\varepsilon(t)) \ni 0, \quad t > 0,$$

and we pass to the limit when $\varepsilon \to 0$. Notice that the shocks occurring in $(E_\varepsilon)$ simply follow the Descartes law since the sets $C_\varepsilon$ are regular. For a given corner of the boundary of $C$, the previous method yields a family of (possible) bounce laws, depending on the approximating scheme $(C_\varepsilon)$. When particularizing the previous abstract scheme, we obtain remarkable shock laws. For example, denoting by $\mathbb{B}_p$ ($p > 1$) the unit ball associated with the canonical $p$-norm $| \cdot |_p$ of $H$, let us define the set $C_\varepsilon$ by $C_\varepsilon = C + \varepsilon \mathbb{B}_p$. In the case $p = 2$, the associated bounce law coincides with the Moreau-Yosida one.

At this stage, it is crucial to wonder if, among all these possible bounce laws, one of them is more natural or adequate than the other ones (because of the mechanical or physical aspects of the problem,...). To answer this question, we turn to the geometrical point of view. The geometrical approach of convex billiards has given rise to an abundant literature (see for example Kozlov-Treschev [6]). To simplify the problem, we restrict ourselves to the two-dimensional setting and we consider an angular sector of vertex $O$ and angle $\theta \in [0, \pi]$. If an incident ray arrives at $O$, any “small” perturbation of this ray will hit first one of the two boundary half-lines.
An elementary computation of angles shows that the ray always escapes to horizon after a finite number of bounces. When the angle $\theta$ varies, the expression of the final reflected direction makes appear two extreme bounce laws: either the incident and reflected rays are the same or they are symmetric with respect to the bisecting line. It is remarkable that these two behaviours correspond to a set regularization with $p$-balls $\mathbb{B}_p$, respectively $p = 2$ in the first case and $p \to +\infty$ in the second one.

The paper is organized as follows. Section 2 is devoted to the geometrical approach evoked above in the case of a plane angular sector. In particular, we explicit calculations of incident and reflected angles. In Section 3, we recall useful facts about shock solutions and the Moreau-Yosida approximation. In Section 4, we define a general approximation scheme of a set $C$ by regular sets $C_\varepsilon$. The main result of the paper consists in exhibiting the shock law associated with this scheme. Finally, we end Section 4 by giving explicit bounce laws associated with standard regularization schemes in dimension two.

2 Bounce law on the vertex of a plane sector. Geometrical approach

In the whole section, we work in $H = \mathbb{R}^2$ endowed with its canonical euclidean structure. Given a subset $C \subset H$, we are interested in the problem of the bounce of an incident ray on the boundary of $C$. When the ray meets the boundary at a regular point, it undergoes a symmetry with respect to the normal direction: this is the classical Descartes law. The problem is much more complicated when the shock occurs at a corner of the boundary. Precisely, let us turn to the case of an angular sector of vertex $O$, delimited by the half-lines $D_1$ and $D_2$. We are interested in the shock law of a ray hitting the boundary of the sector at $O$. Our approach consists in considering that, because of the physical nature of the problem, the ray first meets either $D_1$ or $D_2$. We then compute the successive reflected angles by using the classical Descartes law at each bounce on $D_1$ (resp. $D_2$). An elementary calculation shows that the trajectory of the ray always escapes to horizon after a finite number of bounces on $D_1$ and $D_2$. We then express the final reflected angle as a function of the incident one. This is subject of the following proposition.

Before stating our result, we need to introduce several useful arithmetical functions. We will denote by $\text{int} : \mathbb{R} \to \mathbb{Z}$ the integer part function which associates to each $x \in \mathbb{R}$ the largest integer less or equal to $x$. We will also use the function $\text{rnd} : \mathbb{R} \to \mathbb{Z}$ defined by $\text{rnd}(x) = \text{int}(x + 1/2)$; the number $\text{rnd}(x)$ is the rounded integer of $x$, i.e. the nearest integer to $x$. The fractional part $[x]$ of $x$ is defined by $[x] = x - \text{int}(x)$.

**Proposition 2.1** Let us consider an angular sector of vertex $O$ and angle $\theta \in [0, \pi]$, delimited by the half-lines $D_1$ and $D_2$. Let $\Delta$ be the bisecting half-line of the sector. Consider an incident ray making an angle $i \in [-\theta/2, \theta/2]$ with $\Delta$ and hitting the boundary $D_1 \cup D_2$ outside $O$. The trajectory of the ray is assumed to be a straight
line between two consecutive shocks and to follow the classical Descartes law at the
shocks. In such conditions, the following assertions hold:
(i) The trajectory of the ray escapes to horizon after \( N = \text{rnd} (\frac{\pi}{\theta}) \) bounces.
(ii) If the incident ray first hits \( D_1 \) (resp. \( D_2 \)), then the final reflected direction
makes an angle \( r_1 \) (resp. \( r_2 \)) with the bisecting line, given by the formula

\[
r_1 = (-1)^{\text{int}(\pi/\theta)} \left( \left| i - \theta \left( \left\lfloor \pi/\theta \right\rfloor - 1/2 \right) \right| - \theta/2 \right),
\]

\[
\text{resp.} \quad r_2 = (-1)^{\text{int}(\pi/\theta)} \left( - \left| i + \theta \left( \left\lfloor \pi/\theta \right\rfloor - 1/2 \right) \right| + \theta/2 \right).
\]

The proof relies on elementary geometrical considerations and it is postponed to the
appendix at the end of the paper.

**Remark 2.1** Let us notice that formulas (2.1) and (2.2) are considerably simplified
when \( \theta \) is of the form \( \pi/n, \ n \geq 1 \). When \( \theta = \pi/2n \), we have \( r_1 = r_2 = i \), which
means that the directions of the incident and reflected rays are the same. On the
opposite, when \( \theta = \pi/(2n + 1) \), the bounce law reduces to \( r_1 = r_2 = -i \), i.e. the
incident and reflected rays are symmetric with respect to the bisecting line of the
sector.

