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Abstract

In this paper we construct new extremal and optimal unimodular lattices in dimensions 36, 38, 42, 45, 52, 54, 60 and 68. We construct them in two ways: first in the case of dimensions congruent to 4 modulo 8 by construction B_3 followed by density doubling, generalizing the constructions of [2, p.148] and [9] and second by applying the well known construction A to self-dual codes over $GF(5)$ and to codes over the ring $Z/25Z$. In particular the lattice in dimension 60, P_{60q} , generalizes the construction of the lattice P_{48q} . We also give the complete weight enumerator of the extended ternary quadratic residue code of length 60.

1 Introduction

There are many links between codes and lattices (cf [2]), and especially between self-dual codes and unimodular lattices. In particular Type II codes can be related to Type II lattices and the Leech lattice can be constructed from the Golay code. The well known Construction A is a simple way to associate a lattice to a code, in particular a unimodular lattice is associated to a self-dual code. But applying construction A directly to the fields $GF(2)$ and $GF(3)$ is not interesting if one is to obtain unimodular lattices with norm higher than 2 or 3. A way to obtain a lattice with a higher norm is to construct lattices by construction A and then consider a particular lattice associated to the first one through a neighboring construction. The method was first used over $GF(2)$ to construct unimodular lattices of norm 4, in particular the Leech lattice is obtained from the Golay code and other lattices of norm 4 are constructed in dimension 32 and 40 (cf [2] for references).

The same method was also applied by Leech and Sloane ([2]) over $GF(3)$ in dimensions 24 and 48 to obtain, in dimension 24, a new construction of the Leech lattice and in dimension

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48, two extremal Type II lattices of norm 6 namely P_{48q} and P_{48p} , constructed respectively from two self-dual ternary codes: the extended ternary quadratic residue code of length 48 (XQ_{47}) and the Pless symmetry code PS_{48} . Later this construction was generalized by Ozeki in [9] to construct for dimensions congruent to 0 mod 8, Type II lattices of norm 4 or 6. Ozeki constructed in particular Type II lattices of dimensions 56 and 64 associated to extremal ternary self-dual of lengths 56 and 64. His construction was simplified by Conway and Sloane in [2].

In this paper we generalize the previous construction over $GF(3)$ to the case of dimensions congruent to 4 mod 8 and we construct new lattices of dimensions 52, 60 and 68 associated to ternary self-dual codes. In particular the lattice P_{60q} is obtained from the complete weight enumerator (*cwe*) of XQ_{59} which was previously unknown. We also propose methods to construct self-dual codes over $GF(5)$ and over the ring $Z/25Z$ that we use with Construction A to construct new extremal unimodular lattices of dimensions 36, 38, 42, 45 and 54. The shadow theory of [3] is used to compute, when it is possible, the theta series of these lattices.

Note that most of these lattices were previously announced in the on-line tables of [8]. In particular the results concerning ternary construction for dimensions 52, 60 and 68 were previously announced and explained in [8] in June 2001 and at the Workshop on Coding and Cryptography at Singapore in Sept 2001, at the same time or earlier than the results of [6].

The paper is organized as follows: Section 2 collects basic definitions and notation, Section 3 considers the construction from self-dual ternary codes, Section 4 deals with Construction A and Appendix gives explicit generator matrices of some of the codes described in the paper.

2 Notation and Definitions

In this section we recall some basic notions on codes, lattices and the basic construction of lattice from codes. We refer to [12] and [2] for a more general context regarding self-dual codes and lattices. We deal with codes over rings and codes over fields. A code of length n and dimension k over a field K is a subspace of K^n of dimension k . A code C over a ring R is a R -submodule of R^n . In the following we will consider indifferently that the code C is over a ring (or over a field) of the form Z/mZ for m an integer. A codeword is an element of a code. Let us define the (Euclidean) scalar product of two codewords $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ by $x \cdot y = \sum_{i=1}^n x_i \cdot y_i$. The dual of C is defined as $C^\perp = \{y \in (Z/mZ)^n \mid x \cdot y = 0 \text{ for all } x \in C\}$. A code is said self-dual if and only if $C = C^\perp$. The Hamming weight of a codeword corresponds to the number of non zero components, the Euclidean weight $w_E(a)$ of an element a of Z/mZ ($=\{0, \dots, m-1\}$) is the minimum of $\{a^2, (m-a)^2\}$, by extension the Euclidean weight of a codeword x is $\sum_{i=1}^n w_E(x_i)$. The Euclidean minimum weight $d_E(C)$

of a code C is the minimum Euclidean weight among all the non zero codewords of C .

