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Abstract

Let F be an abelian number field and S the set of primes of F that are either ramified or over p, with p an odd prime. In this paper we compute the (first) Fitting ideal of $K_{2i-2}^{\text{ét}}(\mathcal{O}_F^S)(\phi)$ for $i \geq 2$, where $\mathcal{O}_F^S$ is the ring of S-integers of F and ϕ is a character of $\text{Gal}(F/\mathbb{Q})$ of order prime to p different from the ith power of the Teichmüller character. This Fitting ideal proves to be principal and generated by a Stickelberger element.

keywords: étale K-groups, Fitting ideals, Iwasawa modules, Stickelberger elements.

Introduction

Let F be an abelian number field. The classical Stickelberger's Theorem states that the first Stickelberger ideal annihilates the ideal class group of F (see [W]). Inspired by Stickelberger, Coates and Sinnott made similar guesses about annihilator ideals for higher even K-groups. Namely, they conjectured in [CS] that the ith "twisted" Stickelberger element annihilates $K_{2i}(\mathcal{O}_F)$. Adopting a p-adic approach, one can try to annihilate $K_{2i}(\mathcal{O}_F) \otimes \mathbb{Z}_p$, with p a fixed odd prime. The Quillen-Lichtenbaum conjecture (which is true for $i = 2$) affirms that $K_{2i}(\mathcal{O}_F) \otimes \mathbb{Z}_p$ is canonically isomorphic to the higher étale K-theory group $K_{2i}^{\text{ét}}(\mathcal{O}_F[1/p])$. This group injects into $K_{2i}^{\text{ét}}(\mathcal{O}_F^S)$ where S is the set of primes of F that are either ramified or over p, and $\mathcal{O}_F^S$ is the ring of S-integers of F. Iwasawa theory provides another expression of $K_{2i}^{\text{ét}}(\mathcal{O}_F^S)$ and one can use the Main Conjecture to annihilate it (see [N]). The determination of the annihilator of a Galois module is a presumably difficult

problem. A first step in this task consists in computing the first Fitting ideal of this module, since it is contained in its annihilator ideal.
Let G denote $\mathrm{Gal}(F/\mathbb{Q})$ which we decompose in $G = \Delta \times P$ where P is the p-part of G. We suppose that F is linearly disjoint from the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$, so that the étale cohomology group $H^2_{\text{ét}}(\mathcal{O}_F^S; \mathbb{Z}_p(i)) \cong K^{\text{ét}}_{2i-2}(\mathcal{O}_F^S)$ is a $\mathbb{Z}_p[G]$-module. To study its $\mathbb{Z}_p[G]$-structure, we will use the decomposition $\mathbb{Z}_p[G] \cong \oplus_\phi \mathbb{Z}_p(\phi)[P]$, where the ϕ run over the $\mathbb{Q}_p$-irreducible characters of Δ (see section 1).
In this paper we compute the (first) Fitting ideal of $K^{\text{ét}}_{2i-2}(\mathcal{O}_F^S)(\phi)$, where $(.)(\phi)$ means $e_\phi(.)$ and e_ϕ is the usual orthogonal idempotent. When S is exactly the set of primes above p, it coincides with the main result obtained by [CO] for totally real abelian number fields of prime power conductor.

We now formulate the main theorem of this paper in the case where $F/\mathbb{Q}$ is *tamely ramified* over p. Let f denote the conductor of F and let σ_a be the restriction to F of the automorphism of $\mathbb{Q}(\mu_f)$ that maps a primitive fth root of unity to its ath power; define the ith Brumer-Stickelberger element $\Theta_{i,S}$ by

$$\Theta_{i,S} := \sum_{\substack{1 \le a < f \\ (a,f)=1}} \zeta_{f,S}(-i, a)\sigma_a^{-1},$$

where the partial zeta function $\zeta_{f,S}(a, s)$ is given by

$$\zeta_{f,S}(a, s) := \sum_{\substack{k \equiv a \bmod f \\ (k,S)=1}} k^{-s}.$$

Theorem 1. *Let ϕ be a $\mathbb{Q}_p$-irreducible character of Δ, ψ a character of degree one of Δ dividing ϕ and i be an integer greater or equal to 2 such that $\psi(-1) = (-1)^i$. We also require that $\phi^{-1}\omega^i$ is a non-trivial $\mathbb{Q}_p$-irreducible character of Δ', the non-p-part of $G' := \mathrm{Gal}(F(\mu_p)/\mathbb{Q})$. Then,*

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, K^{\acute{e}t}_{2i-2}(\mathcal{O}_F^S)(\phi) = (\Theta_{i-1,S}(\psi)),$$

where $\Theta_{i,S}(\psi) := \sum_{\substack{1 \le a < f \\ (a,f)=1}} \zeta_{f,S}(-i, a)\psi(\delta_a)^{-1}\rho_a^{-1}$ *and* $\sigma_a = \delta_a\rho_a$ *in the decomposition* $G = \Delta \times P$.

A slightly different statement will be formulated in section 3 for the *wildly ramified* case (see Theorem 2 in section 3). Notice that for the excluded character $\phi = \omega^i$, the corresponding Fitting ideal is probably not principal.

