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ON THE CLOSEDNESS OF THE ALGEBRAIC DIFFERENCE OF CLOSED CONVEX SETS

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ABSTRACT. We characterize in a reflexive Banach space all the closed convex sets C_1 containing no lines for which the condition $C_1^\infty \cap C_2^\infty = \{0\}$ ensures the closedness of the algebraic difference $C_1 - C_2$ for all closed convex sets C_2. We also answer a closely related problem: determine all the pairs C_1, C_2 of closed convex sets containing no lines such that the algebraic difference of any sufficiently small uniform perturbation of C_1 and C_2 remains closed. As an application, we state the bradest settting for the strict separation theorem in a reflexive Banach space.

1. INTRODUCTION AND NOTATIONS

It is a well known fact in analysis that in a reflexive Banach space, the algebraic difference $C_1 - C_2$ of two closed convex subsets C_1 and C_2 is closed whenever at least one of the sets C_1 and C_2 is bounded. Accordingly, this difference may fail to be closed, but only when both C_1 and C_2 are unbounded, and the failure of the closedness is always due to the relative position of the sets "at infinity". A first manner to control the interplay of the sets C_1 and C_2 is to impose to the intersection of their recession cones to reduce to the singleton $\{0\}$. This condition ensures the closedness of the algebraic difference, but only in finite dimensional spaces. By imposing, in addition, that at least one of the terms of the algebraic difference to be locally compact, Dieudonné [7] obtained a closedness criterion for the algebraic difference valid in every Hausdorff topological linear space.

Dieudonné's original result knew several refinements. The initial condition of local compactness was successively relaxed to a generalized compactness (in [3] and [13]), or to an asymptotic compactness property (see [10] and [11]).

The first objective of this article is to determine, in a reflexive Banach space, the broadest condition which should be added to the recession condition $C_1^\infty \cap C_2^\infty = \{0\}$ in order to ensure the closedness of the algebraic difference. Namely, we characterize the closed convex sets C_1 containing no lines for which the algebraic difference $C_1 - C_2$ is closed for every closed convex set C_2 fulfilling $C_1^\infty \cap C_2^\infty = \{0\}$. This class is characterized (Theorem 2.1) by a geometrical condition: the set C_1 has the above property if and only if it is well-positioned, that is if there is a continuous linear functional such that the difference between this functional and the norm is bounded from below on C_1.

Thus, the statement of the broadest setting for the Dieudonné Theorem in a reflexive setting is the following: *the algebraic difference of two closed convex sets containing no lines remains closed whenever the both following condition are fulfilled:*

a) $C_1^\infty \cap C_2^\infty = \{0\}$,
and
b) *at least one of the sets C_1, C_2 is well-positioned.*

However, it should be observed that in a reflexive Banach space, the algebraic difference of two closed and convex sets may be closed even if none of the terms of the difference is well-positioned. The closedness of the algebraic difference is, for instance, ensured if the distance between the parts of the two sets lying without the ball of radius r goes to infinity when r goes to infinity:

$$\lim_{r\to\infty} \mathrm{d}(C_1 \setminus r\mathbb{B}_X\,, C_2 \setminus r\mathbb{B}_X\,) = \infty. \tag{1.1}$$

This condition is sufficient but is obviously not necessary - the difference of $C_1 = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$ and $C_2 = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \leq 0\}$ being the closed half upper plane $C_1 - C_2 = \{(x,y) \in \mathbb{R}^2 : y \geq 0\}$ although relation (1.1) is violated.

In the second part of this article (Section 3) we prove that, at least for closed convex sets containing no lines, there is an important difference between the cases when $C_1 - C_2$ is closed while the sets C_1 and C_2 fulfill condition (1.1) and those when $C_1 - C_2$ is closed while condition (1.1) is not verified.

Let us first remark that if condition (1.1) is fulfilled by two closed convex sets C_1 and C_2, then it is also verified by any two uniform perturbations $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ of C_1 and C_2, that is closed convex sets fulfilling

$$C_{1,\varepsilon} \subseteq C_1 + \varepsilon\mathbb{B}_X\,, C_1 \subseteq C_{1,\varepsilon} + \varepsilon\mathbb{B}_X\,,\; C_{2,\varepsilon} \subseteq C_2 + \varepsilon\mathbb{B}_X\,, C_2 \subseteq C_{2,\varepsilon} + \varepsilon\mathbb{B}_X\,. \tag{1.2}$$

Consequently, the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ is closed and therefore relation (1.1) ensures not only the closure of the algebraic difference of the initial sets, but also the closure of the algebraic difference of any small uniform perturbations of the initial sets.

The main result of this part of the paper states that the closedness of the algebraic difference is unstable with respect to small uniform perturbations of the initial sets when condition (1.1) is violated. More precisely, we prove (Theorem 3.1) that if the algebraic difference of two closed convex sets, C_1 and C_2 containing no lines is a closed proper subset of a reflexive Banach space X and if condition (1.1) is violated, then, for every $\varepsilon > 0$ there are $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$, two uniform perturbations of C_1 and C_2 in the sense (1.2), such that $C_{1,\varepsilon} - C_{2,\varepsilon}$ is not a closed set.

Accordingly, condition (1.1), although only a sufficient condition when the mere closedness of the algebraic difference is requested, turns out to be both sufficient and necessary, if we are seeking on stable closedness.

The last section contains the proofs of several technical results needed for establishing Theorems 2.1 and 3.1.

We use Theorem 2.1 to give an application to the strict separation of closed convex sets in reflexive Banach spaces.

is presented in the last section. Throughout the paper, we suppose, unless otherwise specified, that X is a reflexive Banach space with continuous dual X^*. By $\mathbb{B}_X$, $\mathbb{B}^*_X$ we denote the primal and dual unit ball, respectively. As usual:

i) $|\cdot|$, $|\cdot|_*$ are the norms on X and X^*, and $\langle\cdot,\cdot\rangle$ is the duality pairing between X^* and X,

ii) $j : X^* \to X$ is the *duality mapping* given by $\langle f, j(f)\rangle = |f|^2_*$ and $|j(f)| = |f|_*$, (see for example [15]). Due to a well known renorming Theorem of Troyanski (see e.g.

[6]) we can (and will) assume that the norms on X and X^* are locally uniformly rotund. This implies that the duality mapping j is single-valued and norm-to-norm continuous,

iii) $\mathrm{d}(x, y) = |x - y|$ is the distance between two elements x and y of X, and $\mathrm{d}(x, S) = \inf_{y \in S} \mathrm{d}(x, y)$ is the distance between a point x and a subset S of X,

iv) $\mathrm{co}(S)$, $\overline{\mathrm{co}}(S)$, $\mathrm{sp}(S)$ and $\overline{\mathrm{sp}}(S)$ are the convex hull, the closed convex hull, the linear span, and the closed linear span of the set S,

v) $S^\circ = \{f \in X^* : \langle f, w\rangle \leq 0 \ \forall w \in S\}$ is the negative polar cone of the set S of X, which reduces to the orthogonal $S^\perp = \{f \in X^* \ : \ \langle f, w\rangle = 0 \ \forall w \in S\}$ when S is a closed linear subspace,

vi) $\mathcal{B}(S)$, the *barrier cone* of a closed convex subset S of X, is defined as the domain of the support functional $\sigma_S(f) := \sup_{x \in S}\langle f, x\rangle$,

$$\mathcal{B}(S) = \{f \in X^* : \sigma_S(f) < +\infty\} = \mathrm{Dom}\, \sigma_S,$$

vii) C^∞ is the *recession cone* of the closed convex set C, that is the maximal closed convex cone whose translate at every point of C lies in C,

$$C^\infty = \{v \in X : \quad \forall \lambda > 0, x_0 \in C, x_0 + \lambda v \in C\},$$

(see Rockafellar [12] as a reference book); C is called linearly bounded if $C^\infty = \{0\}$,

viii) $l(C) = C^\infty \cap (-C^\infty)$ is the maximal linear subspace contained in the closed convex set C,

ix) $P_C : X \to C$ is the projection operator onto the closed convex set C,

$$P_C(x) = \mathrm{argmin}_{y \in C}|x - y|.$$

Finally, we use the symbol “ $\rightharpoonup$ ” to denote the weak convergence on X.

2. WELL-POSITIONED SETS AND CLOSEDNESS OF THE ALGEBRAIC DIFFERENCE

In a recent paper Adly-Ernst and Théra [2] introduced the concept of *well-positioned convex sets.* Recall that $C \subset X$ is well-positioned if there exists $x_0 \in X$ and $g \in X^*$ such that:

$$\langle g, x - x_0\rangle \geq |x - x_0|, \quad \text{for all } x \in C.$$

We give below a new characterization of such sets.

Theorem 2.1. *Let X be a reflexive Banach space and C_1 be a closed convex subset of X containing no lines. The two following statements are equivalent:*

i) *C_1 is well-positioned;*
ii) *for every closed convex set C_2 such that $C_1^\infty \cap C_2^\infty = \{0\}$, the algebraic difference $C_1 - C_2$ is closed.*

Proof of Theorem 2.1: $i) \Longrightarrow ii)$

Let C_1 be a well-positioned closed convex set, $x_0 \in X$ and $g \in X^*$ such that

$$\langle g, x - x_0\rangle \geq |x - x_0| \text{ for all } x \in C_1, \tag{2.1}$$

and let C_2, denote a closed convex set such that $C_1^\infty \cap C_2^\infty = \{0\}$. Take a sequence $\{z_n\}_{n \in \mathbb{N}^*}$ in $C_1 - C_2$ which strongly converges to a limit point z and let us prove that $z \in C_1 - C_2$.

As $z_n \in C_1 - C_2$, select $x_n \in C_1$ and $y_n \in C_2$, such that

$$z_n = x_n - y_n = (x_n - x_0) - (y_n - x_0).$$

For the purpose of obtaining a contradiction, let us suppose that the sequence $\{x_n\}$ is unbounded. Accordingly, by using a subsequence if necessary, we can (and will) suppose that $\lim_{n\to\infty} |x_n - x_0| = +\infty$. Since the sequence $\{z_n\}$ is convergent, there exists some $M > 0$ such that $|z_n| \leq M$; accordingly, $|y_n - x_0| \geq |x_n - x_0| - M$. Thus, for n large enough we deduce that $|y_n - x_0| > 0$, and therefore we can write:

$$\frac{y_n - x_0}{|y_n - x_0|} - \frac{x_n - x_0}{|x_n - x_0|} = \frac{y_n - x_0}{|y_n - x_0|}\left(1 - \frac{|y_n - x_0|}{|x_n - x_0|}\right) + \frac{y_n - x_n}{|x_n - x_0|}.$$

As $x_n - y_n = (x_n - x_0) - (y_n - x_0) = z_n$, it follows that $|y_n - x_n| \leq |z_n| \leq M$ and $||y_n - x_0| - |x_n - x_0|| \leq |z_n| \leq M$; consequently,

$$\left|\frac{y_n - x_0}{|y_n - x_0|} - \frac{x_n - x_0}{|x_n - x_0|}\right| \leq \frac{||y_n - x_0| - |x_n - x_0||}{|x_n - x_0|} + \frac{|y_n - x_n|}{|x_n - x_0|} \leq \frac{2M}{|x_n - x_0|}.$$

This inequality yields

$$\lim_{n\to\infty}\left(\frac{y_n - x_0}{|y_n - x_0|} - \frac{x_n - x_0}{|x_n - x_0|}\right) = 0. \tag{2.2}$$

Let w denote a weak cluster point of the bounded sequence $\left\{\frac{x_n - x_0}{|x_n - x_0|}\right\}$; as $x_n \in C_1$, it follows that $w \in C_1^\infty$, while from relation (2.1) it follows that $\langle g, w\rangle \geq 1$ and therefore that $w \neq 0$. Relation (2.2) implies that the element w of $C_1^\infty \setminus \{0\}$ also belongs to C_2^∞, contradicting in this way the Dieudonné condition $C_1^\infty \cap C_2^\infty = \{0\}$.

The sequence $\{x_n\}$ is therefore necessarily bounded. Let x denote one of its weak cluster points. As the set C_1 is convex and closed, it is also weakly closed, so x belongs to C_1. Since the sequence $\{z_n\}$ strongly converges to z, and $y_n = z_n - x_n$, it follows that $z - x$ is a weak cluster point for the sequence $\{y_n\}$ and since C_2 is weakly closed belongs to C_2. Accordingly, $z = x - (z - x) \in C_1 - C_2$, and the set $C_1 - C_2$ is closed, completing the desired implication.

$ii) \Longrightarrow i)$

We will complete the proof of Theorem 2.1 by establishing that, for every non well-positioned closed convex set C_1 containing no lines, there is a closed convex set C_2 such that $C_1^\infty \cap C_2^\infty = \{0\}$ and the set $C_1 - C_2$ is not closed.