The dual of a lattice Λ is denoted by Λ^* . A lattice Λ is integral if $\Lambda \subset \Lambda^*$ and unimodular if $\Lambda = \Lambda^*$. A unimodular lattice whose all vectors have even norm is said to be even or Type II. The minimum norm of a unimodular lattice of dimension n is bounded by [11]:

$$\mu \leq 2\left[\frac{n}{24}\right] + 2,$$

except for $n = 23$ where the norm is bounded by 3. A lattice of dimension n with norm $2\left[\frac{n}{24}\right] + 2$ (and $n \neq 23$) is said to be extremal. A unimodular lattice with the highest possible norm is said to be optimal.

The well known Construction A is used to associate a unimodular lattice to a self-dual code over a ring Z/mZ :

Theorem 2.1 *Let C be a self-dual code over the ring Z/mZ with minimum Euclidean weight d_E then:*

$$A_m(C) = \frac{1}{\sqrt{m}} \{(x_1, \dots, x_n) \in Z^n \mid (x_1 \bmod m, \dots, x_n \bmod m) \in C\},$$

is a unimodular lattice with norm $\min\{\frac{d_E}{m}, m\}$.

The theta series of a lattice Λ is

$$(1) \quad \theta_\Lambda(\tau) := \sum_{x \in \Lambda} q^{(x \cdot x)}$$

where $\tau \in \mathfrak{h}$ the upper half complex plane and $q := e^{\pi i \tau}$, satisfies an invariance property under the transformation $\tau \rightarrow -1/\tau$.

Let Λ be a unimodular lattice. The shadow S of Λ is $S := (\Lambda_0)^* \setminus \Lambda$, where Λ_0 denotes the even sublattice of Λ . If Λ is an odd lattice, its theta series has the following expression

$$(2) \quad \theta_\Lambda(\tau) = \sum_{j=0}^{\lfloor n/8 \rfloor} a_j \Delta_8(q)^j \theta_3(q)^{n-8j}$$

and the theta series of the shadow S is

$$(3) \quad \theta_S(\tau) = \sum_{j=0}^{\lfloor n/8 \rfloor} \frac{(-1)^j}{16^j} a_j \theta_4(q^2)^{8j} \theta_2(q)^{n-8j}.$$

where $q := e^{\pi i \tau}$, $\Delta_8(q) = q \prod_{m=1}^{\infty} (1 - q^{2m-1})^8 (1 - q^{4m})^8$, and $\theta_2, \theta_3, \theta_4$ are the usual Jacobi theta series (see [2, Chap. 4, § 4]).

In all the following the notation (x^a, y^b) stands for a vector whose first a coordinates are x and the b last coordinates are y .

3 Unimodular lattices and self-dual codes over $GF(3)$

3.1 Constructions

In this section we generalize the construction of [9] to lengths of n congruent to 4 modulo 8. Note that in the latter paper the author remarks that his method can be generalized to lengths congruent to 4 modulo 8 without any change. It appears indeed that this affirmation is partly false since some conditions concerning the complete weight enumerators of the codes and the minimum weight have to be precised.

We recall that for C a ternary code, construction $B_3(C)$ (cf [2]) is obtained by considering only the vectors: $x = \frac{1}{\sqrt{3}}(x_1, \dots, x_n)$ of $A_3(C)$ such that $6 \mid \sum_{i=1}^n x_i$ (6 divides $\sum_{i=1}^n x_i$). Now, for C a self-dual ternary code of length n and v a vector of R^n , such that $2v$ is in $B_3(C)$, we denote by $N_3(C, v)$ the lattice obtained by taking the sublattice $B_3(C)$ of index 2 of $A_3(C)$ and by doubling its density by considering the lattice $B_3(C) \cup \{v + B_3(C)\}$. The lattice $N_3(C, v)$ is a neighbour of $A_3(C)$. This construction is described in [2] for the case of lengths a multiple of 12.