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1 ϕ-components and semi-simple decomposition

Let us denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of $\mathbb{Q}_p$. For a finite abelian group Δ, we denote by $\mathcal{D}_\Delta$ the group of $\overline{\mathbb{Q}}_p$-irreducible characters of Δ. One knows that the $\mathbb{Q}_p$-irreducible characters of Δ (which we shall denote by Ψ_Δ) can be expressed by means of elements of $\mathcal{D}_\Delta$. Namely, any $\phi \in \Psi_\Delta$ can be written

$$\phi = \sum_{u \in D_\psi} \psi^u,$$

where $\psi \in \mathcal{D}_\Delta$ is of order g_ψ, has image μ_{g_ψ}, the group of g_ψ-th roots of unity, and D_ψ is the decomposition group of p in $\mathbb{Q}(\mu_{g_\psi})/\mathbb{Q}$. We say that ψ divides ϕ and write $\psi \mid \phi$.

When $p \nmid |\Delta|$, there is a *semi-simple decomposition* of the ring $\mathbb{Z}_p[\Delta]$ which can be indexed by the elements of Ψ_Δ. More precisely,

$$\mathbb{Z}_p[\Delta] \cong \bigoplus_{\phi \in \Psi_\Delta} e_\phi \mathbb{Z}_p[\Delta],$$

where the $e_\phi = \frac{1}{|\Delta|} \sum_{\delta \in \Delta} \phi(\delta^{-1})\delta$ are the usual orthogonal idempotents. Define $\mathbb{Z}_p(\phi) := \mathbb{Z}_p(\mu_{g_\psi})$ for a $\psi \mid \phi$ (hence for all such ψ). Then $e_\phi \mathbb{Z}_p[\Delta] \cong \mathbb{Z}_p(\phi)$. This is an isomorphism of $\mathbb{Z}_p[\Delta]$-modules if $\mathbb{Z}_p(\phi)$ is endowed with the Δ-action $\delta.s = \psi(\delta)s$ for any $s \in \mathbb{Z}_p(\phi)$ and any $\delta \in \Delta$. If M is a $\mathbb{Z}_p[\Delta]$-module, then

$$M \cong \bigoplus_{\phi \in \Psi_\Delta} M(\phi),$$

with $M(\phi) := e_\phi M$, $M(\phi)$ being canonically a $\mathbb{Z}_p(\phi)$-module with action $\psi(\delta).m = \delta.m$ for any $\delta \in \Delta$ and any $m \in M(\phi)$.
When ϕ is a character of a group G whose order is divisible by p, one cannot define a ϕ-component as a direct summand of $\mathbb{Z}_p[G]$. For this reason we introduce another $\mathbb{Z}_p(\phi)$-module called "the ϕ-quotient" of M and defined

by $M_\phi := M \otimes_{\mathbb{Z}_p[G]} \mathbb{Z}_p(\phi)$, where G acts on $\mathbb{Z}_p(\phi)$ by $g.\phi = \phi(g)$ for any $g \in G$. The $\mathbb{Z}_p(\phi)$-module $M_\phi = M_\psi$ (where $\psi \mid \phi$) is isomorphic to the largest quotient-module of $M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(\phi)$ on which G acts via ψ (see [Ts]). If $p \nmid |G|$, and $\psi \mid \phi$, we have the following isomorphism of $\mathbb{Z}_p(\phi)$-modules

$$e_\phi M \cong M_\psi,$$

and the functor ψ-quotient $M \mapsto M_\psi$ is exact.

2 Fitting ideals of étale cohomology groups

The abelian group $G' = \mathrm{Gal}(F(\mu_p)/\mathbb{Q})$ decomposes into $\Delta' \times P'$ with $P' \cong P$. In the following, we will identify P and P'. Define $\mu_{p^\infty} := \cup_{n \geq 1} \mu_{p^n}$ as the union of all p-power roots of unity; we will call $\mathfrak{X}_\infty$ (resp. $\mathfrak{X}_\infty^S$) the Galois group over $F(\mu_{p^\infty})$ of the maximal p-ramified (resp. S-ramified) abelian pro-p extension of $F(\mu_{p^\infty})$. Let $\mathbb{Q}_\infty/\mathbb{Q}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$; we will assume throughout this section that F is *tamely ramified over p*. This implies in particular that $F \cap \mathbb{Q}_\infty = \mathbb{Q}$, hence $\Gamma := \mathrm{Gal}(F(\mu_{p^\infty})/F(\mu_p))$ identifies by restriction to $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and we have a decomposition $\mathrm{Gal}(F(\mu_{p^\infty})/\mathbb{Q}) = \Gamma \times \mathrm{Gal}(F(\mu_p)/\mathbb{Q})$. This ensures us that $\mathfrak{X}_\infty^S$ is a $\mathbb{Z}_p[[\Gamma]][G']$-module.