Let us remind the following classical result (see [1] or [4]).

Theorem 2.2. *For every closed and convex unbounded linearly bounded set C of a reflexive Banach space X, there is g in X^* such that*

$$\sup_{x\in C}\langle g, x\rangle = 1 \text{ and } \langle g, x\rangle < 1 \text{ for all } x \in C.$$

Applying Theorem 4.2 (Section 4) to the set C_1, we deduce that there is a closed linear manifold L of X such that $C_1 \cap L$ is unbounded and linearly bounded. Let V denote the subspace of X parallel to L. Applying Theorem 2.2 to $C_1 \cap L$, it follows that there is $g \in X^*$ such that

$$\sup_{x\in(C_1\cap L)}\langle g, x\rangle = 1 \text{ and } \langle g, x\rangle < 1 \text{ for all } x \in (C_1 \cap L). \tag{2.3}$$

Relation (2.3) shows that the linear functional g is not constant on L, and therefore the subspace V is not contained in Ker $g := \{x \in X \,:\, \langle g, x\rangle = 0\}$. Accordingly, there is $v \in V$ such that $\langle g, v\rangle = 1$. From relation (2.3), for every $n \in \mathbb{N}^*$, select $x_n \in (C_1 \cap L)$ such that

$$(n-1)/n \leq \langle g, x_n\rangle < 1$$

and define

$$C_2 := \overline{\text{co}}\left(\{y_n : n \in \mathbb{N}^*\}\right),$$

where $y_n := x_n + (1 - \langle g, x_n\rangle)\, v$. As $\mathrm{d}(y_n, C_1 \cap L) \leq |y_n - x_n| \leq |v|$, we have $C_2 \subseteq (C_1 \cap L) + |v|\mathbb{B}_X$ and consequently,

$$C_2^\infty \subseteq ((C_1 \cap L) + |v|\mathbb{B}_X)^\infty = (C_1 \cap L)^\infty = \{0\}.$$

The condition $C_1^\infty \cap C_2^\infty = \{0\}$ is therefore fulfilled. From the definition of y_n it follows that $\langle g, y_n\rangle = 1$ for every $n \in \mathbb{N}^*$. Thus, $C_2 \subseteq \{x \in L : \langle g, x\rangle = 1\}$. Accordingly, $C_2 \cap C_1 \subseteq \{x \in (C_1 \cap L) : \langle g, x\rangle = 1\}$, and relation (2.3) implies that $C_2 \cap C_1 = \emptyset$, that is $0 \notin C_1 - C_2$. Taking into account that for every n we have $|x_n - y_n| \leq \frac{1}{n}|v|$, it follows that the set $C_1 - C_2$ is not closed since $\lim_{n\to+\infty}(x_n - y_n) = 0$. $\square$

3. ANALYSIS OF THE CLOSURE OF THE DIFFERENCE FOR ARBITRARY UNIFORM PERTURBATIONS

In this section we determine the conditions under which the closed algebraic difference of two closed and convex sets C_1 and C_2 remains closed when C_1 and C_2 are subjected to small uniform perturbations.

More precisely, we prove that if condition (1.1) is not fulfilled, then, even if $C_1 - C_2$ happens to be closed, there is at least an uniform perturbation $C_{1,\varepsilon}$, $C_{2,\varepsilon}$, of C_1, C_2 (in the sense of (1.2)) such that the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ is no longer a closed subset of X.

Theorem 3.1. *Let C_1 and C_2 be two closed convex subsets containing no lines of a reflexive Banach space X, such that $C_1 - C_2$ is a proper closed subset of X. If*

$$\mathrm{d}(C_1 \setminus r\mathbb{B}_X, C_2 \setminus r\mathbb{B}_X) \leq d < +\infty \quad \forall r > 0, \tag{3.1}$$

then, for every $\varepsilon > 0$ there are $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$, two closed and convex uniform perturbations of C_1 and C_2 (in the sense of (1.2)) such that the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ fails to be closed.

General idea of the construction and proof of Theorem 3.1: Let $f \in \mathcal{B}(C_1 - C_2)$, with $|f|_* = 1$. As $\mathcal{B}(C_1 - C_2) = \mathcal{B}(C_1) \cap \mathcal{B}(-C_2)$, we have

$$m_1 = \sup_{x \in C_1} \langle f, x\rangle < +\infty \tag{3.2}$$

and

$$m_2 = \inf_{x \in C_2} \langle f, x\rangle > -\infty, \tag{3.3}$$

so there are $x_1 \in C_1$ and $x_2 \in C_2$ such that

$$\langle f, x_1\rangle + \frac{\varepsilon}{2} \geq m_1,\ \langle f, x_2\rangle - \frac{\varepsilon}{2} \leq m_2. \tag{3.4}$$

Set $y_1 = x_1 + \varepsilon j(f)$ and $y_2 = x_2 - \varepsilon j(f)$, and introduce the points z_1 and z_2 belonging to the segments $[x_1, y_1]$ and $[x_2, y_2]$ and satisfying $\langle f, z_1\rangle = m_1$, $\langle f, z_2\rangle = m_2$. More precisely,

$$\begin{aligned} z_1 &= \frac{m_1 - \langle f, x_1\rangle}{\langle f, y_1\rangle - \langle f, x_1\rangle} y_1 + \frac{\langle f, y_1\rangle - m_1}{\langle f, y_1\rangle - \langle f, x_1\rangle} x_1 \\ &= x_1 + (m_1 - \langle f, x_1\rangle) j(f), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} z_2 &= \frac{\langle f, x_2\rangle - m_2}{\langle f, x_2\rangle - \langle f, y_2\rangle} y_2 + \frac{m_2 - \langle f, y_2\rangle}{\langle f, x_2\rangle - \langle f, y_2\rangle} x_2 \\ &= x_2 - (\langle f, x_2\rangle - m_2) j(f). \end{aligned} \tag{3.6}$$

As j is an isometric map and $|f|_* = 1$, we have

$$\begin{aligned} &|y_1 - x_1| = |x_2 - y_2| = |\varepsilon j(f)| = \varepsilon, \\ &\langle f, y_1 - x_1\rangle = \langle f, x_2 - y_2\rangle = \langle f, \varepsilon j(f)\rangle = \varepsilon. \end{aligned} \tag{3.7}$$

This yields $\mathrm{d}(y_1, C_1) \leq \varepsilon$ and $\mathrm{d}(y_2, C_2) \leq \varepsilon$.

Accordingly, given two closed convex sets F_1 and F_2 such that

$$C_1 \subseteq F_1 \subseteq \overline{\mathrm{co}}(y_1, C_1) \text{ and } C_2 \subseteq F_2 \subseteq \overline{\mathrm{co}}(y_2, C_2)$$

we have

$$\begin{aligned} &F_1 \subseteq C_1 + \varepsilon\mathbb{B}_X\,, C_1 \subseteq F_1 + \varepsilon\mathbb{B}_X \\ &F_2 \subseteq C_2 + \varepsilon\mathbb{B}_X\,, C_2 \subseteq F_2 + \varepsilon\mathbb{B}_X\,. \end{aligned} \tag{3.8}$$

In other words, condition (1.2) is satisfied for every closed convex sets F_1 and F_2 containing C_1 and C_2 and contained in $\overline{\mathrm{co}}(y_1, C_1)$ and $\overline{\mathrm{co}}(y_2, C_2)$.

Therefore, the uniform perturbations $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ will be chosen as closed convex sets containing C_1 and C_2 and contained in $\overline{\mathrm{co}}(y_1, C_1)$ and $\overline{\mathrm{co}}(y_2, C_2)$.

Finally, from relation (3.4) and (3.7) we derive

$$m_1 + \frac{\varepsilon}{2} \leq \langle f, y_1\rangle \leq m_1 + \varepsilon \text{ and } m_2 - \frac{\varepsilon}{2} \geq \langle f, y_2\rangle \geq m_2 - \varepsilon. \tag{3.9}$$

For technical reasons, the proof of Theorem 3.1 will be divided into three parts:

(i) $C_1^\infty \cap C_2^\infty \neq \{0\}$

(ii) $l(C_1 - C_2) \neq \{0\}$ and $C_1^\infty \cap C_2^\infty = \{0\}$,

(iii) condition (3.1) is fulfilled and $l(C_1 - C_2) = \{0\}$.

3.1. Case i): $C_1^\infty \cap C_2^\infty \neq \{0\}$.

Let us remark that the example given in the introduction fits this case. Indeed, if

$$C_1 = \{(x, y) \in \mathbb{R}^2 \ :\ x \geq 0, y \geq 0\} \text{ and } C_2 = \{(x, y) \in \mathbb{R}^2 \ :\ x \geq 0, y \leq 0\},$$

then $\bar{w} = (1, 0)$ belongs both to $C_1^\infty = C_1$ and to $C_2^\infty = C_2$.

• *Construction of the uniform perturbations $C_{1,\varepsilon}$ and $C_{2,\varepsilon}$.*

Fix $w \in C_1^\infty \cap C_2^\infty$, $w \neq 0$, and set

$$D = \left\{ z_1 + \nu w + \mu j(f) \ :\ 0 \leq \nu, 0 \leq \mu \leq \frac{\nu}{1+\nu}\frac{\varepsilon}{2} \right\}.$$

Let us prove that $C_{1,\varepsilon} = \overline{\mathrm{co}}(D, C_1)$ and $C_{2,\varepsilon} = \overline{\mathrm{co}}(y_2, C_2)$ can be used as uniform perturbation to prove Theorem 3.1 in case (i).

• *$C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ satisfy relation (1.2).*

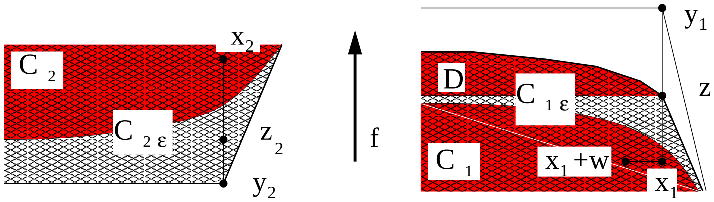

FIGURE 1. Uniform perturbations in case i)

Using (3.5), for each $x = (z_1 + \nu w + \mu j(f)) \in D$, we have

$$x = x_1 + (m_1 - \langle f, x_1\rangle + \mu) j(f) + \nu w.$$

Hence,

$$x = \frac{\varepsilon - (m_1 - \langle f, x_1\rangle + \mu)}{\varepsilon}(x_1 + \nu w) + \frac{m_1 - \langle f, x_1\rangle + \mu}{\varepsilon}(x_1 + \nu w + \varepsilon j(f))$$

and therefore

$$x = \frac{\varepsilon - (m_1 - \langle f, x_1\rangle + \mu)}{\varepsilon}(x_1 + \nu w) + \frac{m_1 - \langle f, x_1\rangle + \mu}{\varepsilon}(y_1 + \nu w).$$

Noticing that $\mu \leq \frac{\varepsilon}{2}$, and using (3.4) we observe that $m_1 - \langle f, x_1\rangle + \mu \leq \varepsilon$ and therefore x is a convex combination of $y_1 + \nu w$ and $x_1 + \nu w$.

Since $w \in C_1^\infty$ and $C_1 \subseteq \overline{\mathrm{co}}(y_1, C_1)$, then $w \in (\overline{\mathrm{co}}(y_1, C_1))^\infty$, and since $x_1, y_1 \in \overline{\mathrm{co}}(y_1, C_1)$, it follows that $(x_1 + \nu w)$ and $(y_1 + \nu w)$ are two elements of $\overline{\mathrm{co}}(y_1, C_1)$. Consequently, $D \subseteq \overline{\mathrm{co}}(y_1, C_1)$. Hence, $C_1 \subseteq C_{1,\varepsilon} = \overline{\mathrm{co}}(D, C_1) \subseteq \overline{\mathrm{co}}(y_1, C_1)$. As $C_2 \subseteq C_{2,\varepsilon} = \overline{\mathrm{co}}(y_2, C)$, relation (3.8) implies that relation (1.2) is satisfied.

- *$C_{1,\varepsilon} - C_{2,\varepsilon}$ is not closed.*

In order to prove that the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ is not a closed set, we shall use the following result.

Lemma 3.1. *For every $x \in C_{1,\varepsilon} = \overline{\mathrm{co}}(D, C_1)$, we have $\langle f, x\rangle < m_1 + \frac{\varepsilon}{2}$.*

Proof of Lemma 3.1: By contradiction, take $x \in \overline{\mathrm{co}}(D, C_1)$ such that $\langle f, x\rangle = m_1 + \frac{\varepsilon}{2}$. As $x \in \overline{\mathrm{co}}(D, C_1)$, select a sequence $\{a_n\}_{n \in \mathbb{N}^*}$ in $\mathrm{co}(D, C_1)$ norm-converging to x. As obviously D is closed and convex, for every $a_n \in \mathrm{co}(D, C_1)$ select $d_n \in D$ with $d_n = z_1 + \nu_n w + \mu_n j(f)$, $c_n \in C_1$ and $\lambda_n \in [0, 1]$ such that $a_n = \lambda_n c_n + (1 - \lambda_n) d_n$.