We now obtain the following construction theorem, where the maximal codewords are the words with maximal hamming weight:

Theorem 3.1 *Let C be a self-dual ternary code of length n such that C contains: a codeword of the form (1^n) if $n = 0 \pmod{3}$, a codeword of the form $(1^{n-1}, 0)$ if $n = 2 \pmod{3}$, or a codeword of the form $(1^{n-2}, 0^2)$ if $n = 1 \pmod{3}$. The following result holds:*

- 1) *if C is $[28, 14, 9]$ code then the lattice $N_3(C, v)$ with $v = ((\frac{1}{2})^{27}, \frac{3}{2})$ is a unimodular lattice with minimum norm 3,*
- 2) *if C is $[36, 18, d]$ code with $d \geq 9$ and such that the maximal codewords of C have only an even number of '1' then the lattice $N_3(C, v)$ for $v = ((\frac{1}{2})^{35}, -\frac{5}{2})$ is a unimodular lattice with minimum norm 4,*
- 3) *if C is $[44, 22, d]$ code with $d \geq 9$ then the lattice $N_3(C, v)$ for $v = ((\frac{1}{2})^{42}, \frac{3}{2}, \frac{3}{2})$ is a unimodular lattice with minimum norm 4,*
- 4) *if C is $[52, 26, 15]$ code then the lattice $N_3(C, v)$ for $v = ((\frac{1}{2})^{51}, \frac{3}{2})$ is a unimodular lattice with minimum norm 5,*
- 5) *if C is $[60, 30, d]$ code with $d \geq 15$ and such that the maximal codewords of C have only an even number of '1' then the lattice $N_3(C, v)$ for $v = ((\frac{1}{2})^{59}, -\frac{5}{2})$ is a unimodular lattice with minimum norm 6,*
- 6) *if C is $[68, 34, d]$ code with $d \geq 15$ then the lattice $N_3(C, v)$ for $v = ((\frac{1}{2})^{66}, \frac{3}{2}, \frac{3}{2})$ is a unimodular lattice with minimum norm 6.*

Proof.

The proofs are true modulo 24 for $28 \leq n \leq 68$, therefore we only prove the cases $n = 44, 52$ and 60 .

For $n = 44$ let us consider the vectors $w = \frac{1}{\sqrt{3}}(1^{42}, 3^2)$, $v = \frac{w}{2}$. For $x = \frac{1}{\sqrt{3}}(x_1, \dots, x_{44})$ a vector of $A_3(C)$ we want to prove that $L := N_3(C, v)$ is unimodular with minimum norm 4. First we prove that L is integral. Since $x \in B_3(C)$, by divisibility one gets:

$$6 \mid \sum_{i=1}^{44} x_i \quad (4)$$

and since x and w are in $B_3(C)$, which is integral, then $x.w \in Z$ and

$$3 \mid (3(x_{43} + x_{44}) + \sum_{i=1}^{42} x_i) \quad (5).$$

From (5) one deduces that $3 \mid \sum_{i=1}^{42} x_i$ and therefore by (4), $3 \mid (x_{43} + x_{44})$, which implies that $v.x \in Z$, and therefore since $2v \in B_3(C)$ this shows that $N_3(C, v)$ is unimodular. Moreover two vectors of $B_3(C)$ are distant from at least $\frac{12}{3}$ since C has minimum distance 9 or more, and since L is unimodular the distance between two vectors is an integer. Now since the coordinates of $\sqrt{3}B_3(C)$ are integers and since the coordinates of $\sqrt{3}w$ are half integers, d is greater than $\frac{44}{4 \times 3}$ and hence is 4.

For $n = 52$, we follow the same procedure with $w = \frac{1}{\sqrt{3}}(1^{51}, 3)$ and $v = \frac{w}{2}$. For x any vector of $B_3(C)$ we show that $v.x \in Z$. The minimum norm is an integer greater than $\frac{52}{4 \times 3}$ and therefore is 5.

