Exponential solutions of parameterized linear differential equations.

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1 Introduction

Let us consider a linear differential equation

$$L(y(x)) = a_n(x)\, y^{(n)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0$$

with coefficients $a_k(x)$ in $\overline{\mathbb{Q}}[x]$. The generalized exponents at the singular points of the equation (i.e. the roots of $a_n(x)$ or infinity) give both a local and a global information on the solutions of the equation. Indeed they are the starting point to the description of the formal solutions ([Beke], [Hoei96], [To87]) but also to the computation of the exponential solutions and to the factorization of linear differential operators ([Hoei96], [Pfu98]).

Definition 1 *A* formal solution *of $L(y(x)) = 0$ at the point x_0 is a solution $z(x)$ of the type*

$$z(x) = \exp\left(\int \frac{e_{x_0}(x)}{x - x_0}\right) \sigma(x)$$

where

$$e_{x_0}(x) \in \bigcup_{r \in \mathbb{N}} \overline{\mathbb{Q}}[(x - x_0)^{\frac{-1}{r}}],$$

$$\sigma \in \overline{\mathbb{Q}}((x - x_0))[e_{x_0}(x), \ln(x - x_0)],$$

$$\sigma(x) = \sum_{finie} \sigma_i(x) \ln(x - x_0)^i$$

and

$$\exists i,\ \sigma_i(x) = \sum_{k \geq 0} \sigma_{i,k} x^k \text{ with } \sigma_{i,k} \neq 0.$$

The part $e_{x_0}(x)$ is called generalized exponent *at x_0.*

Remark 1 *If the generalized exponent $e_{x_0}(x)$ belongs to $\overline{\mathbb{Q}}$ ($e_{x_0}(x) = \alpha_{x_0}$), then it is called an* exponent. *In this case the formal solution can be written $(x - x_0)^{\alpha_{x_0}} \sigma(x)$.*

Definition 2 *Let $L(y(x)) = 0$ be a linear homogeneous differential equation with coefficients in $\overline{\mathbb{Q}}(x)$. A solution z of $L(y(x)) = 0$ is* exponential *if its logarithmic derivative $\frac{z'(x)}{z(x)}$ belongs to $\overline{\mathbb{Q}}(x)$.*

The exponential solutions correspond to the right factors of order one of the operator L. They can be written in the following form ([Sin81]):

$$z(x) = \exp\left(\int S(x)\right) q(x)$$

where $q(x) \in \overline{\mathbb{Q}}[x]$,

$$S(x) = \sum_{x_0} \frac{e_{x_0}(x)}{x - x_0} + p(x)$$

and $e_{x_0}(x)$ is a generalized exponent at the point x_0.
The exponent at infinity $e_\infty(x)$ belongs to $K[x]$ and satisfies:

$$\sum_{x_0} \alpha_{x_0} + e_\infty(x) + deg(q) + x\, p(x) = 0$$

where α_{x_0} is the constant term of $e_{x_0}(x)$.
To compute the exponential solutions one needs to compute the part $S(x)$, which contains local information given by the exponents, and then the polynomial $q(x)$ which also satisfies a linear differential equation.
The question one may then ask is : what happens when the coefficients $a_k(x)$ depend on a finite number of parameters ?

The *aim* of this paper is to focus on the computation of the generalized exponents in the parameterized situation and to prove that it can be achieved in the same way as in the non parameterized case. To afford this we need to handle a finite number of *algebraic* or *semi-algebraic* conditions on the parameters.

Theorem 1 *Let*

$$L_{\mathrm{M}}(y(x)) = a_n(x)\, y^{(n)}(x) + a_{n-1}(\mathrm{M}, x)\, y^{(n-1)}(x) + \cdots + a_0(\mathrm{M}, x)\, y(x) = 0$$

be a parameterized linear differential equation with $a_k(\mathrm{M}, x) \in \overline{\mathbb{Q}}[\mathrm{M}][x]$, $a_n(x) \in \overline{\mathbb{Q}}[x]$, $a_n(0) = 0$ *and* $\mathrm{M}=(\mathrm{M}_1, \ldots, \mathrm{M}_s)$ *is a vector of parameters.*
One can construct a finite number of pairs $(\mathbf{C}, (e_1(\mathrm{M}, x), \ldots, e_n(\mathrm{M}, x)))$ *where* $\mathbf{C}$ *is a constructible set,* $e_k(\mathrm{M}, x)$ *belongs to* $\overline{\mathbb{Q}}[\mathrm{M}][\frac{1}{x}]$ *and for all* m *in* $\mathbf{C}$ *the generalized exponents at zero for* $L_m(y(x)) = 0$ *are* $e_1(m, x), \ldots, e_n(m, x)$.