Let us now come back to the problem of the bounce law at the vertex \( O \). The physical
nature of the bounce is different whether we study the trajectory of a point (like a
luminous ray) or of a material ball with a positive radius. In the following, we focus
on the case of a material point (with null radius), hitting the sector at \( O \). A “small”
perturbation of any incident trajectory will meet first the boundary either on \( D_1 \) or
\( D_2 \). Clearly these two eventualities have the same probability \( p = 1/2 \). Then it is
quite natural to define the bounce law in \( O \) by the formula \( r = (r_1 + r_2)/2 \). From
Proposition 2.1, the average reflected angle \( r \) is equal to

\[
r = \frac{(-1)^{\text{int}(\pi/\theta)}}{2} \left( \left| i - \theta \left( \left\lfloor \pi/\theta \right\rfloor - 1/2 \right) \right| - \left| i + \theta \left( \left\lfloor \pi/\theta \right\rfloor - 1/2 \right) \right| \right).
\]

Let us denote by \( i_0 \) the angle \( i_0 := \left| \pi/\theta - 1/2 \right| \theta \in [0, \theta/2] \). From (2.3), we have

\[
r = \frac{(-1)^{\text{int}(\pi/\theta)}}{2} \begin{cases} 
\left| i - i_0 \right| - \left| i + i_0 \right| & \text{if } [\pi/\theta] \geq 1/2 \\
\left| i + i_0 \right| - \left| i - i_0 \right| & \text{if } [\pi/\theta] \leq 1/2.
\end{cases}
\]

These two equalities can be summarized as

\[
r = \frac{(-1)^{\text{rnd}(\pi/\theta)}}{2} \left( \left| i + i_0 \right| - \left| i - i_0 \right| \right).
\]
When \( \text{rnd}(\pi/\theta) \) is even, the previous expression becomes

\[
  r = \begin{cases} 
  -i_0 & \text{for } i \in [-\theta/2, -i_0] \\
  i & \text{for } i \in [-i_0, i_0] \\
  i_0 & \text{for } i \in [i_0, \theta/2].
  \end{cases} \quad \text{(Shock law: \( \text{rnd}(\pi/\theta) \) even)}
\]

In the particular case \( \theta = \pi/2n \), we exactly have \( r = i \), for every \( i \in [-\theta/2, \theta/2] \). On the other hand, when \( \text{rnd}(\pi/\theta) \) is odd, one has

\[
  r = \begin{cases} 
  i_0 & \text{for } i \in [-\theta/2, -i_0] \\
  -i & \text{for } i \in [-i_0, i_0] \\
  -i_0 & \text{for } i \in [i_0, \theta/2].
  \end{cases} \quad \text{(Shock law: \( \text{rnd}(\pi/\theta) \) odd)}
\]

When \( \theta = \pi/(2n+1) \), the previous expression of \( r \) reduces to \( r = -i \), for every \( i \in [-\theta/2, \theta/2] \).

Figure 1 shows the plotting of the function \( i \mapsto r(i) \) when \( \text{rnd}(\pi/\theta) \) is even (left) and \( \text{rnd}(\pi/\theta) \) is odd (right). We observe two fundamentally different behaviours of the

![Figure 1](image-url)  

Fig. 1. Plotting of the function \([-1/2, 1/2] \ni i \mapsto r(i)/\theta \) for \( \theta = 3\pi/5 \) (left) and \( \theta = 3\pi/4 \) (right).

... the bounce law \( i \mapsto r(i) \), depending on the evenness of the quantity \( \text{rnd}(\pi/\theta) \). Roughly speaking, the directions of the incident and reflected rays are the same in the even case and are symmetric with respect to the bisecting line \( \Delta \) in the odd case. In the next sections, we will try to recover this duality by an adequate regularization of the sector around its vertex \( O \).
3 Modelization of shocks by a differential inclusion. Regularization approach

3.1 Differential inclusion (E). Shock solutions

Let $H$ be a finite dimensional Hilbert space and $C$ be a closed convex subset of $H$. We are interested in the bounce of a material point on the boundary of $C$. For that purpose, we modelize the motion by the following second-order in time differential inclusion:

$$(E) \quad \ddot{x}(t) + N_C(x(t)) \ni 0, \quad t > 0,$$

where $N_C(x)$ denotes the exterior normal cone to $C$ at point $x$. Because of the bounce phenomenon, one cannot expect the differential inclusion (E) to have a regular solution. We have to accept, as possible solutions, functions whose second derivatives are measures defined on $\mathbb{R}_+$ with values in $H$. This remark leads us to the notion of shock solution.

Before defining this notion, let us precise our notations. In the whole section, we fix some positive $T > 0$ and we will study the trajectories of (E) on the interval $I := [0,T]$. If $x$ is a function from $I$ into $H$, then $\dot{x}$, $\ddot{x}$ denote its distributional derivatives. The set $C(I, H)$ is the space of continuous functions from $I$ into $H$. The set $\mathcal{M}(I, H)$ is the space of Radon measures on $I$ with values in $H$, that is the space dual to $C(I, H)$ equipped with its usual inductive limit topology; it may be identified with the space of regular Borel measures on $I$ with values in $H$ that are of finite variation (Dinculeanu [5, §19]).

The set $BV(I, H)$ is the space of functions from $I$ into $H$ that are of bounded variation. Every $u \in BV(I, H)$ has left and right limits, $u^-(t)$ and $u^+(t)$ at any point $t \in \text{int}(I)$. It is well known that $BV(I, H)$ is the space of functions whose distributional derivative belongs to $\mathcal{M}(I, H)$, see [1].

Let us now state the definition of shock solution, which may already be found in [2,7,12,14].

**Definition 3.1** A Lipschitz continuous function $x : I \rightarrow H$ is said to be a shock solution to (E) if it verifies:

1. The map $\dot{x}$ has a bounded variation, i.e. $\dot{x} \in BV(I, H)$.
2.a For every $t \in I$, $x(t) \in C$.
2.b For every $y \in C(I, H)$,

$$\forall t \in I, \quad y(t) \in C \implies \langle \ddot{x}, y - x \rangle_{(\mathcal{M}, C)} \geq 0,$$

where $\langle ., . \rangle_{(\mathcal{M}, C)}$ denotes the duality bracket between $\mathcal{M}(I, H)$ and $C(I, H)$.

**Definition 3.2** The function $x$ is an elastic shock solution of (E) if it is a shock
solution satisfying for every \( t \) in \( I \)

\[
| \dot{x}^+(t) |^2 = | \dot{x}^-(t) |^2.
\]

The following proposition shows the link between classical and shock solutions.