For $n = 60$ one takes $v = \frac{1}{\sqrt{3}}(1^{59}, -\frac{5}{2})$, and the proof follows [2, 149] for dimension 48, the minimum norm is guaranteed by the even number of '1' in the maximal weight codewords. \square

Note that two [60, 30, 18] self-dual ternary codes are known, namely PS_{60} and XQ_{59} . The complete weight enumerator of PS_{60} has been computed in [7] but does not satisfy the condition on the even number of '1' in its extremal words. In the next subsection we compute the complete weight enumerator of XQ_{59} which satisfies it.

3.2 Complete weight enumerator of self-dual codes over $GF(3)$

We saw in the preceding section that when the dimension of the code is a multiple of 12 (and therefore may contain the all-ones vector $\mathbf{1}$), it was of interest to know the complete weight enumerator of the code. In [7] the complete weight enumerators of the codes XQ_{23} , PS_{24} , PS_{36} , XQ_{47} , PS_{48} and PS_{60} are computed. These results show in particular that PS_{36} and PS_{60} contain codewords of the form $0^a 1^b$ with a and b odd. We prove next the unicity of the cwe of any [48, 24, 15] self-dual ternary code with the all-ones vector inside and we give the cwe of XQ_{59} which was not considered in [7].

We first recall a theorem in [7] which describes the complete weight enumerator of self-dual codes over $GF(3)$ which contain the all-ones vector $\mathbf{1}$:

Theorem 3.2 *Let C be a ternary self-dual code which contains the all-ones vector $\mathbf{1}$. Let $W_C(x, y, z)$ be the complete weight enumerator of C . Then:*

$$W_C(x, y, z) \in \mathcal{C}[\alpha_{12}, \beta_6^2, \delta_{36}] \oplus \beta_6 \gamma_{18} \mathcal{C}[\alpha_{12}, \beta_6^2, \delta_{36}],$$

where $\alpha_{12} = a(a^3 + 8p^3)$, $\beta_6^2 = a^2 - 12b$, $\gamma_{18} = a^6 - 20a^3p^3 - 8p^6$, $\delta_{36} = p^3(a^3 - p^3)^3$, and $a = x^3 + y^3 + z^3$, $p = 3xyz$, $b = x^3y^3 + x^3z^3 + y^3z^3$.

In [9] it is claimed that for dimension 48, all the $[48, 24, 15]$ self-dual ternary codes lead to Type II lattice of dimension 48 and norm 6. Although this claims is incomplete since one also needs the condition on the maximal codewords, it turns out that it is true *a posteriori* by the following proposition which fixes the complete weight enumerator of any $[48, 24, 15]$ self-dual ternary code:

Proposition 3.3 *All extremal $[48, 24, 15]$ ternary self-dual codes, with the all-ones vector inside, have the same complete weight enumerator.*

Proof. The number of independent polynomials from theorem 1 is 10. Now the condition of minimum weight 15 leads to 9 conditions. It is known from the weight enumerator that the number of codewords of maximal weight (*ie* 48) is 94. Therefore one may check for all the possibilities between 0 and 94 for the number N of maximal codewords in the code with 24 '1' and 24 ' - 1'. The computation shows that the only valid possibility for a weight enumerator is obtained for $N = 94$ the others leading to weight enumerators with fractional or negative terms. This gives a 10th condition and fix the complete enumerator of any extremal self-dual ternary $[48, 24, 15]$ code. \square

Now if one considers the problem of finding the cwe of XQ_{59} (with the vector $\mathbf{1}$ inside) one has to find sufficiently many coefficients of the cwe to be able to solve the system of equations (given by the preceding theorem) which generate the cwe of the code. The minimum weight 18 gives several coefficients but it is not enough and some more have to be found. We follow [7] in searching for the codewords of weight 60.

It is known from the weight enumerator of extremal ternary self-dual $[60, 30, 18]$ codes that the number of codewords of weight 60 is 41184, it is also known that the automorphism group of XQ_{59} is $PSL_2(59)$. Using a simple program in Magma it is possible to find some codewords of weight 60 and then apply $PSL_2(59)$ to these codewords to construct their orbits under this group. It appears that there are exactly 2 orbits: one of order 120 generated by $\mathbf{1}$ and another of order 41064, moreover it turns out that all the codewords we find have an even number of '1' and ' - 1'. Computing the exact number of codewords for each possible distribution gives sufficiently many coefficients to compute the cwe of XQ_{59} of Table 1.