As usual, Λ will denote the Iwasawa algebra $\mathbb{Z}_p[[\Gamma]]$ which is isomorphic to $\mathbb{Z}_p[[T]]$ by mapping a topological generator γ_0 of Γ to $1+T$. We will call κ the restriction of the cyclotomic character of $\mathrm{Gal}(F(\mu_{p^\infty})/F)$ giving the action on all p-power roots of unity to Γ and by ω (the Teichmüller character) its restriction to $F(\mu_p)$. For any Γ-module M, $M(i)$ will mean M twisted i-times.
The triviality of $H^2_{\text{ét}}(\mathcal{O}_F^S; \mathbb{Q}_p/\mathbb{Z}_p(i))$ may be seen as a higher analog of the Leopoldt's conjecture (which is the case $i = 0$) and it is known to be true for any $i \geq 2$, because of the finiteness of the group $K_{2i-2}(\mathcal{O}_F)$ (and thus of $H^2_{\text{ét}}(\mathcal{O}_F^S; \mathbb{Z}_p(i))$); consequently we can use Theorem 4.1 of [KNF] (stated for $\mathfrak{X}_\infty$ but which holds also for $\mathfrak{X}_\infty^S$ as one can readily verify) and formulate the following lemma which gives the relation between $\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma$ and $H^2_{\text{ét}}(\mathcal{O}_F^S; \mathbb{Z}_p(i))(\phi)$, where ϕ is a p-adic character of $\Psi_{\Delta'}$ of same parity as $i \geq 2$.

Lemma 1.
$$((\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma)^\sharp \cong H^2_{ét}(\mathcal{O}_F^S; \mathbb{Z}_p(i))(\phi),$$
where $\sharp$ denotes the Pontrjagin dual.

Note that since $\mathfrak{X}_\infty^S$ is a Galois group over $F(\mu_{p^\infty})$, the $\Lambda[G']$-modules $\mathfrak{X}_\infty^S(-i)(\phi^{-1})$ and $\mathfrak{X}_\infty^S(\phi^{-1}\omega^i)(-i)$ coincide. It is also well-known that the latter is a $\Lambda(\phi^{-1})$-torsion-module (since $\phi^{-1}\omega^i$ is *even*) without non-trivial finite submodules (see [I]). Another important property of $\mathfrak{X}_\infty^S$, which will be crucial for our goal, is that, provided $\phi^{-1}\omega^i$ is not trivial, $\mathfrak{X}_\infty^S(\phi^{-1}\omega^i)$ is a cohomologically trivial $\Lambda(\phi^{-1})[P]$-module, or, equivalently, that its projective dimension over $\Lambda(\phi^{-1})[P]$ is lower or equal to 1. We record this important result in a lemma and we include a proof (the arguments can be found in [N2] and [BN]) for the convenience of the reader.

Lemma 2. *Suppose $\phi^{-1}\omega^i$ is a non-trivial character of $\Psi_{\Delta'}$. Then*

$$\mathrm{pd}(\mathfrak{X}_\infty^S(\phi^{-1}\omega^i)) \leq 1$$

over the algebra $\Lambda(\phi^{-1}\omega^i)[P]$.

Proof. Set $\xi := \phi^{-1}\omega^i$. By a result of Greither (Theorem 2.2 in [G]), if a $\Lambda(\xi)[P]$-module M is cohomologically trivial for P and has no $\mathbb{Z}_p$-torsion, then $\mathrm{pd}(M) \leq 1$ over $\Lambda(\xi)[P]$. Since $\mathfrak{X}_\infty^S$ has no non-trivial finite submodule and its μ-invariant is zero by Ferrero-Washington, it is $\mathbb{Z}_p$-torsion free; consequently, there remains to show that $\mathfrak{X}_\infty^S(\xi) = \varprojlim \mathfrak{X}_n^S(\xi)$ is cohomologically trivial over $\mathbb{Z}_p(\xi)[P]$, and for this, it suffices to show that it is true for each $\mathfrak{X}_n^S(\xi)$, $n \geq 0$. Prop 1.7 of [N2] gives an exact sequence of $\mathbb{Z}_p[P]$-modules

$$0 \to \mathfrak{X}_n^S \to Y_n^S \to I(P) \to 0,$$

where $I(P)$ is the kernel of the augmentation map $\mathbb{Z}_p[P] \to \mathbb{Z}_p$ and Y_n^S is a cohomologically trivial module if Leopoldt's conjecture holds for F_n. Therefore $\hat{H}^i(P, \mathfrak{X}_n^S) \cong \hat{H}^{i-2}(P, \mathbb{Z}_p)$ for any $i \in \mathbb{Z}$. Next, as $\hat{H}^{-1}(P, \mathbb{Z}_p) = 0$ and $\hat{H}^0(P, \mathbb{Z}_p) = \mathbb{Z}_p/|P|\mathbb{Z}_p$, it follows that for ξ non-trivial, $\hat{H}^i(P, \mathfrak{X}_n^S(\xi)) = \hat{H}^i(P, \mathfrak{X}_n^S)(\xi) = 0$ for $i = 1, 2$ and we are done. □