**Proposition 3.1** Let \( H \) be a finite dimensional Hilbert space and \( C \) be a closed convex subset of \( H \) such that \( \text{int}(C) \neq \emptyset \). For any shock solution \( x : I \to H \) of (E), the following properties hold:

(a) \( \tilde{x}_a(t) \in -N_C(x(t)) \) for almost every \( t \) in \( I \), where \( \tilde{x}_a \) is the density of \( \tilde{x} \) with respect to the Lebesgue measure,

(b) \( \dot{x}^+(t) - \dot{x}^-(t) \in -N_C(x(t)) \) for all \( t \) in \( I \).

The main ingredient of the proof results from Corollary 5.A of [13]. For more details, we refer the reader to [2,7,14]. From (a) we recover the fact that the trajectory \( x(.) \) is a straight line as soon as it leaves the boundary of \( C \). On the other hand, from (b) we deduce the classical Descartes law in the case of elastic shocks at regular points of the boundary. Indeed, we have:

**Corollary 3.1 (Descartes law)** Let \( C \) be a closed convex subset of \( H \) such that \( \text{int}(C) \neq \emptyset \). Consider an elastic shock solution \( x(.) \) of (E). Let \( t > 0 \) such that the normal cone \( N_C(x(t)) \) is reduced to the half-line \( \mathbb{R}_+ n \). Then the shock law at time \( t \) is given by

\[
\dot{x}^+(t) = \text{sym}_{n^\perp} (\dot{x}^-(t)),
\]

where \( \text{sym}_{n^\perp} \) denotes the symmetry with respect to the hyperplane \( n\perp = \{ \nu \in H, \langle \nu, n \rangle = 0 \} \).

For the sake of completeness, we give the proof of this elementary result.

**Proof.** From Proposition 3.1 (b), there exists \( \lambda \geq 0 \) such that

\[
\dot{x}^+(t) - \dot{x}^-(t) = -\lambda \ n.
\]

If \( \lambda = 0 \), then \( \dot{x}^+(t) = \dot{x}^-(t) \) and these two elements respectively belong to the tangent cone to \( C \) at \( x(t) \) and its opposite, whose intersection is reduced to \( n\perp \). Consequently \( \dot{x}^+(t) = \dot{x}^-(t) \in n\perp \) and hence relation (3.1) holds.

Now assume that \( \lambda > 0 \). Using (3.2), we obtain

\[
\langle \dot{x}^+(t) + \dot{x}^-(t), n \rangle = -\frac{1}{\lambda} \langle \dot{x}^+(t) + \dot{x}^-(t), \dot{x}^+(t) - \dot{x}^-(t) \rangle \\
= -\frac{1}{\lambda} (|\dot{x}^+(t)|^2 - |\dot{x}^-(t)|^2) \\
= 0,
\]

from the definition of an elastic shock solution. As a consequence, \( \dot{x}^+(t) + \dot{x}^-(t) \in n\perp \), which combined with (3.2) clearly implies relation (3.1). \( \square \)
3.2 Approximation by epiconvergence

A classical approach to study the differential inclusion (E) consists in using an approximation method. The idea is to approximate the indicator function $\delta_C$ of $C$ by a sequence of "smooth" convex functions $(\Phi_n)_{n \in \mathbb{N}}$, and study the convergence of the sequence $(x_n)_{n \in \mathbb{N}}$ of the corresponding classical solutions of the equations

$$\dot{x}_n(t) + \nabla \Phi_n(x_n(t)) = 0. \quad (3.3)$$

In order to recover various types of approximation methods (like penalty and barrier methods, Moreau- Yosida approximations,...), Attouch, Cabot and Redont have introduced in [2] the following general approximation scheme ($\mathcal{H}_{\text{app}}$):

(i) For each $n \in \mathbb{N}$, $\Phi_n : H \to \mathbb{R} \cup \{+\infty\}$ is a closed convex function whose domain is open in $H$ and $\Phi_n$ is of class $C^1$ on its domain.

(ii) $\Phi_n \to \delta_C$ in the sense of epi-convergence as $n \to +\infty$.

They have proved that any trajectory $x(.)$ obtained as a limit of approximate trajectories $x_n(.)$ of (3.3) is an elastic shock solution of (E). In particular, from Corollary 3.1 the shock law at a regular point of the boundary of $C$ is prescribed by the Descartes law. However, when the trajectory $x(.)$ meets a corner of the boundary, then the shock law actually depends on the approximation scheme (see [2]). In the particular case of the Moreau-Yosida approximation, Shatzman [14] has exhibited the shock law at the corners. Calling $t$ the shock instant, it is given by

$$\dot{x}^+(t) = 2 \text{ proj}_T(\dot{x}^-(t)) - \dot{x}^-(t), \quad \text{(M.Y. approximation)}$$

where $\text{proj}_T$ is the projection on the tangent cone $T$ to the set $C$ at $x(t)$. In Buttazzo-Percivale [3,4], different types of approximation of the bounce problem are considered and they essentially correspond to exterior penalizations.

In the next section, we propose a different method, still based on approximation, to regularize the differential inclusion (E).

4 Shock law obtained by $\varepsilon$–regularization of the set $C$

Let $H$ be a finite dimensional Hilbert space and $C$ be a closed convex subset of $H$. Our purpose in this section is to study the differential inclusion (E)

$$(E) \quad \ddot{x}(t) + N_C(x(t)) \ni 0, \quad t > 0,$$

by means of a set regularization of $C$. More precisely, we are going to define a sequence $(C_{\varepsilon})$ of regular sets tending toward $C$ in the sense of Painlevé-Kuratowski.
Then we will study the trajectory \( x_\varepsilon(t) \) of the following differential inclusion:

\[
(E_\varepsilon) \quad \dot{x}_\varepsilon(t) + N_{C_\varepsilon}(x_\varepsilon(t)) \ni 0, \quad t > 0,
\]

and finally we will pass to the limit when \( \varepsilon \) tends to 0. Notice that, since the sets \( C_\varepsilon \) are regular, the shocks appearing in \( (E_\varepsilon) \) are simply governed by the Descartes law.

### 4.1 Construction of regularized sets \( C_\varepsilon \)

Let \( f : H \to \mathbb{R}_+ \) a \( C^1 \) function which satisfies the following assumptions \( (\mathcal{H}_f) \):

(i) \( f \) is strictly convex on \( H \).