$A_{ijk}/59$	i	j	k	N. of terms
$\frac{1}{59}$	60	0	0	3
0	42	18	0	6
580	42	15	3	6
14210	42	12	6	6
36540	42	9	9	3
0	39	21	0	6
8120	39	18	3	6
316680	39	15	6	6
1721440	39	12	9	6
145	36	24	0	6
52780	36	21	3	6
3601220	36	18	6	6
34972260	36	15	9	6
72310630	36	12	12	3
0	33	27	0	6
190820	33	24	3	6
19305300	33	21	6	6
305502820	33	18	9	6
1133255620	33	15	12	6
408	30	30	0	3
353220	30	27	3	6
52030930	30	24	6	6
1253638100	30	21	9	6
7579015150	30	18	12	6
13591942720	30	15	15	3
72243640	27	27	6	3
2514544760	27	24	9	6
23134484940	27	21	12	6
67623713220	27	18	15	6
33433941660	24	24	12	3
148722919080	24	21	15	6
242406463530	24	18	18	3
368890513080	21	21	18	3

Table 1: Complete weight enumerator of the $[60, 30, 18]$ extended quadratic residue code XQ_{59}

3.3 Applications

We now apply Theorem 3.1 to construct new lattices.

- dimension 28:

many $[28, 14, 9]$ self-dual codes over $GF(3)$ ([12]) are known and they all give rise to an optimal unimodular lattice of norm 3.

- dimension 36:

as far as we know no $[36, 18, d]$ code with $d \geq 9$ and even number of '1' in its maximal words is known, such a code would lead to an extremal unimodular lattice of norm 4.

- dimension 44:

in this case also many $[44, 22, 12]$ self-dual codes are known ([12]) which all lead to extremal unimodular lattices of norm 4.

- dimension 52:

recently the first extremal $[52, 26, 15]$ self-dual code over $GF(3)$ was found by Gaborit and Otmani in [5]. This code leads to an optimal unimodular lattice of norm 5. Note that this lattice is optimal since the case of minimum norm 6 is ruled out by shadow arguments. The theta series of the lattice and its shadow are:

$$\theta_{\Lambda} = 1 + 130624q^5 + \dots \quad \theta_S = 104q^3 + \dots$$

- dimension 60:

as previously said there are two known self-dual $[60, 30, 18]$ codes over $GF(3)$. We prove in the preceding section that the complete weight enumerator of XQ_{59} has the good property on its words of maximal weight and therefore leads to an extremal unimodular lattice of norm 6: P_{60q} . And since the automorphismgroup of XQ_{59} is $PSL_2(59)$, the automorphism group of P_{60q} contains $SL_2(59)$. The theta series of the lattice and its shadow are as follows:

$$\theta_{\Lambda} = 1 + 3908160q^6 + \dots \quad \theta_S = 120q^3 + \dots$$

rem : the condition of minimum distance 18 is not a necessity and could be reduced to 15 but without additional information on the number of codewords of given weight or without the automorphism group of the code, it is difficult to compute the cwe to check for 0^a1^b vectors with a and b odd.

- dimension 68: it is simple to construct a $[68, 34, 15]$ self-dual code using the method described in [5] or by subtraction of a $[4, 2, 3]$ self-dual code from the $[72, 36, 18]$ extended quadratic residue code of length 72, XQ_{72} . This code leads to an extremal lattice of norm 6.

4 Some new unimodular lattices obtained by Construction A

In this section we present two methods to construct self-dual codes over rings which lead to unimodular lattices using Construction A.