We briefly recall the definition of the (first) Fitting ideal of a finitely generated module M over a commutative ring R (for more details, see the appendix of [MW]). For some $r \in \mathbb{N}$ and $B \subseteq R^r$, there exists an exact sequence

$$0 \to B \xrightarrow{f} R^r \xrightarrow{g} M \to 0.$$

Consider the $r \times r$ matrices of the form

$$A = \begin{bmatrix} f(b_1) \\ \vdots \\ f(b_r) \end{bmatrix},$$

where $(b_1, \cdots, b_r)$ runs through all r-tuples of elements of B. The *Fitting ideal* $\mathrm{Fitt}_R(M)$ is defined to be the R-ideal generated by the elements $\det(A)$ for all such A. In fact, $\mathrm{Fitt}_R(M)$ depends only on M and one can show that

$$\mathrm{Fitt}_R(M) \subseteq \mathrm{Ann}_R(M).$$

Since $R := \Lambda(\phi^{-1})[P]$ is a local ring and $\mathfrak{X}^S_\infty(\phi^{-1}\omega^i)$ is a torsion module, Lemma 2 gives us a free resolution

$$0 \longrightarrow R^m \longrightarrow R^m \longrightarrow \mathfrak{X}^S_\infty(\phi^{-1}\omega^i) \longrightarrow 0$$

which ensures us that $\mathrm{Fitt}_{\Lambda(\phi^{-1})[P]}(\mathfrak{X}^S_\infty(\phi^{-1}\omega^i))$ is *principal.* To find a generator of this Fitting ideal, we will use Iwasawa's Main Conjecture which we now recall.

The Main Conjecture for abelian fields has been proven by Mazur and Wiles in [MW] . For any even primitive Dirichlet character $\tilde{\xi}$ (attached to an abelian field) of the first kind, i.e., whose conductor is not divisible by p^2, there exists a unique power series $G_{S,\tilde{\xi}}(T) \in \mathbb{Z}_p(\tilde{\xi})[[T]]$ such that

$$L_{p,S}(1-s, \tilde{\xi}) = G_{S,\tilde{\xi}}(u^s - 1),$$

where $u := \kappa(\gamma_0)$ and $L_{p,S}(s, \tilde{\xi})$ is the p-adic L-function associated to S and $\tilde{\xi}$. The p-adic L-function $L_{p,S}(s, \tilde{\xi})$ is the unique continuous function defined for $s \in \mathbb{Z}_p$ such that for all integer $k \geq 1$,

$$L_{p,S}(1-k, \tilde{\xi}) = L_S(1-k, \tilde{\xi}\omega^{-k}),$$

and $L_S(s, \chi)$ is given by the Euler product

$$L_S(s, \chi) = \prod_{(l,S)=1} (1 - \chi(l) l^{-s})^{-1},$$

where the product is taken over all rational primes l with $(l, S) = 1$.

Since we assume in this section that $F/\mathbb{Q}$ is tamely ramified in p, any character of $\mathcal{D}_{G'}$, viewed as a Dirichlet character, is of the first kind. Besides $\tilde{\xi}$ is even, so by the result of Ferrero-Washington on the μ-invariant, $W(\tilde{\xi}) := (\mathfrak{X}^S_\infty)_{\tilde{\xi}} \otimes_{\mathbb{Z}_p(\tilde{\xi})} \overline{\mathbb{Q}}_p$ is a *finite* dimensional vector space and the characteristic ideal of $(\mathfrak{X}^S_\infty)_{\tilde{\xi}}$ over $\Lambda(\tilde{\xi})$ (denoted $\mathrm{char}(\mathfrak{X}^S_\infty)_{\tilde{\xi}}$) is defined as the ideal generated by the characteristic polynomial of $(\gamma_0 - 1)$ acting on this space. By the structure theory of finitely generated torsion $\Lambda(\tilde{\xi})$-modules, there is a pseudo-isomorphism

$$(\mathfrak{X}_\infty^S)_{\tilde{\xi}} \sim \Lambda(\tilde{\xi})/(f_1) \oplus \cdots \oplus \Lambda(\tilde{\xi})/(f_r)$$

for some integer r, where each $f_i \in \Lambda(\tilde{\xi})$. The ideal $(f_1 \cdots f_r)$ is an invariant of $(\mathfrak{X}_\infty^S)_{\tilde{\xi}}$ and is equal to $\mathrm{char}(\mathfrak{X}_\infty^S)_{\tilde{\xi}}$. The Main Conjecture (now a theorem, see [MW] or [W1]) asserts that

$$\mathrm{char}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = (G_{S,\tilde{\xi}}(T)).$$

Let us denote by H_ϕ a generator of $\mathrm{Fitt}_{\Lambda(\phi^{-1})[P]}(\mathfrak{X}_\infty^S(\phi^{-1}\omega^i))$. Let us call ξ the $\mathbb{Q}_p$-character of G' induced by $\phi^{-1}\omega^i$ (i.e., ξ is the trace of the linear representation of G' obtained from $e_{\phi^{-1}\omega^i}\mathbb{Q}_p[G']$) and consider a $\tilde{\xi} \in \mathcal{D}_{G'}$ dividing ξ. Note that $\tilde{\xi}$ is even since p is odd.
Lemma 3.5 of [G] tells us that $\mathrm{Fitt}_{\Lambda(\tilde{\xi})}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = \mathrm{char}(\mathfrak{X}_\infty^S)_{\tilde{\xi}}$. Furthermore, if we take the resolution

$$0 \longrightarrow R^m \stackrel{f}{\longrightarrow} R^m \longrightarrow \mathfrak{X}_\infty^S(\phi^{-1}\omega^i) \longrightarrow 0$$