(ii) \( f \) is coercive, i.e. \( \lim_{|x| \to +\infty} f(x) = +\infty \).

(iii) \( f(x) = 0 \iff x = 0 \).

For any closed convex set \( C \subset H \), let us define the \( f \)-distance to the set \( C \) by

\[
d^f(x, C) = \inf_{y \in C} f(x - y).
\]

In this definition of \( d^f(x, C) \), the infimum is achieved at a unique point, that we will denote by \( \text{proj}^f_C(x) \). The existence of \( \text{proj}^f_C(x) \) is due to the coercivity of \( f \) and the uniqueness is an immediate consequence of the strict convexity of \( f \). For every \( \varepsilon > 0 \), let us define

\[
\mathbb{B}^f_\varepsilon := \{ x \in H, \quad f(x) \leq \varepsilon \}.
\]

The set \( \mathbb{B}^f_\varepsilon \) is closed, convex and bounded since it is the \( \varepsilon \)- sublevel set of the function \( f \), which is continuous, convex and coercive. The following proposition states several results which will be of interest in the sequel.

**Proposition 4.1** Let \( H = \mathbb{R}^n \) and \( f : H \to \mathbb{R}_+ \) a \( C^1 \) function satisfying \( (\mathcal{H}_f) \). For any closed convex set \( C \) and \( \varepsilon > 0 \), we define the convex set \( C_\varepsilon := C + \mathbb{B}^f_\varepsilon \). Then the following properties hold

(i) \( \bigcap_{\varepsilon > 0} C_\varepsilon = C \).

(ii) For \( x_0 \in C \), we denote by \( \gamma_{C_\varepsilon - x_0} \) the gauge function of the set \( C_\varepsilon - x_0 \)

\[
\gamma_{C_\varepsilon - x_0}(u) := \inf\{ \lambda \geq 0, \quad u \in \lambda(C_\varepsilon - x_0) \}.
\]

If \( x_0 \in \text{bd}(C) \) and \( u \notin T_C(x_0) \), then \( \lim_{\varepsilon \to 0} \gamma_{C_\varepsilon - x_0}(u) = +\infty \).
(iii) The set $C_\varepsilon$ is regular and for every $x \in \partial C_\varepsilon$, the normal direction to the set $C_\varepsilon$ at $x$ is given by the vector \( \nabla f(x - \text{proj}_C(x)) \).

**Proof.** (i) Let $x \in \bigcap_{\varepsilon > 0} C_{\varepsilon}$. For every $\varepsilon > 0$, we have $x \in C + \mathbb{B}_\varepsilon^f$ and hence $d^f(x, C) \leq \varepsilon$. This being true for every $\varepsilon > 0$, we deduce $d^f(x, C) = 0$. In view of $(\mathcal{H}_f - \text{iii})$ and the fact that $C$ is closed, this implies $x \in C$.

(ii) Let $x_0 \in \partial C$ and $u \not\in T_C(x_0)$. Since the map $\varepsilon \mapsto C_{\varepsilon}$ is increasing (for the set inclusion), we clearly deduce that the map $\varepsilon \mapsto \gamma_{C_{\varepsilon} - x_0}(u)$ is decreasing. As a consequence, $\lim_{\varepsilon \to 0} \gamma_{C_{\varepsilon} - x_0}(u)$ exists. Let us argue by contradiction and assume that $\lim_{\varepsilon \to 0} \gamma_{C_{\varepsilon} - x_0}(u) < +\infty$, i.e. there exists $m > 0$ such that $\gamma_{C_{\varepsilon} - x_0}(u) \leq m$ for every $\varepsilon > 0$. This means that, for every $\varepsilon > 0$,

$$u \in m \,(C_{\varepsilon} - x_0).$$

Passing to the limit when $\varepsilon \to 0$ and using (i), we obtain $u \in m \,(C - x_0) \subset T_C(x_0)$, which contradicts the assumption $u \not\in T_C(x_0)$.

(iii) Let us decompose $x$ as

$$x = \text{proj}_C^f(x) + (x - \text{proj}_C^f(x)).$$

Since $x \in C + \mathbb{B}_\varepsilon^f$, we have

$$f(x - \text{proj}_C^f(x)) = d^f(x, C) \leq \varepsilon$$

and hence $x - \text{proj}_C^f(x) \in \mathbb{B}_\varepsilon^f$. On the other hand, $\text{proj}_C^f(x) \in C$ and equality (4.1) then yields

$$N_{C_{\varepsilon}}(x) = N_C(\text{proj}_C^f(x)) \cap N_{\mathbb{B}_\varepsilon^f}(x - \text{proj}_C^f(x))$$

where $N_{C_{\varepsilon}}$ (resp. $N_C$, $N_{\mathbb{B}_\varepsilon^f}$) denotes the outwards normal cone to the set $C_{\varepsilon}$ (resp. $C$, $\mathbb{B}_\varepsilon^f$). We then deduce

$$N_{C_{\varepsilon}}(x) \subset N_{\mathbb{B}_\varepsilon^f}(x - \text{proj}_C^f(x)) \subset \mathbb{R}_+ \cdot \nabla f(x - \text{proj}_C^f(x)).$$

Since $x \in \partial C_{\varepsilon}$, the normal cone $N_{C_{\varepsilon}}(x)$ is not reduced to the singleton \( \{0\} \) and the conclusion follows. \qed

4.2 Limit bounce law associated with the approximation scheme $(C_\varepsilon)$

We keep here the same notations as in the previous paragraph. The function $f$ satisfies $(\mathcal{H}_f)$ and for every $\varepsilon > 0$, we define the $\varepsilon$-ball $\mathbb{B}_\varepsilon^f := \{ x \in H, \quad f(x) \leq \varepsilon \}$ and the $\varepsilon$-regularized set $C_{\varepsilon} := C + \mathbb{B}_\varepsilon^f$ as above. In the following, we will assume that $f$ is positively homogeneous, i.e. there exists some $p > 1$, such that, for every $\lambda \in \mathbb{R}_+$ and $x \in H$,

$$f(\lambda x) = \lambda^p f(x).$$
We are interested in the bounce law of the trajectories \( x(\cdot) \) of (E) hitting the boundary of \( C \) at \( x_0 \in \text{bd}(C) \). For that purpose, let us consider the Cauchy problem

\[
\begin{align*}
\text{(E)} & \quad \begin{cases} 
\ddot{x}(t) + N_C(x(t)) \ni 0, & t > 0, \\
x(0) = x_0, & \dot{x}(0) = u,
\end{cases} \\
\end{align*}
\]

and its approximate version

\[
\begin{align*}
\text{(E}_\varepsilon) & \quad \begin{cases} 
\ddot{x}_\varepsilon(t) + N_{C_\varepsilon}(x_\varepsilon(t)) \ni 0, & t > 0, \\
x_\varepsilon(0) = x_0, & \dot{x}_\varepsilon(0) = u.
\end{cases} \\
\end{align*}
\]

The main result of the paper is summarized by the following theorem.