The main difficulty then encountered during the detection of the logarithmic terms in the formal solutions or during the computation of the polynomial solutions comes from the fact that there may appear *arithmetic* conditions on the parameters. We refer to [DuLR92], [B99] and [B00-2] for these questions.

In section 2 we recall the steps of the computation of the generalized exponents (our presentation is based on the study of the Riccati equation).
In section 3 we prove how to compute the generalized exponents at a point (independant of the parameters) when the coefficients of the linear differential equation depend on parameters. We consider each step of the computation of the generalized exponents of the section 2. We construct a finite number of *constructible sets* ([GD95]) and associate to each of them a set of generalized exponents that we compute in the same way as in the non parameterized case.

2 Non parameterized case

The generalized exponents have been calculated for a long time ([Sch], [Bar88], [Bar87], [DDDCT82], [Hi87], [Hoei97], [Le75], [Ma79], [Som94], [To87]). We propose here a presentation that is based on the study of the Riccati equation and that can be then easily generalized to the parameterized situation (see section 3).

Definition 3 *Let*

$$L(y(x)) = a_n(x)\, y^{(n)}(x) + a_{n-1}(x)\, y^{(n-1)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0$$

be a linear differential homogeneous equation with coefficients in $K[x]$.
The Riccati equation *associated to the equation* $L(y(x)) = 0$ *is the equation satisfied by* $u(x) = \frac{y'(x)}{y(x)}$*:*

$$R(u(x)) = a_n(x)\, f_n(u(x)) + a_{n-1}(x)\, f_{n-1}(u(x)) + \cdots + a_0(x)\, f_0(u(x)) = 0$$

where the functions f_k are defined by the following relations:

$$f_0(u(x)) = 1 \text{ and } f_{k+1}(u(x)) = (f_k(u(x)))' + u(x)\, f_k(u(x)).$$

To look for the generalized exponents at zero, we look for the Puiseux series at zero solutions to the Riccati equation associated to the equation $L(y(x)) = 0$. Indeed

$$z(x) = \exp\left(\int \frac{e(x)}{x}\right)\sigma(x) \Leftrightarrow \frac{z'(x)}{z(x)} = \frac{e(x)}{x} + \frac{\sigma'(x)}{\sigma(x)}.$$

If $u = p_l\, x^{\mu_l} + \cdots + p_1\, x^{\mu_1} + \tilde{u}$ ($\mu_l < \cdots < \mu_1 \leq -1$, $\mu_k \in \mathbb{Q}$, $val(\tilde{u}) > -1$) is a solution to the Riccati equation then a generalized exponent at zero is $e_0(x) = p_l\, x^{\mu_l+1} + \cdots + p_2\, x^{\mu_2+1} + p_1\, x^{\mu_1+1}$.

Remark 2 *Later on, for notational simplicity, we always assume $x_0 = 0$ and we seek the generalized exponents in $\overline{\mathbb{Q}}[(x - x_0)^{-1}]$. We replace the condition $\mu_k \in \mathbb{Q}$ with the condition $\mu_k \in \mathbb{Z}$, which does not restrict our study.*

We develop two parts that we sum up in the table page 7:

1. Computation of the leading term at zero of a solution $u(x)$ of the Riccati equation (steps 1 et 2 of the table page 7)

2. Computation of the coefficients of the Riccati equation satisfied by $u(x) - px^{\mu}$ (step 3 of the table page 7).

For each loop we compute a term $p_j\, x^{\mu_j}$ with $\mu_{j-1} < \mu_j \leq -1$. We stop as soon as we find one μ_j such that $\mu_j > -1$.