Observing that $f \in \mathcal{B}(C_1)$ and $w \in C_1^\infty$, we have $\langle f, w\rangle \leq 0$. Noticing also that $f \in \mathcal{B}(-C_2)$ and $w \in C_2^\infty$, we also have $\langle f, w\rangle \geq 0$. Consequently, $\langle f, w\rangle = 0$, and therefore

$$\langle f, a_n\rangle = \lambda_n \langle f, c_n\rangle + (1 - \lambda_n)(m_1 + \mu_n).$$

Since $\lim_{n \to +\infty} a_n = x$, then $\lim_{n \to +\infty} \big(\langle f, a_n\rangle - m_1\big) = \langle f, x\rangle - m_1 = \frac{\varepsilon}{2}$ and therefore,

$$\lim_{n \to +\infty} \Big(\mu_n + \lambda_n (\langle f, c_n\rangle - m_1 - \mu_n)\Big) = \frac{\varepsilon}{2}.$$

As $\mu_n \leq \frac{\varepsilon}{2}$ and $\lambda_n (\langle f, c_n\rangle - m_1 - \mu_n) \leq 0$, the previous relation implies that

$$\lim_{n \to +\infty} \Big(\lambda_n (\langle f, c_n\rangle - m_1 - \mu_n)\Big) = 0 \text{ and } \lim_{n \to +\infty} \mu_n = \frac{\varepsilon}{2}. \tag{3.10}$$

Using the fact that $\langle f, c_n\rangle - m_1 - \mu_n \leq -\mu_n$, we derive

$$\limsup_{n\to\infty}\Big(\langle f, c_n\rangle - m_1 - \mu_n\Big) \leq -\frac{\varepsilon}{2}. \tag{3.11}$$

Combining relations (3.11) and (3.10) it follows that $\lambda_n \to 0$, while since $\mu_n \to \frac{\varepsilon}{2}$ we deduce that $\nu_n \to \infty$. Let us observe that

$$\lambda_n c_n = -(1-\lambda_n)\nu_n w + a_n - (1-\lambda_n)(z_1 + \mu_n j(f)) \tag{3.12}$$

and that $(1-\lambda_n)\nu_n > 0$, for n large enough ($1-\lambda_n \to 1$ and $\nu_n \to \infty$). Hence, dividing by $(1-\lambda_n)\nu_n$ we obtain

$$\frac{\lambda_n}{(1-\lambda_n)\nu_n} c_n = -w + \frac{a_n - (1-\lambda_n)(z_1 + \mu_n j(f))}{(1-\lambda_n)\nu_n}. \tag{3.13}$$

Being convergent, the sequence $\{a_n\}$ is bounded, so the previous relation implies that

$$\frac{\lambda_n}{(1-\lambda_n)\nu_n} c_n \to -w.$$

As $t_n := \frac{\lambda_n}{(1-\lambda_n)\nu_n} \to 0$, $c_n \in C_1$ and $t_n c_n \to -w$ as $n \to +\infty$, it follows that $-w \in C_1^\infty$, that is $0 \neq w \in l(C_1)$, contradicting the fact that the closed convex set C_1 contains no lines. $\square$

Let us return to the proof of the case i). As $\langle f, y_2\rangle \leq m_2 - \frac{\varepsilon}{2}$ (see (3.9)), we deduce that, for every $x \in C_{2,\varepsilon} = \overline{\text{co}}(y_2, C_2)$,

$$\langle f, x\rangle \geq \min\left(\langle f, y_2\rangle, \inf_{z\in C_2}\langle f, z\rangle\right) = \min\left(\langle f, y_2\rangle, m_2\right) = \langle f, y_2\rangle. \tag{3.14}$$

Lemma 3.1 and relation (3.14) imply that

$$\langle f, x\rangle < m_1 - \langle f, y_2\rangle + \frac{\varepsilon}{2} \quad \forall x \in C_{1,\varepsilon} - C_{2,\varepsilon}. \tag{3.15}$$

Set $b_n = (z_1 + nw + \frac{n}{n+1}\frac{\varepsilon}{2}j(f)) - (y_2 + nw)$; as $w \in C_2^\infty \subseteq (\overline{\text{co}}(y_2, C_2))^\infty$ and $y_2 \in \overline{\text{co}}(y_2, C_2)$, it follows that $(y_2 + nw) \in \overline{\text{co}}(y_2, C_2) = C_{2,\varepsilon}$ and consequently $b_n \in C_{1,\varepsilon} - C_{2,\varepsilon}$. Moreover

$$|b_n| = \left|(z_1 - y_2) + \frac{n}{n+1}\frac{\varepsilon}{2}j(f)\right| \leq |z_1 - y_2| + \frac{\varepsilon}{2}, \tag{3.16}$$

and

$$\langle f, b_n\rangle = m_1 - \langle f, y_2\rangle + \frac{n}{n+1}\frac{\varepsilon}{2},$$

relation which implies

$$\lim_{n\to\infty}\langle f, b_n\rangle = m_1 - \langle f, y_2\rangle + \frac{\varepsilon}{2}. \tag{3.17}$$

Let b be a weak cluster point of the bounded (see (3.16)) sequence $\{b_n\}_{n\in\mathbb{N}^*}$. Using relation (3.17) we obtain

$$\langle f, b\rangle = m_1 - \langle f, y_2\rangle + \frac{\varepsilon}{2}. \tag{3.18}$$

Using relations (3.15) and (3.18) it follows that b does not belong to $C_{1,\varepsilon} - C_{2,\varepsilon}$. Consequently, the algebraic difference $C_{1,\varepsilon} - C_{2,\varepsilon}$ is a convex set which is not closed and the proof of case (i) is thereby completed..

3.2. **Case ii): $l(C_1 - C_2) \neq \{0\}$ and $C_1^\infty \cap C_2^\infty = \{0\}$.** This case can occur as the following construction shows.

Let H be a separable Hilbert space and $(e_n)_{n\in\mathbb{N}}$ be an hilbertian basis. Take

$$u = \sum_{i=1}^{\infty} \frac{e_i}{i^2} \text{ and } v = \sum_{i=1}^{\infty} \frac{e_i}{i},$$

and define

$$K = \Big\{ x \in H \ : \ \langle x, e_i \rangle \geq 0 \ \ \forall i \in \mathbb{N}^*, \ \langle x, u \rangle \leq 1 \text{ and } |\langle x, e_0 \rangle| \leq \langle x, v \rangle \Big\}.$$

Obviously, K is a closed convex set and $K^\infty = \{0\}$, while

$$K - K = \Big\{ x \in H : \langle P_+(x), u \rangle \leq 1 \text{ and } \langle P_-(x), u \rangle \geq -1 \Big\},$$

where P_+ and P_- are the projections on the positive and respectively on the negative cone

$$C_+ = \{x \in H \ : \ \langle x, e_i \rangle \geq 0 \ \ \forall i \in \mathbb{N}\}, \ C_- = \{x \in H \ : \ \langle x, e_i \rangle \leq 0 \ \ \forall i \in \mathbb{N}\}.$$

Set $C_1 = C_2 = K$. Since simultaneously $C_1^\infty \cap C_2^\infty = \{0\} \cap \{0\} = \{0\}$, the algebraic difference $C_1 - C_2 = K - K$ is closed, and

$$l(C_1 - C_2) = \{\lambda e_0 \ : \ \lambda \in \mathbb{R}\},$$

we may conclude that the closed convex sets C_1 and C_2 fulfill conditions ii).

- Construction of the uniform perturbations.

When $l(C_1 - C_2) \neq \{0\}$ and $C_1^\infty \cap C_2^\infty = \{0\}$, the uniform perturbations satisfying Theorem 3.1 are:

$$C_{1,\varepsilon} = \overline{\text{co}}(y_1, C_1) \text{ and } C_{2,\varepsilon} = \overline{\text{co}}(y_2, C_2). \tag{3.19}$$

As obviously $C_{1,\varepsilon}$ contains C_1 and is contained in $\overline{\text{co}}(y_1, C_1)$, and $C_{2,\varepsilon}$ contains C_2 and is contained in $\overline{\text{co}}(y_2, C_2)$, condition (1.2) is verified.

For the purpose of obtaining a contradiction, let us suppose that the algebraic difference $\overline{\text{co}}(y_1, C_1) - \overline{\text{co}}(y_2, C_2)$ is closed, and fix $d \in l(C_1 - C_2)$, $d \neq 0$. Accordingly, $d \in (C_1 - C_2)^\infty \subseteq (C_{1,\varepsilon} - C_{2,\varepsilon})^\infty$, relation which implies, as $(y_1 - y_2) \in (C_{1,\varepsilon} - C_{2,\varepsilon})$, that $(y_1 - y_2 + d), (y_1 - y_2 - d) \in (C_{1,\varepsilon} - C_{2,\varepsilon})$. Thus, there are $a_1, b_1 \in C_{1,\varepsilon}$ and $a_2, b_2 \in C_{2,\varepsilon}$ such that :

$$a_1 - a_2 = y_1 - y_2 + d \text{ and } b_1 - b_2 = y_1 - y_2 - d. \tag{3.20}$$

Since $f \in \mathcal{B}(C_1 - C_2)$ and $d \in l(C_1 - C_2)$, we have $\langle f, d \rangle = 0$. Thus,

$$\langle f, a_1 - a_2 \rangle = \langle f, y_1 - y_2 + d \rangle = \langle f, y_1 - y_2 \rangle,$$
$$\langle f, b_1 - b_2 \rangle = \langle f, y_1 - y_2 - d \rangle = \langle f, y_1 - y_2 \rangle.$$

Then

$$\langle f, a_1 \rangle - \langle f, y_1 \rangle = \langle f, a_2 \rangle - \langle f, y_2 \rangle, \ \langle f, b_1 \rangle - \langle f, y_1 \rangle = \langle f, b_2 \rangle - \langle f, y_2 \rangle. \tag{3.21}$$

As

$$\langle f, a_1 \rangle, \langle f, b_1 \rangle \leq \max\left(\sup_{x \in C_1} \langle f, x \rangle, \langle f, y_1 \rangle \right) = \max(m_1, \langle f, y_1 \rangle) = \langle f, y_1 \rangle,$$

and

$$\langle f, a_2 \rangle, \langle f, b_2 \rangle \geq \min\left(\inf_{x \in C_2} \langle f, x \rangle, \langle f, y_2 \rangle \right) = \min(m_2, \langle f, y_2 \rangle) = \langle f, y_2 \rangle,$$

relation (3.21) implies that $\langle f, a_1\rangle = \langle f, b_1\rangle = \langle f, y_1\rangle$, and $\langle f, a_2\rangle = \langle f, b_2\rangle = \langle f, y_2\rangle$, that is $a_1, b_1 \in N_1$ and $a_2, b_2 \in N_2$, where

$$N_1 = \{x \in \overline{\text{co}}(y_1, C_1)\ :\ \langle f, x\rangle = \langle f, y_1\rangle\}$$

and

$$N_2 = \{x \in \overline{\text{co}}(y_2, C_2)\ :\ \langle f, x\rangle = \langle f, y_2\rangle\}.$$

Lemma 3.2. $N_1 \subseteq (y_1 + C_1^\infty)$ *and* $N_2 \subseteq (y_2 + C_2^\infty)$.

Proof of Lemma 3.2: Let $w_1, w_2 \in X$ such that $y_1 + w_1 \in N_1$ and $y_2 + w_2 \in N_2$; accordingly, $\langle f, w_1\rangle = \langle f, w_2\rangle = 0$.