**Theorem 4.1** Let \( H \) be a finite dimensional space and \( C \) be a closed convex subset of \( H \). Let us fix a boundary point \( x_0 \) of \( C \) and let us denote by \( T \) the tangent cone to \( C \) at \( x_0 \). The function \( f : H \to \mathbb{R}_+ \) is assumed to satisfy \((H_f)\) and to be positively homogeneous. For every \( \varepsilon > 0 \), we define the \( f \)-regularized set \( C_\varepsilon \) like in Proposition 4.1. For every \( u \not\in T \), let \( x_\varepsilon \) denote some elastic shock solution of \((E_\varepsilon)\) with initial conditions \( x_\varepsilon(0) = x_0 \) and \( \dot{x}_\varepsilon(0) = u \). Calling by \( t_\varepsilon \) the shock instant of the trajectory \( x_\varepsilon(\cdot) \) on the boundary of \( C_\varepsilon \), we have

\[
\begin{align*}
\dot{x}_\varepsilon^+(t_\varepsilon) = \lim_{\varepsilon \to 0} \dot{x}_\varepsilon(t_\varepsilon) = \text{sym}_{n(u)\perp}(u),
\end{align*}
\]

where

\[
n(u) := \nabla f \left( u - \text{proj}^f_T(u) \right) \in N_T(\text{proj}^f_T(u))
\]

and \( \text{sym}_{n(u)\perp} \) denotes the orthogonal symmetry with respect to the normal hyperplane to the vector \( n(u) \). When \( u \) belongs to the cone \( (\nabla f)^{-1}(N_C(x_0)) \), the expression of \( n(u) \) reduces to \( n(u) = \nabla f(u) \).

**Proof.** For any shock solution \( x_\varepsilon \) of \((E_\varepsilon)\) satisfying \( x_\varepsilon(0) = x_0 \) and \( \dot{x}_\varepsilon(0) = u \), the velocity vector is constant (equal to \( u \)) before the shock on the boundary of \( C_\varepsilon \), so that the shock instant equals \( t_\varepsilon = \sup \{ \lambda > 0 \mid x_0 + \lambda u \in C_\varepsilon \} \). Since the set \( C_\varepsilon \) is regular everywhere on its boundary, we deduce from Corollary 3.1 that the velocity \( \dot{x}_\varepsilon^+(t_\varepsilon) \) after the shock is given by the Descartes law

\[
\dot{x}_\varepsilon^+(t_\varepsilon) = \text{sym}_{n_\varepsilon\perp}(u),
\]

where \( n_\varepsilon \) denotes a normal vector to \( C_\varepsilon \) at the point \( x_\varepsilon(t_\varepsilon) \in \text{bd}(C_\varepsilon) \) and \( n_\varepsilon\perp \) is the tangent hyperplane to \( C_\varepsilon \) at \( x_\varepsilon(t_\varepsilon) \). From Proposition 4.1 (iii), we can take \( n_\varepsilon = \nabla f\left( x_\varepsilon(t_\varepsilon) - \text{proj}^f_C(x_\varepsilon(t_\varepsilon)) \right) \) and hence

\[
\dot{x}_\varepsilon^+(t_\varepsilon) = u - 2 \frac{\langle u, \nabla f\left( x_\varepsilon(t_\varepsilon) - \text{proj}^f_C(x_\varepsilon(t_\varepsilon)) \right) \rangle}{\| \nabla f\left( x_\varepsilon(t_\varepsilon) - \text{proj}^f_C(x_\varepsilon(t_\varepsilon)) \right) \|^2} \nabla f\left( x_\varepsilon(t_\varepsilon) - \text{proj}^f_C(x_\varepsilon(t_\varepsilon)) \right). \quad (4.4)
\]
Notice that the shock instant \( t_\varepsilon \) and the vectors \( x_\varepsilon(t_\varepsilon) \) and \( \dot{x}_\varepsilon^+(t_\varepsilon) \) are independent of the choice of the elastic shock solution \( x_\varepsilon \). From the definition of \( t_\varepsilon \), we have
\[
\frac{1}{t_\varepsilon} = \inf \{ \mu > 0, \ u \in \mu(C_\varepsilon - x_0) \} = \gamma_{C_\varepsilon - x_0}(u),
\]
and in view of Proposition 4.1 (ii), we deduce
\[
\lim_{\varepsilon \to 0} t_\varepsilon = 0. \tag{4.5}
\]
Lemma 4.6 of Zarantonello [15] allows to write
\[
\text{proj}_\varepsilon^f(x_0 + t_\varepsilon u) = x_0 + \text{proj}_\varepsilon^f(t_\varepsilon u) + \eta(t_\varepsilon u), \tag{4.6}
\]
where \( T \) denotes the tangent cone to the set \( C \) at \( x_0 \) and \( \eta : H \to H \) is a function satisfying \( \lim_{x \to x_0} |\eta(x)|/|x| = 0 \). Since the function \( f \) is positively homogeneous and since the set \( T \) is a cone, we have \( \text{proj}_\varepsilon^f(t_\varepsilon u) = t_\varepsilon \text{proj}_\varepsilon^f(u) \) and hence in view of (4.6),
\[
\dot{x}_\varepsilon(t_\varepsilon) - \text{proj}_\varepsilon^f(x_\varepsilon(t_\varepsilon)) = t_\varepsilon(u - \text{proj}_\varepsilon^f(u)) - \eta(t_\varepsilon u).
\]
From (4.4) combined with the fact that \( f \) is positively homogeneous, we then deduce
\[
\dot{x}_\varepsilon^+(t_\varepsilon) = u - 2 \frac{\left\langle u, \nabla f \left( u - \text{proj}_\varepsilon^f(u) - \eta(t_\varepsilon u)/t_\varepsilon \right) \right\rangle}{\left\| \nabla f \left( u - \text{proj}_\varepsilon^f(u) - \eta(t_\varepsilon u)/t_\varepsilon \right) \right\|^2} \nabla f \left( u - \text{proj}_\varepsilon^f(u) - \eta(t_\varepsilon u)/t_\varepsilon \right).
\]
Taking into account (4.5) and passing to the limit when \( \varepsilon \) tends to 0, we finally obtain
\[
\lim_{\varepsilon \to 0} \dot{x}_\varepsilon^+(t_\varepsilon) = u - 2 \frac{\left\langle u, \nabla f \left( u - \text{proj}_\varepsilon^f(u) \right) \right\rangle}{\left\| \nabla f \left( u - \text{proj}_\varepsilon^f(u) \right) \right\|^2} \nabla f \left( u - \text{proj}_\varepsilon^f(u) \right) = \text{sym}_n(u)^\perp(u),
\]
where \( n(u) := \nabla f \left( u - \text{proj}_\varepsilon^f(u) \right) \) and \( n(u)^\perp \) is the normal hyperplane to the vector \( n(u) \).
The first order optimality condition applied to the problem \( \min \{ f(u - v), \ v \in T \} \) yields \( \nabla f(u - v) \in N_T(v) \). Since \( f \) is convex, the last condition is both necessary and sufficient so that
\[
v = \text{proj}_T^f(u) \iff \nabla f(u - v) \in N_T(v). \tag{4.7}
\]
In particular, we deduce that \( n(u) \in N_T(\text{proj}_T^f(u)) \). On the other hand, from (4.7), we have
\[
\text{proj}_T^f(u) = 0 \iff \nabla f(u) \in N_T(0) = N_C(x_0)
\iff u \in (\nabla f)^{-1}(N_C(x_0)).
\]
As a consequence, \( u \in (\nabla f)^{-1}(N_C(x_0)) \) implies that \( n(u) = \nabla f(u) \), which ends the proof of Theorem 4.1. \( \square \)
Remark 4.1 For simplicity of notations, only shocks at time 0 are considered in Theorem 4.1. However shocks at time $t_0$ can be simply deduced by the change of variables $t = t_0 + \tau$, $\tau \geq 0$.