and tensor it with $\Lambda(\tilde{\xi})$, we obtain an exact sequence

$$0 \longrightarrow Q \longrightarrow \Lambda(\tilde{\xi})^m \stackrel{f\otimes_{\Lambda G}1_{\Lambda(\tilde{\xi})}}{\longrightarrow} \Lambda(\tilde{\xi})^m \longrightarrow (\mathfrak{X}_\infty^S)_{\tilde{\xi}} \longrightarrow 0,$$

and since $(\mathfrak{X}_\infty^S)_{\tilde{\xi}}$ is a $\Lambda(\tilde{\xi})$-torsion module, so is Q; but $Q \subseteq \Lambda(\tilde{\xi})$, hence Q is trivial. As a result, $\mathrm{Fitt}_{\Lambda(\tilde{\xi})}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = \tilde{\xi}(\det f)$.
Using the Main Conjecture we can then state the following equalities

$$(\tilde{\xi}(H_\phi)) = \mathrm{Fitt}_{\Lambda(\tilde{\xi})}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = \mathrm{char}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = (G_{S,\tilde{\xi}}). \qquad (1)$$

Before proceeding further, we make yet another remark.
Since $(\Lambda(\phi^{-1})[P])_\Gamma \cong \mathbb{Z}_p(\phi^{-1})[P]$, we derive from property 4 of the appendix of [MW] that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma = \left(\mathrm{Fitt}_{\Lambda(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(-i)(\phi^{-1})\right)_\Gamma.$$

This, in turn, is equal to

$$\left(\mathrm{Fitt}_{\Lambda(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(\phi^{-1}\omega^i)(-i)\right)_\Gamma.$$

As a result, $\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma$ is principal. Here the action is twisted (-i) times, consequently we must replace the action of γ_0 by that of $\kappa(\gamma_0)^{-i}\gamma_0$, i.e., T is replaced by $\kappa(\gamma_0)^i(1+T)-1$, and taking the coinvariants, i.e., letting $T=0$, we find that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma = (H_\phi(\kappa(\gamma_0)^i(1+T)-1)_{|T=0}) = (H_\phi(\kappa(\gamma_0)^i-1).$$

We denote by α_ϕ the generator $H_\phi(\kappa(\gamma_0)^i-1)$. We now formulate a proposition which gives a first evaluation of the element α_ϕ (recall that the element $\Theta_{i-1,S}(\psi)$ is defined in the statement of Theorem 1).

Proposition 1. *For any character $\tilde{\xi}$ of G' dividing ξ, the induced character of $\phi^{-1}\omega^i$ on G', we have*

$$\tilde{\xi}(\alpha_\phi) = (\tilde{\xi}\omega^{-i})^{-1}(u_\phi\Theta_{i-1,S}(\psi)),$$

where u_ϕ is a unit of $\mathbb{Z}_p(\phi)[P]$.

Proof. Equality (1) yields that

$$(\tilde{\xi}(H_\phi)) = (G_{S,\tilde{\xi}}(T)).$$

Further, we know by [W1] (see also [G2]) that there exists a power series $\tilde{G}_{\psi^{-1}\omega^i}(T)\in\Lambda(\phi^{-1}\omega^i)[P]$ such that

$$\tilde{\xi}(\tilde{G}_{\psi^{-1}\omega^i}(T)) = G_{S,\tilde{\xi}}(T).$$

Thus, for any character $\tilde{\xi}$ dividing the induced character of $\phi^{-1}\omega^i$, we have

$$\mathrm{Fitt}_{\Lambda(\tilde{\xi})}(\mathfrak{X}_\infty^S)_{\tilde{\xi}} = (\tilde{\xi}(H_\phi(T))) = (\tilde{\xi}(\tilde{G}_{\psi^{-1}\omega^i}(T))),$$

and applying Lemma 3.7 of [G] to the $\Lambda(\phi^{-1}\omega^i)[P]$-module $\mathfrak{X}_\infty^S(\phi^{-1}\omega^i)$, we obtain that

$$(H_\phi(T)) = (\tilde{G}_{\psi^{-1}\omega^i}(T))$$

as ideals of $\Lambda(\phi^{-1}\omega^i)[P]$. Hence, there is a unit $U_\phi(T)\in(\Lambda(\phi^{-1}\omega^i)[P])^\times$ such that

$$H_\phi(T) = U_\phi(T)\tilde{G}_{\psi^{-1}\omega^i}(T).$$

Recall that our aim is to evaluate $(\alpha_\phi) = \mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\,\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma$. We successively find

$$\begin{aligned}\tilde{\xi}(\alpha_\phi) &= \tilde{\xi}(H_\phi(\kappa(\gamma_0)^i-1))\\ &= \tilde{\xi}(U_\phi(\kappa(\gamma_0)^i-1))G_{S,\tilde{\xi}}(\kappa(\gamma_0)^i-1)\\ &= \tilde{\xi}(U_\phi(\kappa(\gamma_0)^i-1))L_{p,S}(1-i,\tilde{\xi}).\end{aligned}$$