4.1 Construction with quadratic double circulant codes

In all the following q is a power of an odd prime number. We now introduce some definitions. Let R be a commutative ring containing 1 and let r, s and t be elements of R . Following the notation of [10], we let a be a one-one mapping of the integers $0, 1, \dots, q-1 \cup \infty$ over the elements of $GF(q) \cup \infty$ with $a(0) = 0, a(1) = 1$ and $a(\infty) = \infty$. Then the inverse mapping a^{-1} is a mapping from $GF(q) \cup \infty$ onto the integers $0, 1, \dots, q-1 \cup \infty$. In the case where q is a prime, we let a be the identity. We now set the matrix $\mathcal{Q}_q(r, s, t)$ to be the $q \times q$ matrix on R labeled on its rows and its columns by elements of $GF(q)$: $a_0 = 0, a_1 = 1, a_2 = a(2), \dots, a_{q-1} = a(q-1)$, and with components q_{ij} . The entries q_{ij} are defined in terms of quadratic residues and by the function χ (which is not necessarily a character) defined on $GF(q)$ by $\chi(a_0) = r, \chi(a_i) = s$ if a_i is a quadratic residue in $GF(q)$ and otherwise $\chi(a_i) = t$. Then we let q_{ij} equal $\chi(a_j - a_i)$. In the important case where q is a prime we let $a_0 = 0, a_1 = 1, \dots, a_{q-1} = q-1$, which leads to a circulant $\mathcal{Q}_q(r, s, t)$ matrix. We define by I the identity $q \times q$ matrices.

We define a quadratic double circulant code to be a $[2q, q]$ or $[2q+2, q+1]$ code over R with one of the two possibly double circulant form for its generator matrix:

$$\mathcal{P}(r, s, t) = \left(\begin{array}{c|c} I & \mathcal{Q}(r, s, t) \end{array} \right) \quad (6)$$

$$\mathcal{B}(r, s, t) = \left(\begin{array}{c|c|c|c} 1 & 0 \cdots 0 & \alpha & \beta \cdots \beta \\ \hline 0 & & \gamma & \\ 0 & & \gamma & \\ \vdots & I & \vdots & \mathcal{Q}(r, s, t) \\ \vdots & & \vdots & \\ 0 & & \gamma & \end{array} \right) \quad (7)$$

These two forms are also called respectively *pure* and *bordered* quadratic double circulant forms. Unless otherwise specified we always consider $\beta = \gamma = 1$ and $\alpha = r$.

This method was used by Calderbank and Sloane in [1] with $R = Z/4Z$ for $q = 19$ and construction $B(1, 2, 3)$ with $\alpha = 2, \gamma = 1, \beta = 3$, to obtain an extremal doubly even

lattice of dimension 40 and norm 4. But this can also be applied in the case of $GF(5)$ with construction $\mathcal{B}_{19}(0, 1, 4)$ with $\alpha = 0, \gamma = \beta = 1$ or in the case of $GF(7)$ with construction $\mathcal{B}_{19}(0, 5, 2)$ with $\alpha = 0, \gamma = \beta = 5$. In both latter cases, one obtains an odd unimodular lattice of norm 4 and kissing number 39600. In the case of $R = GF(5)$ and $n = 19$ the construction $\mathcal{P}_{19}(2, 1, 3)$ leads to the first extremal unimodular lattice of dimension 38 and norm 4 with theta series:

$$\theta_{\Lambda} = 1 + 29260q^4 + \dots \quad \theta_S = 6080q^{7/2} + \dots$$

Note that construction $\mathcal{P}_{29}(2, 1, 3)$ over $GF(5)$ leads to a lattice for which we could not compute the norm with Magma but whose minimum norm of the LLL-reduced basis has norm 5. If the norm was 5 it would be the unimodular lattice with the highest known norm for that dimension.

4.2 Experimental constructions of self-dual codes

The preceding method permits to construct the first extremal unimodular lattice of dimension 38. One of the advantage of this method is that by construction the code constructed with construction \mathcal{P} or \mathcal{B} have special permutations in their automorphism groups ([4]), which therefore are also in the automorphism group of the associated lattice. Unfortunately this construction only works with some values of q for a given ring or field.

Now if one considers the problem of constructing lattice through construction A and self-dual codes for any dimension and any ring, it is of interest to use the method of [5] that we now recall to construct self-dual codes over rings.