As $y_1 + w_1 \in \overline{\text{co}}(y_1, C_1)$ and $y_2 + w_2 \in \overline{\text{co}}(y_2, C_2)$, select four sequences $\{z_n\}_{n\in\mathbb{N}^*} \in C_1$, $\{t_n\}_{n\in\mathbb{N}^*} \in C_2$, $\{\lambda_n\}_{n\in\mathbb{N}^*} \in [0,1]$, and $\{\mu_n\}_{n\in\mathbb{N}^*} \in [0,1]$, such that

$$\lim_{n\to+\infty}\Big(\lambda_n z_n + (1-\lambda_n)y_1\Big) = y_1 + w_1, \text{ and } \lim_{n\to+\infty}\Big(\mu_n t_n + (1-\mu_n)y_2\Big) = y_2 + w_2.$$

Consequently,

$$\lim_{n\to+\infty}\Big(\lambda_n z_n - \lambda_n y_1\Big) = w_1, \text{ and } \lim_{n\to+\infty}\Big(\mu_n t_n - \mu_n y_2\Big) = w_2. \tag{3.22}$$

Therefore,

$$\lim_{n\to+\infty}\Big(\lambda_n(\langle f, z_n\rangle - \langle f, y_1\rangle)\Big) = \langle f, w_1\rangle = 0, \tag{3.23}$$

$$\lim_{n\to+\infty}\Big(\mu_n(\langle f, t_n\rangle - \langle f, y_2\rangle)\Big) = \langle f, w_2\rangle = 0. \tag{3.24}$$

From relation (3.9) it follows that

$$\langle f, z_n\rangle \le m_1 \le \langle f, y_1\rangle - \frac{\varepsilon}{2} \text{ and } \langle f, t_n\rangle \ge m_2 \ge \langle f, y_2\rangle + \frac{\varepsilon}{2}.$$

This yields,

$$\langle f, z_n\rangle - \langle f, y_1\rangle \le -\frac{\varepsilon}{2} \text{ and } \langle f, t_n\rangle - \langle f, y_2\rangle \ge \frac{\varepsilon}{2}. \tag{3.25}$$

Combining inequalities (3.25) and relation (3.23) yields

$$\lambda_n \to 0 \text{ and } \mu_n \to 0 \text{ as } n \to +\infty, \tag{3.26}$$

Using relations (3.26) and (3.22) we derive

$$\lambda_n z_n \to w_1 \text{ and } \mu_n t_n \to w_2,$$

which, combined to relation (3.26) gives

$$w_1 \in C_1^\infty \text{ and } w_2 \in C_2^\infty. \tag{3.27}$$

□

Applying Lemma 3.2 to a_1, a_2, b_1 and b_2 gives the existence of $z_1, t_1 \in C_1^\infty$ and $z_2, t_2 \in C_2^\infty$ such that $a_1 = y_1 + z_1$, $b_1 = y_1 + t_1$, $a_2 = y_2 + z_2$ and $b_2 = y_2 + z_2$. We thus have $z_1 - z_2 = -t_1 + t_2 = d$, that is $z_1 + t_1 = z_2 + t_2$. As C_1^∞ and C_2^∞ are convex cones, it follows that $z_1 + t_1 \in C_1^\infty$ and $z_2 + t_2 \in C_2^\infty$, and, since $C_1^\infty \cap C_2^\infty = \{0\}$, we deduce that $z_1 + t_1 = z_2 + t_2 = 0$.
Accordingly, $z_1 = -t_1 \in -C_1^\infty$ and $z_2 = -t_2 \in -C_2^\infty$, so $z_1, t_1 \in l(C_1) = \{0\}$ and $z_2, t_2 \in l(C_2) = \{0\}$. Thus, $d = z_1 - z_2 = 0$, a contradiction. The proof of Theorem 3.1 in the case ii) is completed.

3.3. **Case iii): we suppose that** $\mathrm{d}(C_1 \setminus r\mathbb{B}_X\,, C_2 \setminus r\mathbb{B}_X) \leq d < +\infty \quad \forall r > 0$ **and that** $l(C_1 - C_2) = \{0\}$**.** Similarly to the case ii), the fact that case iii) can occur is not obvious. Let us give an example of such a situation.

Let H be a separable Hilbert space and $(e_n)_{n\in\mathbb{N}^*}$ an hilbertian basis. Define

$$C_1 = C_2 = \{x \in H\,:\, -i \leq \langle x, e_i\rangle \leq i \;\; \forall i \in \mathbb{N}^*\}.$$

Obviously, C_1 and C_2 are closed convex sets, and $C_1^\infty = C_2^\infty = \{0\}$. Since $C_2 = -C_1$, the algebraic difference $C_1 - C_2$ equals $2C_1$, that is a closed set, and

$$l(C_1 - C_2) = l(2C_1) = l(C_1) \subseteq C_1^\infty = \{0\}.$$

On the other hand,

$$ne_n \in (C_1 \setminus r\mathbb{B}_X) = (C_2 \setminus r\mathbb{B}_X) \;\; \forall n \in \mathbb{N}, n > r,$$

so

$$\mathrm{d}(C_1 \setminus r\mathbb{B}_X\,, C_2 \setminus r\mathbb{B}_X) = 0, \;\; \forall r > 0.$$

The sets C_1 and C_2 fulfill therefore the conditions of case iii).

- *Construction of the uniform perturbations* $C_{1,\varepsilon}$ *and* $C_{2,\varepsilon}$*.*

As $\lim_{r\to\infty} \mathrm{d}(C_1 \setminus r\mathbb{B}_X\,, C_2 \setminus r\mathbb{B}_X) = d < +\infty$, there are sequences $\{a_n\}_{n\in\mathbb{N}^*} \subset C_1$ and $\{b_n\}_{n\in\mathbb{N}^*} \subset C_2$, such that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} |b_n| = \infty \text{ and } |a_n - b_n| \leq d + 1 \;\; \forall n \in \mathbb{N}^*. \tag{3.28}$$

Applying Proposition 4.1 (see section 4) to the closed convex set $C_1 - C_2$ and to the separable subspace $X_1 = \overline{\mathrm{sp}}(\{a_i\,:\, i \in \mathbb{N}^*\})$, we deduce that the linear continuous functional f of norm one used in defining m_1, m_2, x_1, x_2, y_1 and y_2 (relations (3.2) to (3.6)) may be picked in $\mathcal{B}(C_1 - C_2)$ such that

$$C^f := \{w \in (C_1 - C_2)^\infty \;\;:\;\; \langle f, w\rangle = 0\} \subset j(X_1^\perp).$$

Set

$$B_1 = \{x \in \overline{\mathrm{co}}(y_1, C_1) \;\;:\;\; \langle f, x\rangle = m_1\}. \tag{3.29}$$

As $\langle f, x_1 - x_2\rangle = 0$ for any $x_1, x_2 \in B_1$, it follows that $\langle f, w\rangle = 0$, for all $w \in B_1^\infty$. Moreover, since $B_1 \subset \overline{\mathrm{co}}(y_1, C_1)$, we have

$$B_1^\infty \subseteq (\overline{\mathrm{co}}(y_1, C_1)^\infty) = C_1^\infty \subset (C_1 - C_2)^\infty.$$

Hence, $B_1^\infty \subseteq j(X_1^\perp)$.
Set

$$\begin{aligned} c_n \;\;&=\;\; \left(1 - \frac{1}{2}\frac{m_2 - \langle f, y_2\rangle}{\langle f, b_n\rangle - \langle f, y_2\rangle}\right) z_1 \\ &+\;\; \frac{1}{2}\frac{m_2 - \langle f, y_2\rangle}{\langle f, b_n\rangle - \langle f, y_2\rangle}\left(\frac{m_1 - \langle f, a_n\rangle}{\langle f, y_1\rangle - \langle f, a_n\rangle} y_1 + \frac{\langle f, y_1\rangle - m_1}{\langle f, y_1\rangle - \langle f, a_n\rangle} a_n\right). \end{aligned} \tag{3.30}$$

Then, for every $n \in \mathbb{N}^*$, c_n is a convex combination of z_1, y_1 and a_n. It is easy to check that $\langle f, c_n\rangle = m_1$. Consequently, c_n belongs to B_1.

Lemma 3.3. *The set* $D = \overline{\mathrm{co}}\left((c_n)_{n\in\mathbb{N}^*}\right)$ *is unbounded, and*

$$d(x, -j((B_1^\infty)^\circ)) \leq |z_1| + \frac{|y_1|}{2} \;\text{ for every } x \in D.$$

Proof of Lemma 3.3: By definition of c_n (see (3.30)), for every $n \in \mathbb{N}^*$ we have,

$$\left| c_n - \frac{1}{2} \frac{\langle f, y_2 \rangle - m_2}{\langle f, y_2 \rangle - \langle f, b_n \rangle} \frac{m_1 - \langle f, y_1 \rangle}{\langle f, a_n \rangle - \langle f, y_1 \rangle} a_n \right| \leq |z_1| + \frac{|y_1|}{2}. \tag{3.31}$$

By the inequality $\langle f, (b_n - a_n) \rangle \leq |f|_* |a_n - b_n| \leq d + 1$, it follows that

$$m_1 \geq \langle f, a_n \rangle \geq m_2 - d - 1 \ \text{ and } \ m_2 \leq \langle f, b_n \rangle \leq m_1 + d + 1.$$

Thus, we have

$$\frac{\langle f, y_2 \rangle - m_2}{\langle f, y_2 \rangle - \langle f, b_n \rangle} \geq \frac{\langle f, y_2 \rangle - m_2}{\langle f, y_2 \rangle - m_1 - d - 1} > 0 \tag{3.32}$$

and

$$\frac{m_1 - \langle f, y_1 \rangle}{\langle f, a_n \rangle - \langle f, y_1 \rangle} \geq \frac{m_1 - \langle f, y_1 \rangle}{m_2 - d - 1 - \langle f, y_1 \rangle} > 0. \tag{3.33}$$

Relations (3.31), (3.32) and (3.33) imply that

$$|c_n| \geq \frac{1}{2} \frac{\langle f, y_2 \rangle - m_2}{\langle f, y_2 \rangle - m_1 - d - 1} \frac{m_1 - \langle f, y_1 \rangle}{m_2 - d - 1 - \langle f, y_1 \rangle} |a_n| - |z_1| - \frac{|y_1|}{2},$$

and, as $|a_n| \to \infty$, it follows that $|c_n| \to \infty$ and the unboundedness of D.
Finally, as $B_1^\infty \subseteq j(X_1^\perp)$ it follows that $X_1 = j(j(X_1^\perp)^\perp) \subseteq j(B_1^\infty)^\circ$, and, as relation (3.31) implies that d$(x, X_1) \leq |z_1| + \frac{|y_1|}{2}$, the proof of Lemma 3.3 is completed. ☐

The conclusions of the previous lemma allows us to construct by induction a decreasing sequence $\{D_n\}$ of closed, convex and unbounded subsets of D, and a sequence $\{g_n\}_{n \in \mathbb{N}^*}$ of elements from $\mathcal{B}(B_1)$ of norm 1, such that

$$\langle g_i, x \rangle \leq -i \quad \forall x \in D_i, \ \ \forall i \in \mathbb{N}^*.$$

Let us define D_1 and g_1. As D is unbounded and as

$$D \subseteq -j((B_1^\infty)^\circ) + \left(|z_1| + \frac{|y_1|}{2} \right) \mathbb{B}_X\,, \tag{3.34}$$

we deduce that $P_{-j((B_1^\infty)^\circ)}(D)$ is unbounded, so there is $e_1 \in D$ such that

$$|P_{-j((B_1^\infty)^\circ)}(e_1)| \geq 4 + |z_1| + \frac{|y_1|}{2}. \tag{3.35}$$

By the bipolar Theorem, $\mathcal{B}(B_1)$ is dense in $(B_1^\infty)^\circ$, so there is $h_1 \in \mathcal{B}(B_1)$ such that

$$|P_{-j((B_1^\infty)^\circ)}(e_1) - j(-h_1)| \leq 1. \tag{3.36}$$

From the relations (3.35) and (3.36), it follows that

$$|h_1|_* = |j(-h_1)| \geq |P_{-j((B_1^\infty)^\circ)}(e_1)| - 1 \geq 3 + |z_1| + \frac{|y_1|}{2} > 0. \tag{3.37}$$

Using (3.34), we have

$$|e_1 - P_{-j((B_1^\infty)^\circ)}(e_1)| \leq |z_1| + \frac{|y_1|}{2}. \tag{3.38}$$

Consequently,

$$\begin{aligned} \langle h_1, e_1\rangle &= \langle h_1, j(-h_1)\rangle + \langle h_1, (e_1 - P_{-j((B_1^\infty)^\circ)}(e_1)) \\ &+ (P_{-j((B_1^\infty)^\circ)}(e_1) + j(h_1))\rangle \\ &\leq -|h_1|_*^2 + |h_1|(1 + |z_1| + \frac{|y_1|}{2}) \\ &= |h_1|_*(-|h_1|_* + 1 + |z_1| + \frac{|y_1|}{2}) \\ &\leq -2|h_1|_*. \end{aligned} \tag{3.39}$$

Set $g_1 = \frac{h_1}{|h_1|_*}$ and $D_1 = \{x \in D \ : \ \langle g_1, x\rangle \leq -1\}$. As

$$-1 > -2 \geq \langle g_1, e_1\rangle \geq \inf_{x \in D} \langle g_1, x\rangle\,,$$

it follows from Lemma 4.2 that D_1 is unbounded, so the first step of the induction is completed.