Next, since $L_S(1-i,\tilde{\xi}) = \tilde{\xi}^{-1}(\Theta_{i-1,S})$ (see for instance [CS], Lemma 1.7), and denoting $u_\phi := U_\phi(\kappa(\gamma_0)^i - 1)^{-1}$, it follows that

$$\begin{aligned}\tilde{\xi}(\alpha_\phi) &= \tilde{\xi}^{-1}\omega^i(u_\phi\Theta_{i-1,S}) \\ &= (\tilde{\xi}\omega^{-i})^{-1}(u_\phi\Theta_{i-1,S}(\psi)),\end{aligned}$$

because $(\tilde{\xi}\omega^{-i})^{-1}$ extends ψ. □

We recall the statement of our main theorem

Theorem 1. *Let ϕ be a $\mathbb{Q}_p$-irreducible character of Δ, ψ a character of degree one of Δ dividing ϕ and i be an integer greater or equal to 2 such that $\psi(-1) = (-1)^i$. We also require that $\phi^{-1}\omega^i$ is a non-trivial $\mathbb{Q}_p$-irreducible character of Δ', the non-p-part of $G' := \mathrm{Gal}(F(\mu_p)/\mathbb{Q})$. Then,*

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, K_{2i-2}^{\acute{e}t}(\mathcal{O}_F^S)(\phi) = (\Theta_{i-1,S}(\psi)),$$

where $\Theta_{i,S}(\psi) := \sum_{\substack{1\le a<f \\ (a,f)=1}} \zeta_{f,S}(-i,a)\psi(\delta_a)^{-1}\rho_a^{-1}$ and $\sigma_a = \delta_a\rho_a$ in the decomposition $G = \Delta \times P$.

Before giving the proof of Theorem 1, we state a lemma we shall use in this proof.

For a $\mathbb{Z}_p[G]$-module M, we have denoted by $M^\sharp$ the Pontrjagin dual of M, i.e., the group $\mathrm{Hom}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ where the action of $g \in G$ is made via $(g.\theta)(m) := \theta(g^{-1}.m)$ for $m \in M$ and $\theta \in \mathrm{Hom}(M,\mathbb{Q}_p/\mathbb{Z}_p)$. We shall also need in the sequel the group $\mathrm{Hom}(M,\mathbb{Q}_p/\mathbb{Z}_p)$ where the action of G is $(g.\theta)(m) := \theta(g.m)$. We call it $(M^\sharp)^*$.

We are thankful to T. Nguyen Quang Do for providing us with the proof of

Lemma 3. *Let N be a finite $\mathcal{O}[P]$-module, where $\mathcal{O}$ is a finite ring extension of $\mathbb{Z}_p$, with projective dimension lower or equal to 1 over $R := \mathcal{O}[P]$. Then,*

$$\mathrm{Fitt}_R\, N = \mathrm{Fitt}_R (N^\sharp)^*.$$

Proof. Apply the functor $\mathrm{Hom}_R(.,R)$ to a resolution $0 \to R^m \xrightarrow{f} R^m \to N \to 0$. We obtain

$$0 \to \mathrm{Hom}_R(N,R) \to \mathrm{Hom}_R(R^m,R) \xrightarrow{{}^t f} \mathrm{Hom}_R(R^m,R) \to \mathrm{Ext}^1_R(N,R) \to 0.$$

We know that $\mathrm{Hom}_R(N, R) = 0$ because N is finite, and $\mathrm{Hom}_R(R^m, R) \cong R^m$ because R^m is free over R. Besides, still because of the finiteness of N, $\mathrm{Ext}^1_R(N, R) \cong (N^\sharp)^*$ by Corollary 2.5 of [J]; whence the lemma. $\square$

Proof of Theorem 1. In order to apply Lemma 3 to $N := \mathfrak{X}^S_\infty(\phi^{-1}\omega^i)(-i)_\Gamma$, we need to check that $\mathrm{pd}(\mathfrak{X}^S_\infty(\phi^{-1}\omega^i)(-i)_\Gamma) \leq 1$ over $R := \mathbb{Z}_p(\phi^{-1}\omega^i)[P]$. Taking the coinvariants in a projective resolution of $\mathfrak{X}^S_\infty(\phi^{-1}\omega^i)(-i)$ yields an exact sequence

$$0 \to Q \to R^m \to R^m \to \mathfrak{X}^S_\infty(\phi^{-1}\omega^i)(-i)_\Gamma \to 0.$$

Since $\mathfrak{X}^S_\infty(\phi^{-1}\omega^i)(-i)_\Gamma$ is finite, the $\mathbb{Z}_p$-rank of Q is 0, hence $Q = 0$.
Thus Lemma 4 gives us

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\, N = \mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}(N^\sharp)^*.$$

We want to compute the Fitting ideal of $K^{\text{ét}}_{2i-2}(\mathcal{O}^S_F)(\phi) \cong H^2_{\text{ét}}(\mathcal{O}^S_F; \mathbb{Z}_p(i))(\phi) \cong N^\sharp$ by Lemma 1. Generally speaking, for an abelian group A, let A^* denote the same group as A but where G acts via g^{-1}. If A is a $\mathbb{Z}_p(\phi^{-1})[P]$-module and if $\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\, A = (\alpha)$, then A^* is a $\mathbb{Z}_p(\phi)[P]$-module and $\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, A^* = (\alpha^*)$, where α^* is an element of $\mathbb{Z}_p(\phi)[P]$ such that $\chi(\alpha^*) = \chi^{-1}(\alpha)$ for any character χ of $\mathcal{D}_G$ dividing the induced character of ϕ.
Consequently, keeping in mind that $(\alpha_\phi) = \mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\, \mathfrak{X}^S_\infty(-i)(\phi^{-1})_\Gamma$, it follows that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, N^\sharp = (\alpha^*_\phi).$$