Let R be a ring. Let C be a self-dual code over R of length n . Let G be a generator matrix of C . Now consider $n \times n$ square matrices M_r satisfying for λ_r invertible in R : $M_r \cdot M_r^t = \lambda_r I_n$. For Π_1, \dots, Π_r permutations of the symmetric group S_n ($r \geq 1$), consider the code C_r with generator matrix G_r :

$$G_r = GM_1\Pi_1 \cdots M_r\Pi_r.$$

The following lemma [5] assures the preservation of the self-duality of the code:

Lemma 4.1 *Let G be a generator matrix of a self-dual code of length n over a ring R , let λ be an element of R and let M be a $n \times n$ matrix over R satisfying $M \cdot M^t = \lambda I$. Then the code with generator matrix GM is self-dual.*

In fact for purpose of application it is simpler to take all matrices M_i to be equal to the same simple matrix M as follows. Let B be a square $b \times b$ matrix satisfying:

$$B \cdot B^T = \lambda I_b,$$

and let β be in R such that $\beta^2 = \lambda$. We consider the block matrix M consisting of k_1 times the matrix B and k_2 times β on the diagonal, with $bk_1 + k_2 = n$.

$$M = \begin{pmatrix} B & & & \\ & \ddots & & \\ & & B & \\ 0 & & & \beta I_{k_2} \end{pmatrix}.$$

Now, self-dual codes over fields have even lengths and therefore cannot be used to construct unimodular lattices of odd dimensions, but this can be done if one considers self-dual codes over rings of the form Z/p^2Z . In the next subsections we apply the preceding method to construct self-dual codes over $GF(5)$ and over the ring $Z/25Z$.

4.2.1 Experimental constructions of self-dual codes over $GF(5)$

We use the previous construction with $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ and $G = \begin{pmatrix} 1 & 2 & & \\ & 1 & 2 & \\ & & & \ddots \end{pmatrix}$.

Then using Magma it is possible to check for the minimum norm of the lattice. We found matrices of lengths 36, 42, 54 which lead to unimodular lattices of respective minimum norms 4, 4, 5. We give in the appendix the generator matrices of the corresponding codes. Note that a computation with Magma showed that the lattice of dimension 36 had an automorphism group of order 544 different from the lattice constructed by G. Nebe in [8]. The theta series for the lattices are as follows:

$$\begin{aligned} \text{For } n = 36 : \quad & \theta_\Lambda = 1 + 42840q^4 + \dots & \theta_S = 960q^3 + \dots \\ \text{For } n = 42 : \quad & \theta_\Lambda = 1 + 11844q^4 + \dots & \theta_S = 265216q^{9/2} + \dots \\ \text{For } n = 54 : \quad & \theta_\Lambda = 1 + 95472q^5 + \dots & \theta_S = 2315520q^{11/2} + \dots \end{aligned}$$

Note that this method can also be used easily to construct extremal lattices of dimensions 40, 44 and 46.

4.2.2 Experimental constructions of self-dual codes over $Z/25Z$

In this case we take for generator matrix G of length n the direct sum of $[2, 1, 2]$ codes with generator matrix $(12, 9)$ to which we add the trivial self-dual code of length one generated by (5). We let:

$$B = \begin{pmatrix} 8 & 9 & 9 \\ 9 & 8 & 9 \\ 9 & 9 & 8 \end{pmatrix}.$$

This construction permits to construct a code over $Z/25Z$ of dimension 45 with minimum Euclidean weight 100 and therefore the first extremal lattice of dimension 45 and norm 4.

It has theta series:

$$\theta_{\Lambda} = 1 + 5610q^4 + \dots \quad \theta_S = 150q^{13/4} + \dots$$

The generator matrix is given in the appendix. We also obtained many lattices of dimension 47 and norm 4 that we do not give here. Note that we also tried to construct lattices with codes over the rings $Z/4Z$ and $Z/9Z$ but that it seems that the best results were obtained through $Z/25Z$.

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APPENDIX

In the following we give generator matrices of some of the codes mentioned earlier. The generator matrices are given in the form (IA) (except for $n = 45$), but to save space we only give the matrices A as a sequence of the rows of A separated by ','.