Remark 4.2 It is important to notice that Theorem 4.1 does not provide the global (or even local) existence of a solution $x : [0, +\infty] \rightarrow H$ of (E) obtained as a limit of solutions $x_\varepsilon$ of $(E_\varepsilon)$ when $\varepsilon \rightarrow 0$. This question is out of the scope of the paper and could be addressed in a next study.

Let us illustrate Theorem 4.1 by two instructive examples.

Example 4.1 A first example is given by the square of euclidean norms

$$f(x) = |x|^2_A = \langle Ax, x \rangle,$$

where $A \in \mathcal{M}_n(\mathbb{R})$ is a symmetric definite positive matrix. In such a case, the vector $n(u)$ of Theorem 4.1 equals to $n(u) = 2A(u - \text{proj}_T(u))$.

Example 4.2 Another example comes from the $p$-norms:

$$f(x) = |x|^p_p = \sum_{i=1}^n |x_i|^p, \quad p \in ]1, +\infty[.$$

Setting $u = [u_1, \ldots, u_n]^t$, the gradient $\nabla f(u)$ is then directed by the vector $v = [u_1|u_1|^{p-2}, \ldots, u_n|u_n|^{p-2}]^t$.

Notice that the particular cases $A = Id$ in the first example and $p = 2$ in the second one lead the same function $f$: $f(x) = |x|^2$, i.e. the square of the canonical euclidean norm. In such a situation the bounce law (4.3) can be simplified as shows the following corollary of Theorem 4.1.

Corollary 4.1 Under the hypotheses of Theorem 4.1, additionally assume that $f = |.|^2$. The bounce law (4.3) can then be rewritten as

$$u^+ = 2\text{proj}_T(u) - u,$$

where $T$ is the tangent cone to the set $C$ at $x_0$ and $\text{proj}_T$ is the projection on the convex set $T$.

Proof. From Theorem 4.1 applied with $f = |.|^2$, we deduce

$$u^+ = u - 2\frac{\langle u, u - \text{proj}_T(u) \rangle}{|u - \text{proj}_T(u)|^2} (u - \text{proj}_T(u)). \quad (4.8)$$

Since $\langle \text{proj}_T(u), u - \text{proj}_T(u) \rangle = 0$, we have

$$\langle u, u - \text{proj}_T(u) \rangle = |u - \text{proj}_T(u)|^2,$$
which combined with (4.8) gives \( u^+ = 2 \operatorname{proj}_T(u) - u \). \( \square \)

It is remarkable that the bounce law obtained in the previous corollary is exactly the same as the one emanating from the Moreau-Yosida approximation (cf. Paragraph 3.2). This is not very surprising since the sublevel sets of the Moreau-Yosida approximates of \( \delta_C \) coincide exactly with the sets \( C_\varepsilon \) in the case \( f = |.|^2 \).

### 4.3 Applications in dimension two

This paragraph is devoted to precise the results of the previous section in the two dimensional case. In particular, we are going to express the reflected angle \( r \) as a function of the incident one \( i \).

**Proposition 4.2** Under the hypotheses and notations of Theorem 4.1, assume that \( H = \mathbb{R}^2 \). Let us denote by \( u \) (resp. \( u^+ \)) the incident (resp. reflected) vector before (resp. after) the shock on \( C \) at \( x_0 \). Let \( \theta \in [0, \pi] \) and \( \Delta \) be the respective angle and bisecting half-line of the angular sector \( T := T_C(x_0) \). We denote by \( i \) (resp. \( r \)) the geometric angle of the vector \( u \) (resp. \( u^+ \)) with \( \Delta \). Let us set

\[
\alpha(i) := \arctan \frac{\partial f/\partial y (1, \tan i)}{\partial f/\partial x (1, \tan i)},
\]

where \( \partial f/\partial x \) and \( \partial f/\partial y \) are the respective coordinates of the vector \( \nabla f \) in the orthogonal basis \((\Delta, \Delta^\perp)\). Then, the angle \( r \) continuously depends on \( i \) as follows

\[
r = \begin{cases} 
-(\pi - \theta) - i & \text{if } \alpha(i) \leq -\frac{\pi-\theta}{2} \\
2 \alpha(i) - i & \text{if } \alpha(i) \in \left[-\frac{\pi-\theta}{2}, \frac{\pi-\theta}{2}\right] \\
(\pi - \theta) - i & \text{if } \alpha(i) \geq \frac{\pi-\theta}{2}
\end{cases}
\]