Let $n > 1$, and suppose that the elements D_i and g_i have been constructed for all $1 \leq i \leq (n-1)$. The set D_{n-1} is unbounded, and

$$D_{n-1} \subseteq D \subseteq \left(-j((B_1^\infty)^\circ) + \left(|z_1| + \frac{|y_1|}{2}\right)\mathbb{B}_X\right)\,;$$

therefore $P_{-j((B_1^\infty)^\circ)}(D_{n-1})$ is unbounded, so there is $e_n \in D_{n-1}$ such that

$$|P_{-j((B_1^\infty)^\circ)}(e_n)| \geq n + 3 + |z_1| + \frac{|y_1|}{2}.$$

Pick $h_n \in \mathcal{B}(B_1)$ such that

$$|P_{-j((B_1^\infty)^\circ)}(e_n) - j(-h_n)| \leq 1$$

(h_n exists since $\mathcal{B}(B_1)$ is dense in $(B_1^\infty)^\circ$). The two previous relations yield

$$|h_n|_* = |j(-h_n)| \geq |P_{-j((B_1^\infty)^\circ)}(e_n)| - 1 \geq n + 2 + |z_1| + \frac{|y_1|}{2} > 0.$$

Thus, similarly to relation (3.39),

$$\langle h_n, e_n\rangle \leq -(n+1)|h_n|_*.$$

Let us define $g_n = \frac{h_n}{|h_n|_*}$ and $D_n = \{x \in D_{n-1} : \langle g_n, x\rangle \leq -n\}$. As

$$-n > -(n+1) = \langle g_n, e_n\rangle \geq \inf_{x \in D_{n-1}} \langle g_n, x\rangle\,,$$

we may use Lemma 4.2 to deduce that D_n is unbounded, and complete the induction.

We have thus constructed two sequences, $\{e_n\} \subset D$ and $\{g_n\} \subset \mathcal{B}(B_1)$, such that $|g_n|_* = 1$ and

$$\langle g_i, e_j\rangle \leq -i, \ \ \forall i, j \in \mathbb{N}^* \text{ such that } i \leq j. \tag{3.40}$$

We can now define the desired uniform perturbations as requested in Theorem 3.1, for the case $\mathrm{d}(C_1 \setminus r\mathbb{B}_X\,, C_2 \setminus r\mathbb{B}_X) \leq d < +\infty \ \ \forall r > 0$ and $l(C_1 - C_2) = \{0\}$: let us take

$$C_{1,\varepsilon} = \overline{\mathrm{co}}\left(\{u_n = e_n + \frac{n}{n+1}\frac{\varepsilon}{4} : n \in \mathbb{N}^*\}, C_1\right) \tag{3.41}$$

and

$$C_{2,\varepsilon} = \{x \in \overline{\mathrm{co}}(y_2, C_2) : \langle f, x\rangle \geq m_2\}. \tag{3.42}$$

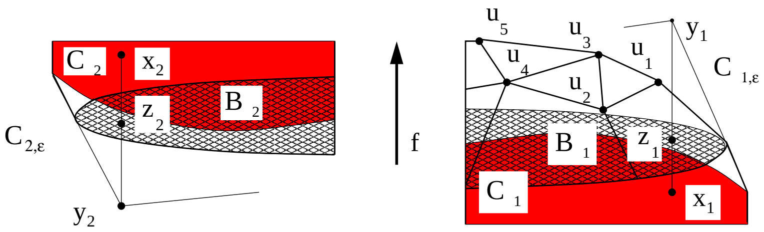

FIGURE 2. Uniform perturbations in case iii)

• *$C_{1,\varepsilon}$ and $C_{2,\varepsilon}$ satisfy relation (1.2).*
By definition of c_n (see 3.30), we have

$$\begin{aligned} z_1 + 2(c_n - z_1) &= \left(1 - \frac{m_2 - \langle f, y_2\rangle}{\langle f, b_n\rangle - \langle f, y_2\rangle}\right) z_1 \\ &+ \frac{m_2 - \langle f, y_2\rangle}{\langle f, b_n\rangle - \langle f, y_2\rangle} \left(\frac{m_1 - \langle f, a_n\rangle}{\langle f, y_1\rangle - \langle f, a_n\rangle} y_1 + \frac{\langle f, y_1\rangle - m_1}{\langle f, y_1\rangle - \langle f, a_n\rangle} a_n\right). \end{aligned}$$

Hence, $z_1 + 2(c_n - z_1) \in \overline{\text{co}}(y_1, C_1)$.
Consequently, $z_1 + 2(D - z_1) \subseteq \overline{\text{co}}(y_1, C_1)$, so $z_1 + 2(e_n - z_1) \in \overline{\text{co}}(y_1, C_1)$ for every $n \in \mathbb{N}^*$.
Taking into account that $y_1 = z_1 + (\varepsilon - m_1 + \langle f, x_1\rangle) j(f)$ (see relation (3.5)) we deduce that

$$e_n + \frac{(\varepsilon - m_1 + \langle f, x_1\rangle)}{2} j(f) = e_n + \frac{1}{2}(y_1 - z_1) = \frac{1}{2}(z_1 + 2(e_n - z_1)) + \frac{1}{2} y_1 \in \overline{\text{co}}(y_1, C_1);$$

Finally,

$$\begin{aligned} u_n &= e_n + \frac{\varepsilon(n-1)}{4n} j(f) \\ &= \left(1 - \frac{\varepsilon(n-1)}{2n(\varepsilon - m_1 + \langle f, x_1\rangle)}\right) e_n \\ &+ \frac{\varepsilon(n-1)}{2n(\varepsilon - m_1 + \langle f, x_1\rangle)} \left(e_n + \frac{1}{2}(\varepsilon - m_1 + (\langle f, x_1\rangle) j(f)\right); \end{aligned}$$

as $\varepsilon(n-1) < 2n\frac{\varepsilon}{2} < 2n(\varepsilon - m_1 + \langle f, x_1\rangle)$ (see 3.4), it follows that u_n is a convex combination of e_n and $e_n + \frac{(\varepsilon - m_1 + \langle f, x_1\rangle)}{2} j(f)$, so $u_n \in \overline{\text{co}}(y_1, C_1)$.

Thus we have established that $C_{1,\varepsilon} \subseteq \overline{\text{co}}(y_1, C_1)$. As obviously $C_1 \subseteq C_{1,\varepsilon}$ and $C_2 \subseteq C_{2,\varepsilon} \subseteq \overline{\text{co}}(y_2, C_2)$, relation (3.8) implies that relation (1.2) holds.

• *$C_{1,\varepsilon} - C_{2,\varepsilon}$ is not closed*:

Set

$$\begin{aligned} d_n &= \left(1 - \frac{1}{2} \frac{\langle f, y_1\rangle - m_1}{\langle f, y_1\rangle - \langle f, a_n\rangle}\right) z_2 \\ &+ \frac{1}{2} \frac{\langle f, y_1\rangle - m_1}{\langle f, y_1\rangle - \langle f, a_n\rangle} \left(\frac{\langle f, b_n\rangle - m_2}{\langle f, b_n\rangle - \langle f, y_2\rangle} y_2 + \frac{m_2 - \langle f, y_2\rangle}{\langle f, b_n\rangle - \langle f, y_2\rangle} b_n\right). \end{aligned}$$

It follows that d_n is a convex combination of z_2, y_2 and b_n, and therefore belongs to $\overline{\text{co}}(y_2, C_2)$. Moreover, $\langle f, d_n\rangle = m_2$, so $d_n \in B_2 = \{x \in C_{2,\varepsilon} : \langle f, x\rangle = m_2\}$. Using the definitions of c_n and d_n and relation (3.28) it follows that

$$|c_n - d_n| \leq |z_1| + |z_2| + \frac{|y_1| + |y_2| + |a_n - b_n|}{2} \leq |z_1| + |z_2| + \frac{|y_1| + |y_2| + d + 1}{2}.$$

Consequently,

$$c_n \in \left(B_2 + \left(|z_1| + |z_2| + \frac{|y_1| + |y_2| + d + 1}{2} \right) \mathbb{B}_X \right),$$

and therefore

$$D \subseteq \left(B_2 + \left(|z_1| + |z_2| + \frac{|y_1| + |y_2| + d + 1}{2} \right) \mathbb{B}_X \right). \tag{3.43}$$

For every $e_n \in D$ the previous relation implies that

$$\mathrm{d}(e_n, B_2) \leq |z_1| + |z_2| + \frac{|y_1| + |y_2| + d + 1}{2},$$

so there is a sequence $\{v_n\}_{n \in \mathbb{N}^*}$ in $B_2 \subseteq C_{2,\varepsilon}$ such that

$$|e_n - v_n| \leq \left(|z_1| + |z_2| + \frac{|y_1| + |y_2| + d + 1}{2} \right).$$

Accordingly, $|u_n - v_n| \leq \left(|z_1| + |z_2| + \frac{|y_1|+|y_2|+d+1}{2} \right) + \frac{\varepsilon}{4}$, and let w be a cluster point with respect to the weak topology of X of the bounded sequence $u_n - v_n$ of $C_{1,\varepsilon} - C_{2,\varepsilon}$.

Since $\langle f, e_n \rangle = m_1$ and $\langle f, v_n \rangle = m_2$, we have :

$$\begin{aligned} \lim_{n \to \infty} \langle f, u_n - v_n \rangle &= \lim_{n \to \infty} \left\langle f, \left(e_n + \frac{n}{n+1} \frac{\varepsilon}{4} j(f) \right) - v_n \right\rangle \\ &= \lim_{n \to \infty} \left(m_1 + \frac{n}{n+1} \frac{\varepsilon}{4} - m_2 \right) = m_1 - m_2 + \frac{\varepsilon}{4}. \end{aligned}$$

Consequently, we obtain

$$\langle f, w \rangle = m_1 - m_2 + \frac{\varepsilon}{4}. \tag{3.44}$$

Similarly to the proof of case i), the following result will play a key role.

Lemma 3.4. *For every $x \in C_{1,\varepsilon}$, we have $\langle f, x \rangle < m_1 + \frac{\varepsilon}{4}$.*

For the purpose of obtaining a contradiction, suppose that there is $\overline{x}$ an element of $C_{1,\varepsilon}$ such that $\langle f, \overline{x} \rangle = m_1 + \frac{\varepsilon}{4}$. Since $\overline{x} \in C_{1,\varepsilon}$, there are $a_n \in \mathrm{co}\left(\{u_i : i \in \mathbb{N}^*\}\right)$, $w_n \in C_1$, and $0 \leq \mu_n \leq 1$ such that $b_n = \mu_n a_n + (1 - \mu_n) w_n \to \overline{x}$ (see (3.41)). Since $\langle f, \overline{x} \rangle = m_1 + \frac{\varepsilon}{4}$, by using a subsequence if necessary, we can suppose that $\langle f, b_n \rangle \geq m_1$ for every $n \in \mathbb{N}^*$.

As $w_n \in C_1$, it follows that $\langle f, w_n \rangle \leq m_1$; for every $n \in \mathbb{N}^*$ such that $\langle f, w_n \rangle = m_1$, set $t_n = w_n$ and $\lambda_n = \mu_n$, and for every $n \in \mathbb{N}^*$ such that $\langle f, w_n \rangle < m_1$, set

$$t_n = \frac{m_1 - \langle f, w_n \rangle}{\langle f, a_n \rangle - \langle f, w_n \rangle} a_n + \frac{\langle f, a_n \rangle - m_1}{\langle f, a_n \rangle - \langle f, w_n \rangle} u_n,$$

and

$$\lambda_n = \frac{\mu_n (\langle f, a_n \rangle - \langle f, w_n \rangle) - (m_1 - \langle f, w_n \rangle)}{\langle f, a_n \rangle - m_1}.$$

Remark that $\langle f, t_n \rangle = m_1$ for every $n \in \mathbb{N}^*$; as $t_n \in C_{1,\varepsilon} \subset \overline{\mathrm{co}}(y_1, C_1)$, it follows that $t_n \in B_1$ (see (3.29) for the definition of B_1). Moreover,

$$\lambda_n - 1 = \frac{(\langle f, w_n \rangle - \langle f, a_n \rangle)(1 - \mu_n)}{\langle f, a_n \rangle - m_1},$$

and, as $\langle f, w_n\rangle \leq m_1 < m_1 + \frac{\varepsilon}{8} \leq \langle f, a_n\rangle$, it follows that $\lambda_n \leq 1$. On the other hand,

$$\lambda_n = \frac{\langle f, b_n\rangle - m_1}{\langle f, a_n\rangle - m_1},$$

and, since $\langle f, b_n\rangle \geq m_1$, we deduce that $0 \leq \lambda_n \leq 1$. Accordingly, b_n can be expressed as a convex combination of t_n and a_n, $b_n = \lambda_n a_n + (1-\lambda_n)t_n$.
Fix $i_0 \in \mathbb{N}^*$; as

$$\mathrm{co}(\{u_i : i \in \mathbb{N}^*\}) = \mathrm{co}\left(\mathrm{co}(\{u_i \,:\, 1 \leq i \leq i_0\}), \mathrm{co}(\{u_i \,:\, i_0 < i\})\right),$$

any element $a_n \in \mathrm{co}\left(\{u_i : i \in \mathbb{N}^*\}\right)$ may be written as a convex combination $a_n = \tau_n p_n + (1-\tau_n)q_n$ of some elements $p_n \in \mathrm{co}\left(\{u_i \,:\, 1 \leq i \leq i_0\}\right)$, and $q_n \in \mathrm{co}\left(\{u_i \,:\, i_0 < i\}\right)$, where $0 \leq \tau_n \leq 1$.
Consequently,

$$\begin{aligned}
&\langle f, b_n\rangle - m_1 - \frac{\varepsilon}{4} = \\
&\quad \lambda_n \left(\tau_n \left(\langle f, p_n\rangle - m_1 - \frac{\varepsilon}{4}\right) + (1-\tau_n)\left(\langle f, q_n\rangle - m_1 - \frac{\varepsilon}{4}\right)\right) + \\
&\quad (1-\lambda_n)\left(\langle f, t_n\rangle - m_1 - \frac{\varepsilon}{4}\right).
\end{aligned}$$

As $\lim\left(\langle f, b_n\rangle - m_1 - \frac{\varepsilon}{4}\right) = 0$, and as $\langle f, p_n\rangle, \langle f, q_n\rangle, \langle f, t_n\rangle \leq m_1 + \frac{\varepsilon}{4}$, it follows that the three sequences $\lambda_n\tau_n(\langle f, p_n\rangle - m_1 - \frac{\varepsilon}{4})$, $\lambda_n(1-\tau_n)(\langle f, q_n\rangle - m_1 - \frac{\varepsilon}{4})$ and $(1-\lambda_n)(\langle f, t_n\rangle - m_1 - \frac{\varepsilon}{4})$ converge separately to zero.