Such characters χ of $\mathcal{D}_G$ dividing the induced character of ϕ can be written under the form $(\tilde{\xi}\omega^{-i})^{-1}$, where $\tilde{\xi}$ is still a character dividing the induced character of $\xi = \phi^{-1}\omega^i$. Now, recall that

$$\tilde{\xi}(\alpha_\phi) = (\tilde{\xi}\omega^{-i})^{-1}(u_\phi \Theta_{i-1,S}(\psi)),$$

by Proposition 1; but $\tilde{\xi}(\alpha_\phi) = (\tilde{\xi}\omega^{-i})(\alpha_\phi)$ since α_ϕ is an element of $\mathbb{Z}_p(\phi^{-1})[P]$. Hence, $\chi^{-1}(\alpha_\phi) = \chi(u_\phi \Theta_{i-1,S}(\psi))$ for any character χ of $\mathcal{D}_G$ dividing the induced character of ϕ. Finally, the element α^*_ϕ we want to compute (i.e., a generator of $\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, H^2_{\text{ét}}(\mathcal{O}^S_F; \mathbb{Z}_p(i))(\phi))$) verifies

$$\chi(\alpha^*_\phi) = \chi(u_\phi \Theta_{i-1,S}(\psi)),$$

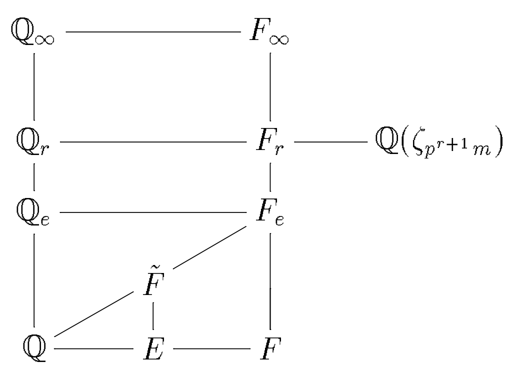
for any character χ of $\mathcal{D}_G$ dividing the induced character of ϕ.
Using the fact that $\mathbb{Q}_p(\phi)[P]$ decomposes into $\oplus_{\chi|\phi}\mathbb{Q}_p(\chi)$, we conclude that $\alpha^*_\phi = u_\phi\Theta_{i-1,S}(\psi))$, so that

$$(\alpha^*_\phi) = (\Theta_{i-1,S}(\psi)).$$

□

3 The wildly ramified case

In this section, we do not suppose that p is tamely ramified in $F/\mathbb{Q}$, but we suppose that F is linearly disjoint from $\mathbb{Q}_\infty$ in order that the whole group G acts on $\mathfrak{X}^S_\infty$, rather than only a subgroup.
To be able to use the Main Conjecture again, which is only valid for Dirichlet characters of the first kind, we shall resort to a construction evoked in [S] and build a field $\tilde{F}$ tamely ramified over p and such that $F.\mathbb{Q}_\infty = \tilde{F}.\mathbb{Q}_\infty$, and thus retrieve the result of the previous section.
Let us define $\tilde{F}$ as the wild inertia field of $F_\infty/\mathbb{Q}$. We prove in the following that this field is the unique field meeting the two previous requirements.
We write the conductor of F under the form $f = p^{r+1}m$, where p and m are coprime, and we call E the wild inertia field of $F/\mathbb{Q}$. We have $[F : E] = p^e$, with $e \leq r$. Consider the diagram :



Since F_r/F is a cyclic extension of prime power order, it is unramified at the start and then totally ramified. On the other hand, the wild ramification index of p in this extension is p^{r-e}; this implies that F_e/F is unramified and F_∞/F_e is totally ramified. Consequently, $\tilde{F} \subseteq F_e$ and considering degrees of the extensions, we get $\tilde{F}.\mathbb{Q}_e = F_e$. Hence, $(\tilde{F})_\infty = F_\infty$. Notice that

$\mathrm{Gal}(F/\mathbb{Q}) \cong \mathrm{Gal}(\tilde{F}/\mathbb{Q}) =: \tilde{G}$. Let $\tilde{P}$ denote the p-Sylow subgroup of $\tilde{G}$. Since $E \subseteq \tilde{F}$, we find that $G/P = \tilde{G}/\tilde{P}$.
There remains to show the uniqueness of $\tilde{F}$. Suppose K is a field tamely ramified over p and verifying $F_\infty = K_\infty$. Clearly, $K \subset \tilde{F}$ for $K/\mathbb{Q}$ is tamely ramified. Besides,

$$[K : \mathbb{Q}] = [K_\infty : \mathbb{Q}_\infty] = [F_\infty : \mathbb{Q}_\infty] = [\tilde{F} : \mathbb{Q}],$$