**Proof.** With the notations of Theorem 4.1, let \( \beta \) be the angle of the vector \( n(u) \) with \( \Delta \). From formula (4.3), it is clear that \((i + r)/2 = \beta\), and hence

\[
r = 2 \beta - i. \tag{4.9}
\]

We need to express \( \beta \) as a function of \( i \). From Theorem 4.1, we have \( n(u) = \nabla f (u - \operatorname{proj}_T^f (u)) \in N_T (\operatorname{proj}_T^f (u)) \). We are going to distinguish the cases \( \operatorname{proj}_T^f (u) = 0 \) and \( \operatorname{proj}_T^f (u) \neq 0 \).

- First assume that \( \operatorname{proj}_T^f (u) = 0 \). In such conditions, we have \( n(u) = \nabla f (u) \) and
hence the angle $\beta$ satisfies
\[ \tan \beta = \frac{\partial f/\partial y (x, y)}{\partial f/\partial x (x, y)}, \]
where $x$ and $y$ are the coordinates of the vector $u$. Since $f$ is positively homogeneous of degree $p$, the derivatives $\partial f/\partial x$ and $\partial f/\partial y$ are positively homogeneous of degree $p - 1$, which implies
\[ \tan \beta = \frac{\partial f/\partial y (1, y/x)}{\partial f/\partial x (1, y/x)} = \frac{\partial f/\partial y (1, \tan i)}{\partial f/\partial x (1, \tan i)}, \]
and hence
\[ \beta = \arctan \frac{\partial f/\partial y (1, \tan i)}{\partial f/\partial x (1, \tan i)} = \alpha(i). \quad (4.10) \]
From (4.9) and (4.10), we deduce $r = 2 \alpha(i) - i$. Notice that the condition $\text{proj}_T^f(u) = 0$ is equivalent to $\nabla f(u) \in N_C(x_0)$. On the other hand, the cone $N_C(x_0)$ is delimited by two straight lines making respectively an angle $\frac{\pi - \theta}{2}$ and $-\frac{\pi - \theta}{2}$ with $\Delta$. Hence the condition $\text{proj}_T^f(u) = 0$ can be rewritten as $\alpha(i) \in [-\frac{\pi - \theta}{2}, \frac{\pi - \theta}{2}]$.

- Now assume that $\text{proj}_T^f(u) \neq 0$, i.e. $\alpha(i) \notin [-\frac{\pi - \theta}{2}, \frac{\pi - \theta}{2}]$. In this case, the vector $\text{proj}_T^f(u)$ makes an angle $\theta/2$ or $-\theta/2$ with $\Delta$. Since $n(u) \in N_T(\text{proj}_T^f(u))$, we have $\beta = \theta/2 - \pi/2$ in the first case and $\beta = -\theta/2 + \pi/2$ in the second one. This combined with (4.9) yields respectively $r = -\left(\pi - \theta\right) - i$ and $r = \left(\pi - \theta\right) - i$, which ends the proof. \( \square \)

Let us now illustrate the previous proposition on examples.

**Example 4.3** Take the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(X) = \langle AX, X \rangle$, where $X = (x, y) \in \mathbb{R}^2$ and $A \in \mathcal{M}_2(\mathbb{R})$ is a symmetric definite positive matrix. There exist $a, b, c \in \mathbb{R}$ such that $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. The quantity $\alpha(i)$ of Proposition 4.2 then equals to
\[ \alpha(i) = \arctan \frac{b + c \tan i}{a + b \tan i}. \]
Let us now focus on the particular case $A = Id$, i.e. $a = c = 1$ and $b = 0$. The quantity $\alpha(i)$ then reduces to $\alpha(i) = i$ and Proposition 4.2 yields the following expression of $r$ as a function of $i$
\[ r = \begin{cases} 
-\left(\pi - \theta\right) - i & \text{if } i \leq -\frac{\pi - \theta}{2} \\
i & \text{if } i \in \left[-\frac{\pi - \theta}{2}, \frac{\pi - \theta}{2}\right] \\
\left(\pi - \theta\right) - i & \text{if } i \geq \frac{\pi - \theta}{2}
\end{cases} \]
When the angle of the sector $T$ is acute, i.e. $\theta < \pi/2$, then $i \in [-\theta/2, \theta/2]$ imposes $i \in [-\frac{\pi - \theta}{2}, \frac{\pi - \theta}{2}]$, so that the shock law becomes $r = i$ for every $i \in [-\theta/2, \theta/2]$. The
directions of the incident and reflected rays are the same, which corresponds to the result given by the geometrical approach in the case \( \text{rnd}(\pi/\theta) \) even (see Section 2).

**Example 4.4** Take the function \( f : \mathbb{R}^2 \to \mathbb{R} \) defined by \( f(x, y) = |x|^p + |y|^p, \ p > 1 \). One easily checks that the function \( i \mapsto \alpha(i) \) of Proposition 4.2 is odd and its restriction to \([0, \theta/2]\) is given by

\[
\alpha(i) = \arctan(\tan^{p-1}(i)). \tag{4.11}
\]

In the particular case \( p = 2 \), the previous formula gives \( \alpha(i) = i \) and we recover the result of the previous example with \( A = \text{Id} \). Let us now consider the case where \( p \) tends toward infinity. Passing to the limit when \( p \to +\infty \), formula (4.11) yields \( \alpha(i) = 0 \) if \( i \in [0, \pi/4[ \) and \( \alpha(i) = \pi/2 \) if \( i \in ]\pi/4, \pi/2[ \), thus implying the following shock law

\[
r = \begin{cases} 
-(\pi - \theta) - i & \text{if } i < -\pi/4 \\
-i & \text{if } i \in ]-\pi/4, \pi/4[ \\
(\pi - \theta) - i & \text{if } i > \pi/4. 
\end{cases}
\]

Notice that the function \( i \mapsto r(i) \) is not continuous here. When the angle of the sector \( T \) is acute, i.e. \( \theta < \pi/2 \), then \( i \in [-\theta/2, \theta/2[ \) imposes \( i \in ]-\pi/4, \pi/4[ \), so that the shock law becomes \( r = -i \) for every \( i \in [-\theta/2, \theta/2[ \). The directions of the incident and reflected rays are symmetric with respect to the bisecting line \( \Delta \), which corroborates the result given by the geometrical approach in the case \( \text{rnd}(\pi/\theta) \) odd (see Section 2).