• $n = 36$:

200420024310042000; 203224402041101320; 020004200243100420; 120242201400432013;
 100410004432402004; 211220202231214410; 213223002030420340; 320242233212320144;
 220002204330341203; 433341343133211021; 421143321020000002; 442324023320202340;
 144323134233003410; 240343214043244023; 310023212032343034; 021001243002100030;
 133114122231403200; 010331141222314032;

• $n = 42$:

000021000243001240002; 300442224204240201340; 400221322104000110320; 030020422430043142010;
 040010213340043121104; 003034204031304241423; 304442321332140132001; 100142230012241014040;
 230012204014131200030; 010022422043123100142; 023042122421143242004; 401242331010002024423;
 302242140423401133210; 140444311332033042320; 030231421211233321330; 014012443232323400424;
 003044314143114213212; 301300343200011000340; 030124003120001300000; 003001240031200013000;
 000300012400312000130;

• $n = 45$: in that case we list hereafter a 23×23 matrix A to which one adds on the left a 22×22 identity matrix plus a last row of zeros.

22,0,10,13,6,5,19,2,14,19,0,10,24,16,20,1,8,13,24,13,16,20,14;
 4,0,15,6,13,4,3,22,5,23,9,2,0,22,10,20,19,23,6,22,20,6,10;
 13,1,10,16,9,4,8,7,3,5,4,12,6,6,16,24,16,5,17,9,20,5,8;
 2,0,22,22,7,3,15,8,24,16,6,13,21,19,2,17,0,14,23,11,18,3,13;
 20,1,2,18,15,1,24,15,7,17,3,15,11,1,1,20,3,22,21,24,9,14,6;
 16,3,12,10,18,24,14,15,2,7,0,4,23,8,17,10,5,1,21,15,1,0,0;
 11,1,21,1,14,18,16,3,23,12,24,3,20,17,24,13,21,8,18,12,9,15,23;
 9,0,22,3,21,4,15,2,8,12,15,4,24,0,11,20,6,12,2,9,24,9,11;
 19,2,11,4,2,17,3,6,22,14,10,18,22,15,19,15,14,10,6,20,23,20,7;
 14,2,13,15,10,8,10,10,3,13,1,15,17,16,9,10,6,11,14,9,3,23,8;
 0,0,9,8,22,22,8,24,14,14,2,17,11,24,4,9,15,3,16,23,10,8,22;
 2,3,20,23,15,22,0,4,0,19,18,23,7,12,17,0,22,20,13,8,12,12,16;
 0,4,9,4,8,23,11,17,9,6,1,5,21,9,16,11,20,6,2,24,10,0,0;
 14,2,0,21,17,22,5,24,1,8,4,24,6,7,15,9,5,21,14,15,13,7,9;
 6,0,12,23,6,0,24,0,22,12,2,16,4,16,14,23,11,0,7,18,9,23,8;
 1,0,24,2,7,9,24,12,10,21,7,18,22,17,24,7,22,18,8,13,18,4,0;
 10,0,2,11,1,10,9,21,18,13,5,21,22,7,21,7,7,2,6,23,24,16,12;

19,2,15,13,23,1,22,10,12,5,21,15,2,23,10,16,20,3,11,8,1,11,6;
9,4,18,23,22,23,22,4,9,13,21,5,1,2,9,14,7,3,9,18,24,15,7;
5,1,24,14,14,4,13,4,22,9,4,6,18,3,0,11,13,15,15,8,5,2,14;
16,1,21,0,13,1,9,24,17,2,12,4,1,20,15,4,12,5,13,5,15,10,21;
1,3,10,10,2,10,1,5,11,21,11,0,0,0,13,14,13,22,24,12,14,14,11;
0,5,0,10,20,5,10,20,10,20,0,15,15,15,20,15,0,10,0,0,5,5;

• $n = 54$:

204221212131124324212121420; 434342231410032322023311343; 104100343414123100323434424;
011143041210233231423421114; 032441310132430410102303342; 230232404241010330120424134;
441411213412434034322022322; 334031424114130013243202324; 343443331102422113433411302;
034240121444430034231001021; 240300021041010132143212032; 223113301142224440400312432;
142211012102144100204002334; 142044140403101243332123333; 333101210243241301444043101;
212441222431233244343130104; 124044331242400141041134143; 431411143134344104411242144;
013040012213002414113104342; 132322004422444032224044134; 441222200433240204312130332;
333002320111444300214243024; 144220234441011401421241012; 034230330404403122101211433;
044142133443332201311024414; 211313134331241213131321030; 230012440113341320324300112