Since $p_n \in \mathrm{co}\left(\{u_i \,:\, 1 \leq i \leq i_0\}\right)$, we know that

$$\langle f, p_n\rangle - m_1 - \frac{\varepsilon}{4} \leq -\frac{\varepsilon}{4(i_0+1)}.$$

Accordingly, since $\lambda_n\tau_n(\langle f, p_n\rangle - m_1 - \frac{\varepsilon}{4}) \to 0$ we deduce that $\lambda_n\tau_n \to 0$. Moreover, $\langle f, t_n\rangle - m_1 - \frac{\varepsilon}{4} = -\frac{\varepsilon}{4}$, so $(1-\lambda_n)(\langle f, t_n\rangle - m_1 - \frac{\varepsilon}{4}) \to 0$ implies that $\lambda_n \to 1$. We may thus conclude that $\lambda_n \to 1$ and $\tau_n \to 0$. Setting $M = \sup_{x \in B_1}\langle g_{i_0}, x\rangle$, we have $\langle g_{i_0}, t_n\rangle \leq M$.
As $\langle g_{i_0}, e_i\rangle \leq M$ for every $1 \leq i \leq i_0$, $p_n \in \mathrm{co}\left(\{u_i \,:\, 1 \leq i \leq i_0\}\right)$, and $|g_{i_0}|_* = 1$, we obtain $\langle g_{i_0}, p_n\rangle \leq M + \frac{\varepsilon}{4}$.
Finally, $\langle g_{i_0}, e_k\rangle \leq -i_0$ for every $k \geq i_0$ and, as $q_n \in \mathrm{co}\left(\{u_i \,:\, i_0 < i\}\right)$ and $|g_{i_0}|_* = 1$, we deduce that $\langle g_{i_0}, q_n\rangle \leq -i_0 + \frac{\varepsilon}{4}$.
From the three previous inequalities we obtain

$$\begin{aligned}
\langle g_{i_0}, b_n\rangle &= \lambda_n\left(\tau_n\langle g_{i_0}, p_n\rangle + (1-\tau_n)\langle g_{i_0}, q_n\rangle\right) + (1-\lambda_n)\langle g_{i_0}, t_n\rangle \\
&\leq \lambda_n\left(\tau_n\left(M + \frac{\varepsilon}{4}\right) + (1-\tau_n)\left(-i_0 + \frac{\varepsilon}{4}\right)\right) + (1-\lambda_n)M \\
&= M(1-\lambda_n+\lambda_n\tau_n) + \frac{\varepsilon}{4}\lambda_n - i_0\lambda_n(1-\tau_n),
\end{aligned}$$

and, as $\lambda_n \to 1$ and $\tau_n \to 0$, it follows that $\limsup_{n\to\infty}\langle g_{i_0}, b_n\rangle \leq -i_0 + \frac{\varepsilon}{4}$.
Since $|g_{i_0}|_* = 1$, from the previous relation we deduce that $\limsup_{n\to\infty}|b_n| \geq i_0 - \frac{\varepsilon}{4}$, relation which, as i_0 is arbitrary in $\mathbb{N}^*$, means that $|b_n| \to \infty$. The last relation contradicts the fact that $b_n \to \overline{x}$. □

Let us return to the proof of case iii). As for every $x \in C_{2,\varepsilon}$, $\langle f, x\rangle \geq m_2$, from Lemma 3.4 we deduce that

$$\langle f, x\rangle < m_1 - m_2 + \frac{\varepsilon}{4} \quad \forall x \in C_{1,\varepsilon} - C_{2,\varepsilon}. \tag{3.45}$$

Relations (3.44) and (3.45) imply that $w \notin C_{1,\varepsilon} - C_{2,\varepsilon}$, which, accordingly, does not contain one weak cluster point of one of its bounded sequences. Thus $C_{1,\varepsilon} - C_{2,\varepsilon}$ is not weakly closed, and, as $C_{1,\varepsilon} - C_{2,\varepsilon}$ is convex, we deduce that $C_{1,\varepsilon} - C_{2,\varepsilon}$ is not closed. $\square$

4. PROOF OF THE TECHNICAL RESULTS

Let us recall the following separation theorem due to Klee (see [9]).

Theorem 4.1. *Let C be a closed convex set in a separable Banach space X. If the set C does not contains lines, there is $h \in \mathcal{B}(C)$ such that*

$$\langle h, w\rangle < 0 \ \ \forall w \in C^{\infty},\ w \neq 0. \tag{4.1}$$

If the ambient Banach space X is not separable, it is impossible, in general, to pick f fulfilling (4.1). The following result proves that, at least, it is possible to manage that the closed convex cone $C^f = \{w \in C^{\infty} : \langle f, w\rangle = 0\}$ be as far as possible from any given separable subspace X_1 of X (in the sense that $C^f \subseteq j(X_1^{\perp})$).

Proposition 4.1. *Let X be a reflexive Banach space and $C \subseteq X$ a closed convex set containing no lines. For every separable closed subspace X_1 of X, there is $f \in \mathcal{B}(C) \setminus \{0\}$ such that $C^f \subseteq j(X_1^{\perp})$.*

Proof of Proposition 4.1: Let us consider in a first step the particular case of subspaces X_1 which are images through the duality mapping j of closed separable subspaces of X^* spanned by elements from $\mathcal{B}(C)$.

Step 1: the subspace X_1 is of the form $j(Y_1)$, where Y_1 is a closed separable subspace of X^* such that $\overline{\text{sp}}\,(\mathcal{B}(C) \cap Y_1) = Y_1$. Consider $K = \overline{\text{co}}(P_{j(Y_1)}C)$. As for every $f \in X^*$ and $w \in X$ we have

$$\langle P_{Y_1}(f), w\rangle = \left\langle P_{Y_1}(f), P_{j(Y_1)}(w)\right\rangle = \left\langle f, P_{j(Y_1)}(w)\right\rangle, \tag{4.2}$$

it follows that

$$P_{Y_1}(f) \in \mathcal{B}(C) \ \Leftrightarrow\ f \in \mathcal{B}(K).$$

Observe from (4.2) that $(j(Y_1))^{\perp} \subseteq \mathcal{B}(K)$. Since also $\mathcal{B}(C) \cap Y_1 = \mathcal{B}(K) \cap Y_1$, we have that $\overline{\text{sp}}\,(\mathcal{B}(K) \cap Y_1) = \overline{\text{sp}}\,(\mathcal{B}(C) \cap Y_1) = Y_1$, and therefore $Y_1 \subseteq \overline{\text{sp}}(\mathcal{B}(K))$. As $X^* = j(Y_1)^{\perp} + Y_1$, it follows that $\overline{\text{sp}}(\mathcal{B}(K)) = X^*$. Consequently $\mathrm{l}(K) = \{0\}$ and therefore the closed convex subset K of the separable reflexive Banach space $j(Y_1)$ contains no lines.

The continuous dual of the separable space $j(Y_1)$ may be identified to the subspace Y_1 of X^*, so the barrier cone of K -when viewed as a subspace of $j(Y_1)$ - is $\mathcal{B}(K) \cap Y_1$. Thus, when applying Theorem 4.1 to the closed convex subset K of the separable Banach space $j(Y_1)$, we deduce that there is f in $\mathcal{B}(K) \cap Y_1$, such that

$$\langle f, w\rangle < 0 \ \ \forall w \in K^{\infty},\ w \neq 0. \tag{4.3}$$

Since $\mathcal{B}(C) \cap Y_1 = \mathcal{B}(K) \cap Y_1$, it follows that $f \in \mathcal{B}(C) \cap Y_1$. Let $w \in C^f$. We have $0 = \langle f, w\rangle = \left\langle f, P_{j(Y_1)}(w)\right\rangle$; as obviously $P_{j(Y_1)}(C^{\infty}) \subseteq K^{\infty}$ we may apply relation (4.3) to f and $P_{j(Y_1)}(w)$ and deduce that $P_{j(Y_1)}(w) = 0$. Thus $C^f \subseteq j(j(Y_1)^{\perp})$.

Step 2: the general case.

Let us prove that for every closed separable subspace X_1 of X there is Y_1, a closed separable subspace of X^* fulfilling $\overline{\text{sp}}\,(\mathcal{B}(C) \cap Y_1) = Y_1$ and $X_1 \subset j(Y_1)$.

As X_1 is separable, select a countable set $\{x_i : i \in \mathbb{N}^*\}$ such that $X_1 = \overline{\text{sp}}(\{x_i : i \in \mathbb{N}^*\})$. The Bipolar Theorem states that $\mathcal{B}(C)$ is dense in $(C^{\infty})^{\circ}$; as relation $\overline{\text{sp}}((C^{\infty})^{\circ}) =$

$\mathrm{l}(C)^{\perp}$ obviously holds for every closed convex set C, we deduce that $\overline{\mathrm{sp}}(\mathcal{B}(C))$ =$(\mathrm{l}(C)^{\perp}$. The set C contains no lines; thus $\overline{\mathrm{sp}}(\mathcal{B}(C)) = X^*$.

Fix $i \in \mathbb{N}^*$; as $j^{-1}(x_i) \in \overline{\mathrm{sp}}(\mathcal{B}(C))$, $j^{-1}(x_i)$ is the strong limit of some sequence $\{q_n\}_{n\in\mathbb{N}^*}$ from $\mathrm{sp}(\mathcal{B}(C))$. For every $k \in \mathbb{N}^*$, select $p_k \in \mathbb{N}^*$ and $h_{k,j} \in \mathcal{B}(C)$ for $1 \le j \le p_k$ such that $q_k \in \mathrm{sp}(\{h_{k,j} : 1 \le j \le p_k\})$. Define $U_i = \{h_{k,j} : k \in \mathbb{N}^*, 1 \le j \le p_k\}$; as $q_k \in \mathrm{sp}(U_i)$ for any $k \in \mathbb{N}^*$, and $q_n \to j^{-1}(x_i)$, it follows that $j^{-1}(x_i) \in \overline{\mathrm{sp}}(U_i)$. Let $Y_1 = \overline{\mathrm{sp}}\left(\bigcup_{i\in\mathbb{N}^*} U_i\right)$. Obviously, $\overline{\mathrm{sp}}\left(\mathcal{B}(C) \cap Y_1\right) = Y_1$; moreover, as $x_i \in j\left(\overline{\mathrm{sp}}(U_i)\right) \subseteq j(Y_1)$ for every $i \in \mathbb{N}^*$, it follows that $X_1 \subseteq j(Y_1)$.

Consequently, $j(Y_1)^{\perp} \subseteq X_1^{\perp}$ and therefore $j(j(Y_1)^{\perp}) \subseteq j(X_1^{\perp})$. We have established (step 1 of the proof) that there is $f \in \mathcal{B}(C)$ such that $C^f \subseteq (j(Y_1))^{\perp}$, so $C^f \subseteq j(X_1^{\perp})$, and the proof of Proposition 4.1 is complete. □

Theorem 4.2. *Let X be a reflexive Banach space, and $C \subseteq X$ be a closed convex set which contains no lines. Then C is not well-positioned if and only if $C \cap L$ is unbounded and linearly bounded for some closed linear manifold L of X.*

Proof of Theorem 4.2: Let us first prove that an unbounded closed convex subset of a well-positioned closed convex set can not be linearly bounded.