whence $K = \tilde{F}$.
We set up a little more notations : $\tilde{\Gamma}$ denotes $\mathrm{Gal}(F_\infty/\tilde{F})$, R the ring $\mathbb{Z}_p(\phi^{-1})[[\Gamma]][P]$, $\tilde{R}$ the ring $\mathbb{Z}_p(\phi^{-1})[[\tilde{\Gamma}]][\tilde{P}]$, and $\tilde{f}$ is the conductor of $\tilde{F}$. The restrictions of $\mathrm{Gal}(F_\infty/\mathbb{Q}_\infty)$ to F and $\tilde{F}$ give a canonical isomorphism between P and $\tilde{P}$. The action of P or $\tilde{P}$ on $\mathfrak{X}_\infty^S$ is made via the p-Sylow subgroup of $\mathrm{Gal}(F_\infty/\mathbb{Q}_\infty)$ and two elements corresponding by this isomorphism give the same action on $\mathfrak{X}_\infty^S$. Besides, both Γ and $\tilde{\Gamma}$ are isomorphic by restriction to $\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ and we denote by τ the isomorphism between $\tilde{R}$ and R obtained from these restriction maps.

Like in the first section, let ϕ be a $\mathbb{Q}_p$-irreducible character of Δ and i be an integer satisfying the assumptions of Theorem 1. Since $\tilde{F}/\mathbb{Q}$ is tamely ramified, we know with Theorem 1 that

$$\tilde{\alpha}_\phi := \mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[\tilde{P}]}(\mathfrak{X}_\infty^S(-i)(\phi^{-1})_{\tilde{\Gamma}})^\sharp = (\tilde{\Theta}_{i-1,S}(\psi)),$$

where

$$\tilde{\Theta}_{i-1,S}(\psi) = \sum_{\substack{1\le a<\tilde{f}\\(a,\tilde{f})=1}} \zeta_{\tilde{f},S}(-n,a)\psi(\delta_a)^{-1}\tilde{\rho}_a^{-1}.$$

There exists an exact sequence of $\tilde{R}$-modules

$$0 \to \tilde{R}^n \xrightarrow{f} \tilde{R}^n \to \mathfrak{X}_\infty^S(-i)(\phi^{-1}) \to 0.$$

The isomorphism τ induces an exact sequence of R-modules

$$0 \to R^n \xrightarrow{\tau f \tau^{-1}} R^n \to \mathfrak{X}_\infty^S(-i)(\phi^{-1}) \to 0,$$

so that

$$\mathrm{Fitt}_R\, \mathfrak{X}_\infty^S(-i)(\phi^{-1}) = \tau(\mathrm{Fitt}_{\tilde{R}}\, \mathfrak{X}_\infty^S(-i)(\phi^{-1})).$$

We consider a topological generator $\tilde{\gamma}$ of $\tilde{\Gamma}$. Then $\gamma := \tau(\tilde{\gamma})$ is a topological generator of Γ, and this is precisely the one we choose to map to $1+T$. Consequently, using property 4 of the appendix of [MW] again, we find that

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}\, \mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma = \tau(\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[\tilde{P}]}\, \mathfrak{X}_\infty^S(-i)(\phi^{-1})_{\tilde{\Gamma}}),$$

and we finally obtain that

$$\begin{aligned}\mathrm{Fitt}_{\mathbb{Z}_p(\phi^{-1})[P]}(\mathfrak{X}_\infty^S(-i)(\phi^{-1})_\Gamma)^\sharp &= \tau(\tilde{\alpha}_\phi)\\ &= \sum_{\substack{1\le a<\tilde{f}\\(a,\tilde{f})=1}} \zeta_{\tilde{f},S}(-n,a)\psi(\delta_a)^{-1}\tau(\tilde{\rho}_a^{-1}).\end{aligned}$$

We thus have proved the following

Theorem 2. *Let F be an abelian number field and let $\tilde{F}$ be the unique field such that $\tilde{F}/\mathbb{Q}$ is tamely ramified over p and such that $\tilde{F}.\mathbb{Q}_\infty = F.\mathbb{Q}_\infty$. Let $\tilde{f}$ denote the conductor of $\tilde{F}$ and let τ be the canonical isomorphism described above that maps $\tilde{\Gamma}\times\tilde{P}$ to $\Gamma\times P$, where $\tilde{\Gamma} = \mathrm{Gal}((\tilde{F})_\infty/\tilde{F})$ and $\tilde{P}$ is the p-part of $\mathrm{Gal}(\tilde{F}/\mathbb{Q})$. Let ϕ be a character of Ψ_Δ verifying the assumption of Theorem 1. Then,*

$$\mathrm{Fitt}_{\mathbb{Z}_p(\phi)[P]}\, K_{2i-2}^{\acute{e}t}(\mathcal{O}_F^S)(\phi) = (\tilde{\Theta}_{i-1,S}(\psi)),$$

where $\tilde{\Theta}_{n,S}(\psi) = \sum_{\substack{1\le a<\tilde{f}\\(a,\tilde{f})=1}} \zeta_{\tilde{f},S}(-n,a)\psi(\delta_a)^{-1}\tau(\tilde{\rho}_a^{-1})$ and $\tilde{\sigma_a} = \delta_a\tilde{\rho_a}$ in the decomposition $\tilde{G} = \Delta\times\tilde{P}$.

Remark. We have recently learnt in a preprint by M. Kurihara that he has obtained similar results by different methods.

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