5 **Appendix: proof of Proposition 2.1**

In the whole proof, we consider the orthogonal basis \((\Delta, \Delta^\perp)\), where the half-line \( \Delta^\perp \) is chosen so that \((\Delta, D_1) = -\theta/2\) and \((\Delta, D_2) = \theta/2\).

(i) Let \( u \in \mathbb{R}^2 \) a vector having the same direction and sense as the incident ray. The oriented angle of the vector \( u \) with \( \Delta \) is equal to \( i - \pi \). At each bounce on \( D_1 \) (resp. \( D_2 \)), the reflected ray undergoes a symmetry \( \text{sym}_{D_1} \) (resp. \( \text{sym}_{D_2} \)) with respect to \( D_1 \) (resp. \( D_2 \)). Without any loss of generality, one can assume that the incident ray first hits \( D_1 \). After \( k \) bounces, the reflected vector \( u_k \) is equal to

- \( u_k = \underbrace{\text{sym}_{D_2} \circ \text{sym}_{D_1}}_{k/2 \text{ terms}} \circ \ldots \circ \underbrace{\text{sym}_{D_2} \circ \text{sym}_{D_1}}_{k/2 \text{ terms}}(u) \) if \( k \) is even.
- \( u_k = \text{sym}_{D_1} \circ \underbrace{\text{sym}_{D_2} \circ \text{sym}_{D_1}}_{(k-1)/2 \text{ terms}} \circ \ldots \circ \underbrace{\text{sym}_{D_2} \circ \text{sym}_{D_1}}_{(k-1)/2 \text{ terms}}(u) \) if \( k \) is odd.
Denoting by $\text{rot}_\phi$ the rotation of center $O$ and angle $\phi$, we classically have

$$\text{sym}_{D_2} \circ \text{sym}_{D_1} = \text{rot}_{2\theta}. $$

First assume that the number $k$ of bounces is even. We then obtain $u_k = \text{rot}_{k\theta}(u)$ and the oriented angle of $u_k$ with $\Delta$ is given by

$$\phi_k = i - \pi + k\theta. $$  \hspace{1cm} (5.1)

When $k$ is odd, the vector $u_k$ is equal to $u_k = \text{sym}_{D_1} \circ \text{rot}_{(k-1)\theta}(u)$ and its oriented angle with $\Delta$ is given by

$$\phi_k = -\theta - (i - \pi + (k - 1)\theta) = -(i - \pi + k\theta). $$  \hspace{1cm} (5.2)

Formulas (5.1) and (5.2) can be unified as follows

$$\phi_k = (-1)^k (i - \pi + k\theta). $$  \hspace{1cm} (5.3)

The total number $N$ of bounces is determined by the condition $\phi_N \in [-\theta/2, \theta/2]$. From (5.3), we have

$$\phi_N \in [-\theta/2, \theta/2] \iff (-1)^N (i - \pi + N\theta) \in [-\theta/2, \theta/2]$$

$$\iff i - \pi + N\theta \in [-\theta/2, \theta/2]$$

$$\iff N \in \left[\frac{\pi - i}{\theta} - \frac{1}{2}, \frac{\pi - i}{\theta} + \frac{1}{2}\right],$$

and we deduce that $N = \text{rnd} \left(\frac{\pi - i}{\theta}\right)$.

\hspace{1cm} (ii) Let us first prove formula (2.1). From (5.3), the angle $r_1 = \phi_N$ is given by

$$r_1 = (-1)^{\text{rnd} \left(\frac{\pi - i}{\theta}\right)} \left(\text{md} \left(\frac{\pi - i}{\theta}\right) - \frac{\pi - i}{\theta}\right) \theta. $$  \hspace{1cm} (5.4)

Let us prove that this last expression is equivalent to (2.1). Since $i \in [-\theta/2, \theta/2]$, we have $\pi/\theta - 1/2 \leq \frac{\pi - i}{\theta} \leq \pi/\theta + 1/2$ and hence either $\text{rnd} \left(\frac{\pi - i}{\theta}\right) = \text{int}(\pi/\theta)$ or $\text{rnd} \left(\frac{\pi - i}{\theta}\right) = \text{int}(\pi/\theta) + 1$. We now distinguish these two cases.

- Case where $\text{rnd} \left(\frac{\pi - i}{\theta}\right) = \text{int}(\pi/\theta)$.

We then have $\frac{\pi - i}{\theta} \leq \text{int}(\pi/\theta) + 1/2$ and expression (5.4) can be rewritten as

$$r_1 = (-1)^{\text{int}(\pi/\theta)} \left(\text{int}(\pi/\theta) - \frac{\pi - i}{\theta}\right) \theta,$$

$$= (-1)^{\text{int}(\pi/\theta)} \left(\left|\text{int}(\pi/\theta) - \frac{\pi - i}{\theta} + 1/2\right| - 1/2\right) \theta,$$

$$= (-1)^{\text{int}(\pi/\theta)} \left(i - \theta \left|\left[\text{int}(\pi/\theta) - 1/2\right] - \theta/2\right|\right).$$
which is exactly (2.1).

- Case where \( \text{rnd} \left( \frac{\pi - i}{\theta} \right) = \text{int}(\pi / \theta) + 1 \).

Since \( \frac{\pi - i}{\theta} \geq \text{int}(\pi / \theta) + 1/2 \), expression (5.4) can be rewritten as

\[
r_1 = (-1)^{\text{int}(\pi / \theta)} \left( -\text{int}(\pi / \theta) + 1 + \frac{\pi - i}{\theta} \right) \theta,
\]

\[
= (-1)^{\text{int}(\pi / \theta)} \left( \left| \text{int}(\pi / \theta) - \frac{\pi - i}{\theta} - 1/2 \right| - 1/2 \right) \theta,
\]

and we find again expression (2.1).

The reflected angle \( r_2 \) is clearly related to \( r_1 \) through the formula \( r_2(i) = -r_1(-i) \), which immediately yields (2.2). \( \Box \)

References