Let C be a well-positioned closed convex set, and D a closed convex unbounded subset of C. As C is well-positioned, there are x_0 and g, two elements of X and X^*, such that

$$\langle g, x - x_0\rangle \ge |x - x_0|, \quad \forall x \in C, \tag{4.4}$$

and, as D is unbounded, it contains an unbounded sequence $\{a_n\}_{n\in\mathbb{N}^*}$.

For every $|a_n| > 0$, we obtain from relation (4.4) that

$$\left\langle g, \frac{a_n}{|a_n|}\right\rangle \ge 1 + \frac{\langle g, x_0\rangle - |x_0|}{|a_n|}.$$

Therefore,

$$\liminf_{n\to\infty} \left\langle g, \frac{a_n}{|a_n|}\right\rangle \ge 1. \tag{4.5}$$

Let w denote one of the weak cluster points of the bounded sequence $\left(\frac{a_n}{|a_n|}\right)_{n\in\mathbb{N}^*}$. As $a_n \in D$, w belongs to D^{∞}; from relation (4.5) it follows that $\langle g, w\rangle \ge 1$. Hence, $w \ne 0$. Accordingly, the set D is not linearly bounded.

In order to complete the proof of Theorem 4.2 we have to prove that for every not well-positioned closed convex set C containing no lines there is at least a linear manifold L such that $C \cap L$ is an unbounded linearly bounded closed convex subset.

Lemma 4.1. *Let C be a closed convex set containing no lines, and $y \in C$. For every $R > 0$, let us define*

$$M^C_{y,R} = \left\{ \frac{x - y}{|x - y|} : x \in C, \ |x - y| \ge R \right\}.$$

The following two affirmations are equivalent:

i) C is well-positioned;

ii) there is $R > 0$ such that $0 \notin \overline{\mathrm{co}}(M^C_{y,R})$.

Proof of Lemma 4.1: $i) \Longrightarrow ii)$

Let C be a well-positioned closed convex set, y an element of C, and x_0 and g two elements of X and X^* such that

$$\langle g, x - x_0\rangle \ge |x - x_0|, \quad \forall x \in C. \tag{4.6}$$

The first part of the proof will be accomplished by proving that, for every $R > 2(1 + |g|_*)|y - x_0|$, 0 does not belong to $\overline{\mathrm{co}}(M^C_{y,R})$.

Indeed, for every $x \in C$ such that $|x - y| > 2(1 + |g|_*)|y - x_0|$, relation (4.6) implies that

$$\begin{aligned} \left\langle g, \frac{x-y}{|x-y|} \right\rangle &\geq \frac{|x-x_0|}{|x-y|} - \left\langle g, \frac{y-x_0}{|x-y|} \right\rangle \\ &\geq 1 - \frac{|y-x_0|}{|x-y|} - |g|_* \frac{|y-x_0|}{|x-y|} \\ &= 1 - (1+|g|_*)\frac{|y-x_0|}{|x-y|} \\ &\geq \frac{1}{2}. \end{aligned}$$

This yields

$$\langle g, x \rangle \geq \frac{1}{2} \quad \forall x \in \overline{\mathrm{co}}(M^C_{y,R}),$$

and consequently $0 \notin \overline{\mathrm{co}}(M^C_{y,R})$.

$ii) \Longrightarrow i)$

Take now a closed convex set C and fix $y \in C$ such that $0 \notin \overline{\mathrm{co}}(M^C_{y,R})$ for some $R > 0$. Let us prove that

$$\left\langle \frac{3h}{|z|^2}, x - \left(y - \frac{2R}{|z|}z\right) \right\rangle \geq \left| x - \left(y - \frac{2R}{|z|}z\right) \right|, \quad \forall x \in C, \tag{4.7}$$

where $z = P_{\overline{\mathrm{co}}(M^C_{y,R})}0$ and $h = j^{-1}(z)$.

Indeed, for every $x \in C$ such that $|x - y| \geq R$, the vector $\frac{x-y}{|x-y|}$ belongs to $\overline{\mathrm{co}}(M^C_{y,R})$, and, as z is the element of minimal norm in $\overline{\mathrm{co}}(M^C_{y,R})$ we have,

$$\left\langle h, \frac{x-y}{|x-y|} \right\rangle \geq |z|^2.$$

Accordingly, for every $x \in C$ such that $|x - y| \geq R$, we have

$$\begin{aligned} \left\langle \frac{3h}{|z|^2}, x - \left(y - \frac{2R}{|z|}z\right) \right\rangle &\geq 3|x-y| + 3\frac{2R}{|z|} \\ &\geq 3\left(|x-y| + \frac{2R}{|z|}|z|\right) \\ &\geq \left| x - \left(y - \frac{2R}{|z|}z\right) \right|. \end{aligned} \tag{4.8}$$

If $|x - y| \leq R$, then

$$\left\langle \frac{3h}{|z|^2}, x - y \right\rangle \geq -\frac{3}{|z|^2}|h|_*|x-y| \geq -\frac{3R}{|z|}.$$

Thus, for every $x \in y + R\mathbb{B}_X$, we have

$$\begin{aligned} \left\langle \frac{3h}{|z|^2}, x - \left(y - \frac{2R}{|z|}z\right)\right\rangle &\geq -\frac{3R}{|z|} + \frac{6R}{|z|} \\ &\geq R + \frac{2R}{|z|}|z| \\ &\geq \left| x - \left(y - \frac{2R}{|z|}z\right)\right|. \end{aligned} \tag{4.9}$$

Combining relations (4.8) and (4.9) yields (4.7), completing thus the proof of Lemma 4.1. ☐

Let us now return to the proof of Theorem 4.2. In order to define the linear manifold L, we consider first the particular case of a separable reflexive Banach space X.

Step 1: the case of a separable reflexive Banach space X.

As X is separable, from Klee's theorem (Theorem 4.1) we infer the existence of $h \in \mathcal{B}(C)$ such that

$$|h|_* = 1, \ \langle h, w\rangle < 0 \ \ \forall w \in C^\infty, \ w \neq 0. \tag{4.10}$$

For every $p \in \mathbb{R}$, let us define $L_p = \{x \in X : \langle h, x\rangle = p\}$. From relation (4.10) it follows that

$$(C \cap L_p)^\infty = C^\infty \cap L_0 = \{w \in C^\infty : \langle h, w\rangle = 0\} = \{0\},$$

and therefore all sets $C_p = C \cap L_p$ are linearly bounded. The first step of Theorem 4.2 will thus be achieved by proving that at least one of the linearly bounded sets C_p is unbounded.

For the purpose of obtaining a contradiction, suppose that C_p is bounded for every $p \in \mathbb{R}$, that is $\rho_p := \sup_{x \in C_p} |x| < \infty$ for every $p \in \mathbb{R}$. Let us prove that the above assumption necessarily implies that $\sup_{p \leq m} \frac{\rho_p}{m+2-p} < \infty$, where $m = \sup_{x \in C} \langle h, x\rangle$.

Indeed, suppose that $\sup_{p \leq m} \frac{\rho_p}{m+2-p} = \infty$, that is that there is a sequence $(p_n)_{n \in \mathbb{N}^*} \subset (-\infty, m]$, such that $\rho_{p_n} \geq n(m+2-p_n)$. Accordingly, we may pick $x_n \in C_{p_n}$ such that

$$|x_n| \geq n(m+1-p_n) + n - \frac{1}{2} = n\left(m + 1 - \langle h, x_n\rangle\right) + n - \frac{1}{2}. \tag{4.11}$$

As $\rho_{p_n} \mapsto 0$, the sequence p_n can not be constant, so relation $\rho_{p_k} = \lim_{n\to\infty} \rho_{p_n}$ fails for at least one value of $k \in \mathbb{N}^*$, say $\bar{k}$. Accordingly, there are $\eta > 0$ and a subsequence $(p_{k_n})_{n \in \mathbb{N}^*}$ of $(p_n)_{n \in \mathbb{N}^*}$ such that

$$p_{k_n} \notin (p_{\bar{k}} - \eta, p_{\bar{k}} + \eta) \ \ \forall n \in \mathbb{N}^*. \tag{4.12}$$

Set

$$y_n = \left(1 - \frac{\eta}{|p_{k_n} - p_{\bar{k}}|}\right) x_{\bar{k}} + \frac{\eta}{|p_{k_n} - p_{\bar{k}}|} x_{k_n};$$

relation (4.12) implies that $0 < \frac{\eta}{|p_{k_n} - p_{\bar{k}}|} \leq 1$. This imply that y_n is a convex combination of $x_{\bar{k}}$ and x_{k_n} and therfore belongs to C. Moreover, $|\langle h, y_n\rangle - p_{\bar{k}}| = \eta$, so $y_n \in (C_{p_{\bar{k}} - \eta} \cup C_{p_{\bar{k}} + \eta})$.

If $p_{k_n} \leq p_{\bar{k}} \leq m$, then

$$(m + 1 - p_{k_n}) \max(1, m - p_{\bar{k}}) \geq m + 1 - p_{k_n} > p_{\bar{k}} - p_{k_n} = |p_{\bar{k}} - p_{k_n}|,$$

while if $p_{\bar{k}} \leq p_{k_n} \leq m$, then

$$(m+1-p_{k_n}) \max(1, m-p_{\bar{k}}) \geq (m+1-p_{k_n})(m-p_{\bar{k}}) \geq m-p_{\bar{k}} \geq p_{k_n}-p_{\bar{k}} = |p_{\bar{k}}-p_{k_n}|.$$

In both cases, $(m+1-p_{k_n})\max(1, m-p_{\bar{k}}) \geq |p_{\bar{k}} - p_{k_n}|$, so

$$\frac{m+1-p_{k_n}}{|p_{\bar{k}} - p_{k_n}|} \geq \frac{1}{\max(1, m-p_{\bar{k}})} \quad \forall n \in \mathbb{N}^*. \tag{4.13}$$

From relations (4.11) and (4.13) we deduce that

$$\begin{aligned}
|y_n| &= \left| \left(1 - \frac{\eta}{|p_{k_n} - p_{\bar{k}}|}\right) x_{\bar{k}} + \frac{\eta}{|p_{k_n} - p_{\bar{k}}|} x_{k_n} \right| \\
&\geq -|x_{\bar{k}}| + \frac{\eta}{|p_{k_n} - p_{\bar{k}}|} |x_{k_n}| \\
&\geq -|x_{\bar{k}}| + \frac{\eta k_n (m+1-p_{k_n})}{|p_{k_n} - p_{\bar{k}}|} \\
&\geq \frac{\eta}{\max(1, m-p_{\bar{k}})} k_n - |x_{\bar{k}}|.
\end{aligned}$$

As $k_n \to \infty$, the sequence y_n is unbounded, and, as $y_n \in (C_{p_{\bar{k}}-\eta} \cup C_{p_{\bar{k}}+\eta})$, it follows that at least one of the sets $C_{p_{\bar{k}}-\eta}$ and $C_{p_{\bar{k}}+\eta}$ is unbounded.

This contradiction proves that, if all sets C_p are bounded, then there is $M > 0$ such that

$$\frac{\rho_p}{m+2-p} \leq M \quad \forall p \leq m,$$

inequality which implies that

$$|x| \leq M(m+2-\langle h, x\rangle) \quad \forall x \in C.$$

Consequently,

$$\begin{aligned}
\langle -(M+m+3)h, x-(m+3)j(h)\rangle &= (M+m+3)(m+2-\langle h, x\rangle) \\
&+ (M+m+3) \\
&\geq M(m+2-\langle h, x\rangle) + m + 3 \\
&\geq |x| + m + 3 \\
&\geq |x-(m+3)j(h)| \quad \forall x \in C,
\end{aligned}$$

relation which contradicts the fact that C is not well-positioned.

We have thus proved that our initial assumption is false, so at least one of the linearly bounded sets C_p, say $C_{\bar{p}}$, must be unbounded. Thus, the linear manifold $L_{\bar{p}}$ satisfies to the requirements of Theorem 4.2, and the proof of the first step is complete.

Step 2: the general case.

Let us first prove that there is a separable subspace X_1 of X such that $C \cap X_1$ is not well-positioned.

Set $y \in C$ and $k \in \mathbb{N}^*$; as C is not well-positioned, Lemma 4.1 implies that $0 \in \overline{\mathrm{co}}(M_{y,k}^C)$. Consequently for every $n \in \mathbb{N}^*$ we may pick $x_n \in \mathrm{co}(M_{y,k}^C)$ such that $|x_n| \leq 1/n$. As x_n belongs to $\mathrm{co}(M_{y,k}^C)$, there is $p_n \in \mathbb{N}^*$ and $\{y_{n,i} : 1 \leq i \leq p_n\} \subset M_{y,k}^C$ such that $x_n \in \mathrm{co}(\{y_{n,i} : 1 \leq i \leq p_n\})$, and define $U_k = \{y_{n,i} : n \in \mathbb{N}^*, 1 \leq i \leq p_n\}$. From the definition of $M_{y,k}^C$ we deduce that there is a set $R_k \subset C \setminus k\mathbb{B}_X$ such that $U_k = \{\frac{x}{|x|} : x \in R_k\}$.

Let $X_1 = \overline{\mathrm{sp}}\left(y, \bigcup_{k\in\mathbb{N}^*} R_k\right)$ and $K = C \cap X_1$. Since $x_n \in \mathrm{co}(U_k)$ and $x_n \to 0$, it follows that $0 \in \overline{\mathrm{co}}(U_k)$ for every $k \in \mathbb{N}^*$. The definitions of X_1 and K imply that, for every $k \in \mathbb{N}^*$, $U_k \subseteq M_{y,k}^K$. Hence, $0 \in \overline{\mathrm{co}}(U_k) \subseteq \overline{\mathrm{co}}(M_{y,k}^K)$, for every $k \in \mathbb{N}^*$.

Since $M^K_{y,R_1} \subseteq M^K_{y,R_2}$ when $R_1 \geq R_2$, and as $0 \in \overline{\text{co}}(M^K_{y,k})$ for every $k \in \mathbb{N}^*$, we deduce that $0 \in \overline{\text{co}}(M^K_{y,R})$ for every $R > 0$. From Lemma 4.1 it follows that the closed convex set $K = C \cap X_1$ is not well-positioned.

As X_1 is separable, we have established (step 1 of the proof) that there is L, a closed linear manifold of X_1 (and thus of X) such that $K \cap L$ is not well-positioned. As $L \subseteq X_1$, $C \cap L = (C \cap X_1) \cap L$, so $C \cap L = K \cap L$. Accordingly, L is a closed linear manifold of X such that $C \cap L$ is not well-positioned, and the proof of Theorem 4.2 is complete. □

Lemma 4.2. *Let A be an unbounded convex set of a normed linear space X, $h \in \mathcal{B}(A)$ and $m > \inf_{x\in A} \langle h, x\rangle$. Then the set $A_{h,m} = \{x \in A : \langle h, x\rangle \leq m\}$ is unbounded.*

Proof of Lemma 4.2. Let us suppose that for some $m > \inf_{x\in A} \langle h, x\rangle$, the set $A_{h,m} = \{x \in A : \langle h, x\rangle \leq m\}$ is bounded, that is there is $\rho > 0$ such that $A_{h,m} \subseteq \rho\mathbb{B}_X$.

As $m > \inf_{x\in A} \langle h, x\rangle$, there is $x_0 \in A_{h,m}$ such that $\langle h, x_0\rangle < m$. Consequently, for every $x \in A \setminus A_{h,m}$ we have $0 < \frac{\langle h,x\rangle - m}{\langle h,x\rangle - \langle h,x_0\rangle} < 1$, so the element

$$y = \frac{\langle h, x\rangle - m}{\langle h, x\rangle - \langle h, x_0\rangle} x_0 + \frac{m - \langle h, x_0\rangle}{\langle h, x\rangle - \langle h, x_0\rangle} x$$

belongs to A. Moreover, $\langle h, y\rangle = m$, so $y \in A_{h,m}$, that is

$$\left| \frac{\langle h, x\rangle - m}{\langle h, x\rangle - \langle h, x_0\rangle} x_0 + \frac{m - \langle h, x_0\rangle}{\langle h, x\rangle - \langle h, x_0\rangle} x \right| \leq \rho.$$

Thus

$$\left| \frac{m - \langle h, x_0\rangle}{\langle h, x\rangle - \langle h, x_0\rangle} x \right| \leq \left| \frac{\langle h, x\rangle - m}{\langle h, x\rangle - \langle h, x_0\rangle} x_0 \right| + \rho. \tag{4.14}$$

Since $h \in \mathcal{B}(A)$, we have $M = \sup_{x\in A} \langle h, x\rangle < +\infty$; then

$$0 \leq \frac{\langle h, x\rangle - \langle h, x_0\rangle}{m - \langle h, x_0\rangle} \leq \frac{M - \langle h, x_0\rangle}{m - \langle h, x_0\rangle},$$

and

$$0 \leq \frac{\langle h, x\rangle - m}{\langle h, x\rangle - \langle h, x_0\rangle} \leq \frac{M - m}{m - \langle h, x_0\rangle}.$$

It results from relation (4.14) that

$$|x| \leq \frac{(M - \langle h, x_0\rangle)(M - m)}{(m - \langle h, x_0\rangle)^2} |x_0| + \frac{\rho(M - \langle h, x_0\rangle)}{m - \langle h, x_0\rangle}, \quad \forall x \in A \setminus A_{h,m}.$$

Accordingly, the set $A \setminus A_{h,m}$ is bounded; therefore, A, as the union of two bounded sets would be bounded, a contradiction. □

5. APPLICATION: A CONVEX SEPARATION THEOREM

In any finite dimensional space, any two closed convex sets C_1 and C_2 fulfilling

$$C_1 \cap C_2 = \emptyset,\ C_1^\infty \cap C_2^\infty = \{0\} \tag{5.1}$$

can be strictly separated by means of a closed hyperplane, that is, there exist $f \in X^*$ and $t \in \mathbb{R}$ such that

$$\langle f, x_1\rangle > t > \langle f, x_2\rangle,\ \forall x_1 \in C_1,\ x_2 \in C_2.$$

This result is no longer true in a general reflexive Banach space. An aditional condition, describing the behaviour at the infinity of at least one of the closed convex sets involved, is nedeed. On the basis of Theorem 2.1 we will completely characterize, a reflexive Banach space, the class of all closed convex sets C_1 that can be strictly separated with a closed hyperplane from all the closed convex sets C_2 fulfilling conditions (5.1). We thus determine the broadest condition which should be added to (5.1) in order to ensure the existence of a closed hyperplane strictly separating C_1 and C_2.

Theorem 5.1. *Let X be a reflexive Banach space and let C_1 be a closed convex subset of X containing no lines. The two following statements are equivalent:*

i) *C_1 is well-positioned;*

ii) *every closed convex set C_2 such that conditions (5.1) are fulfilled may be strictly separated from C_1, that is, there exist $f \in X^*$ and $t \in \mathbb{R}$ such that*

$$\langle f, x_1 \rangle > t > \langle f, x_2 \rangle, \ \forall x_1 \in C_1, \ x_2 \in C_2.$$

Proof of Theorem 5.1: $i) \Longrightarrow ii)$

If C_1 and C_2 are two closed convex sets such that C_1 is well-positioned and $C_1^\infty \cap C_2^\infty = \{0\}$, then, making use of Theorem 2.1, it follows that $C_1 - C_2$ is closed.

Let f denote the projection of 0 on the closed convex set $C_1 - C_2$, i.e., the element of minimal norm in $C_1 - C_2$:

$$f = \mathrm{P}_{C_1 - C_2} 0.$$

Accordingly,

$$\left\langle j^{-1}(f), x \right\rangle \geq \left\langle j^{-1}(f), f \right\rangle = ||f||^2, \ \forall x \in C_1 - C_2.$$

Therefore,

$$\left\langle j^{-1}(f), x_1 \right\rangle \geq \left\langle j^{-1}(f), x_2 \right\rangle + ||f||^2, \ \forall x_1 \in C_1, \ x_2 \in C_2.$$

The above inequality implies that

$$\inf_{x \in C_1} \left\langle j^{-1}(f), x \right\rangle \geq \sup_{x \in C_2} \left\langle j^{-1}(f), x \right\rangle + ||f||^2.$$

Set

$$t = \frac{\inf_{x \in C_1} \left\langle j^{-1}(f), x \right\rangle + \sup_{x \in C_2} \left\langle j^{-1}(f), x \right\rangle}{2}.$$

As $C_1 \cap C_2 = \emptyset$, we deduce that $0 \notin C_1 - C_2$, then$f \neq 0$, and thus

$$\inf_{x \in C_1} \left\langle j^{-1}(f), x \right\rangle \geq t + \frac{||f||^2}{2} > t > t - \frac{||f||^2}{2} \geq \sup_{x \in C_2} \left\langle j^{-1}(f), x \right\rangle.$$

Accordingly, for every $x_1 \in C_1$ and $x_2 \in C_2$,

$$\left\langle j^{-1}(f), x_1 \right\rangle \geq \inf_{x \in C_1} \left\langle j^{-1}(f), x \right\rangle > t > \sup_{x \in C_2} \left\langle j^{-1}(f), x \right\rangle \geq \left\langle j^{-1}(f), x_2 \right\rangle,$$

completing thus the desired implication.

$ii) \Longrightarrow i)$

Let C_1 be a closed convex set containing no lines which is not well-positioned. By the virtue of Theorem 4.2 (Section 4), there is a closed linear manifold L of X such that $C_1 \cap L$ is unbounded and linearly bounded. Applying Theorem 2.2 to $C_1 \cap L$, it follows that there is $g \in X^*$ such that

$$\sup_{x \in (C_1 \cap L)} \langle g, x \rangle = 1 \text{ and } \langle g, x \rangle < 1 \text{ for all } x \in (C_1 \cap L). \tag{5.2}$$

Set $C_2 = \{x \in L : \langle g, x \rangle = 1\}$. As relation (5.2) shows that the linear functional g is not constant on L, it follows that the closed and convex set C_2 is unbounded, and thus nonempty.

The closed and convex set C_2 lies within L. Acordingly,

$$C_1 \cap C_2 = (C_1 \cap L) \cap C_2, \tag{5.3}$$

and

$$C_1^\infty \cap C_2^\infty = (C_1^\infty \cap L) \cap C_2^\infty = (C_1 \cap L)^\infty \cap C_2^\infty. \tag{5.4}$$

The definition of C_2 and relations (5.2) and (5.3) imply that $C_1 \cap C_2 = \emptyset$, while the fact that $C_1 \cap L$ is linearly bounded and relation (5.4) imply that $C_1^\infty \cap C_2^\infty = \{0\}$.

Suppose that there is $f \in X^*$ and $t \in \mathbb{R}$ such that

$$\langle f, x_1 \rangle > t > \langle f, x_2 \rangle, \ \forall x_1 \in C_1, \ x_2 \in C_2.$$

Acordingly,

$$\inf_{x \in C_1} \langle f, x \rangle \geq t \geq \sup_{x \in C_2} \langle f, x \rangle. \tag{5.5}$$

As C_2 is an unbounded linear manifold of X, it follows that every linear functional bounded from below on C_2 is constant on C_2. Accordingly, there is $t_2 \in \mathbb{R}$, $t > t_2$ such that $\langle f, x \rangle = t_2$ for every $x \in C_2$.

Let V be the linear subspace of X parallel to L. The linear functional g is not constant on V, so there is $v \in V$ such that $\langle g, v \rangle = 1$. As $\sup_{x \in C_1 \cap L} \langle g, x \rangle = 1$, for every $n \in \mathbb{N}^*$ there is $x_n \in (C_1 \cap L)$ such that $\langle g, x_n \rangle \geq (n-1)/n$. Set $y_n = x_n + (1 - \langle g, x_n \rangle)v$; we have $y_n \in L$ and $\langle g, y_n \rangle = 1$, that is $y_n \in C_2$.

Consequently, $\langle f, y_n \rangle = t_2$. Thus,

$$|t_2 - \langle f, x_n \rangle| = |\langle f, y_n \rangle - \langle f, x_n \rangle| = |(1 - \langle g, x_n \rangle) \langle f, v \rangle| \leq \frac{|\langle f, v \rangle|}{n}.$$

Hence, $\inf_{x \in C_1} \langle f, x \rangle \leq \liminf_{n \to +\infty} \langle f, x_n \rangle = t_2 < t$, and therefore,

$$\inf_{x \in C_1} \langle f, x \rangle \leq t_2 < t. \tag{5.6}$$

The proof of Theorem 5.1 is thus completed as the contradiction between relations (5.5) and (5.6) shows that for every closed convex set C_1 which is not well-positioned, there is at least a closed convex set C_2 such that $C_1 \cap C_2 = \emptyset$ and $C_1^\infty \cap C_2^\infty = \{0\}$ that can not be strictly separated from C_1. □

We are now in position to state the broadest setting for the strict convex separation theorem in a reflexive Banach space: *two closed convex sets containing no lines may be strictly separated by means of a closed hyperplane if the both following conditions are fulfilled:*

a) $C_1 \cap C_2 = \emptyset, \ C_1^\infty \cap C_2^\infty = \{0\}$

and

b) *at least one of the sets C_1, C_2 is well-positioned.*

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