1. Computation of the leading term at zero of a solution of the Riccati equation.

   Proposition 1 *Let $L(y(x)) = a_n(x)\, y^{(n)}(x) + \cdots + a_1(x)\, y'(x) + a_0(x)\, y(x) = 0$ and $R(u(x)) = 0$ be the associated Riccati equation.*
   Let us denote $c_k\, x^{v_k}$ the leading term at zero of $a_k(x)$ (for all k in $\{0, \ldots, n\}$) and $p\, x^{\mu}$ the leading term at zero of $u(x)$ ($\mu \in \mathbb{Z}$ and $\mu \leq -1$, $p \neq 0$).
   Then the leading term at zero of $R(u(x))$ is equal to

   $$P_{\mu}(p)\, x^{e(\mu)}$$

   where

   $$P_{\mu}(p) = \begin{cases} \sum\limits_{j \in I(\mu)} c_j p^j & \text{if } \mu < -1 \\ \sum\limits_{j \in I(\mu)} c_j\, p\,(p-1)\cdots(p-j+1) & \text{if } \mu = -1 \end{cases}$$

   $$I(\mu) = \{k;\ e(\mu) = v_k + k\,\mu\}$$

   and $e(\mu) = min\{v_k + k\,\mu, k = 0 \ldots n\}$.

Proof

For all k in $\{0,\ldots,n-1\}$, let $F_k(x)$ denote the leading term at zero of $f_k(u(x))$ and let $\tilde{R}(u(x))$ denote the sum of the leading terms of $a_k(x)\,f_k(u(x))$:

$$\tilde{R}(u(x)) = \sum_{k=0}^{n} F_k(x)\,c_k\,x^{v_k}.$$

We prove that if μ is less than or equal to -1, then the leading terms of $R(u(x))$ and of $\tilde{R}(u(x))$ are equal. It suffices to prove that the leading term of $\tilde{R}(u(x))$ is not identically equal to zero.

- Let us assume that the integer μ is less than -1 then

$$\tilde{R}(u(x)) = \sum_{k=0}^{n} c_k\,p^k\,x^{v_k+k\mu}.$$

  Indeed the leading term at zero $F_1(x)$ of $f_1(u(x)) = u(x)$ is equal to $p\,x^{\mu}$.
  Let k be in $\mathbb{N}^*$ and let us assume $F_k(x) = p^k\,x^{k\mu}$. According to the definition 3,

$$f_{k+1}(u(x)) = (f_k(u(x)))' + u(x)\,f_k(u(x)) = k\,\mu\,p^k\,x^{k\mu-1} + hot + p^{k+1}\,x^{k\mu+\mu} + hot$$

  but μ is less than -1, so

$$f_{k+1}(u(x)) = p^{k+1}\,x^{(k+1)\mu} + hot$$

  and $F_{k+1}(x) = p^{k+1}\,x^{(k+1)\mu}$.
  Now the leading term at zero of $\tilde{R}(u(x))$ is the term with exponent $e(\mu) = min\{v_k + k\mu, k = 0,\ldots,n\}$, that is to say the term

$$\left(\sum_{j\in I(\mu)} c_j p^j\right) x^{e(\mu)}.$$

  As the coefficients c_j are non zero, the polynomial $P_\mu(p) = \sum_{j\in I(\mu)} c_j\,p^j$ is not identically equal to zero. So it both represents the leading coefficient of $\tilde{R}(u(x))$ and the leading coefficient of $R(u(x))$.

- Let us assume that the integer μ is equal to -1.
  Then

$$\tilde{R}(u(x)) = \sum_{k=0}^{n} c_k\,p\,(p-1)\ldots(p-k+1)x^{v_k-k}.$$

  Indeed, if μ is equal to -1, an easy induction on k shows the following equality: $F_k(x) = p\,(p-1)\cdots(p-k+1)\,x^{-k}$.
  The coefficient associated to the exponent $e(-1)$ is

$$P_{-1}(p) = \sum_{j\in I(-1)} c_j\,p\,(p-1)\cdots(p-j+1).$$

  It cannot be identically equal to zero and so it represents the leading coefficient of $R(u(x))$. □

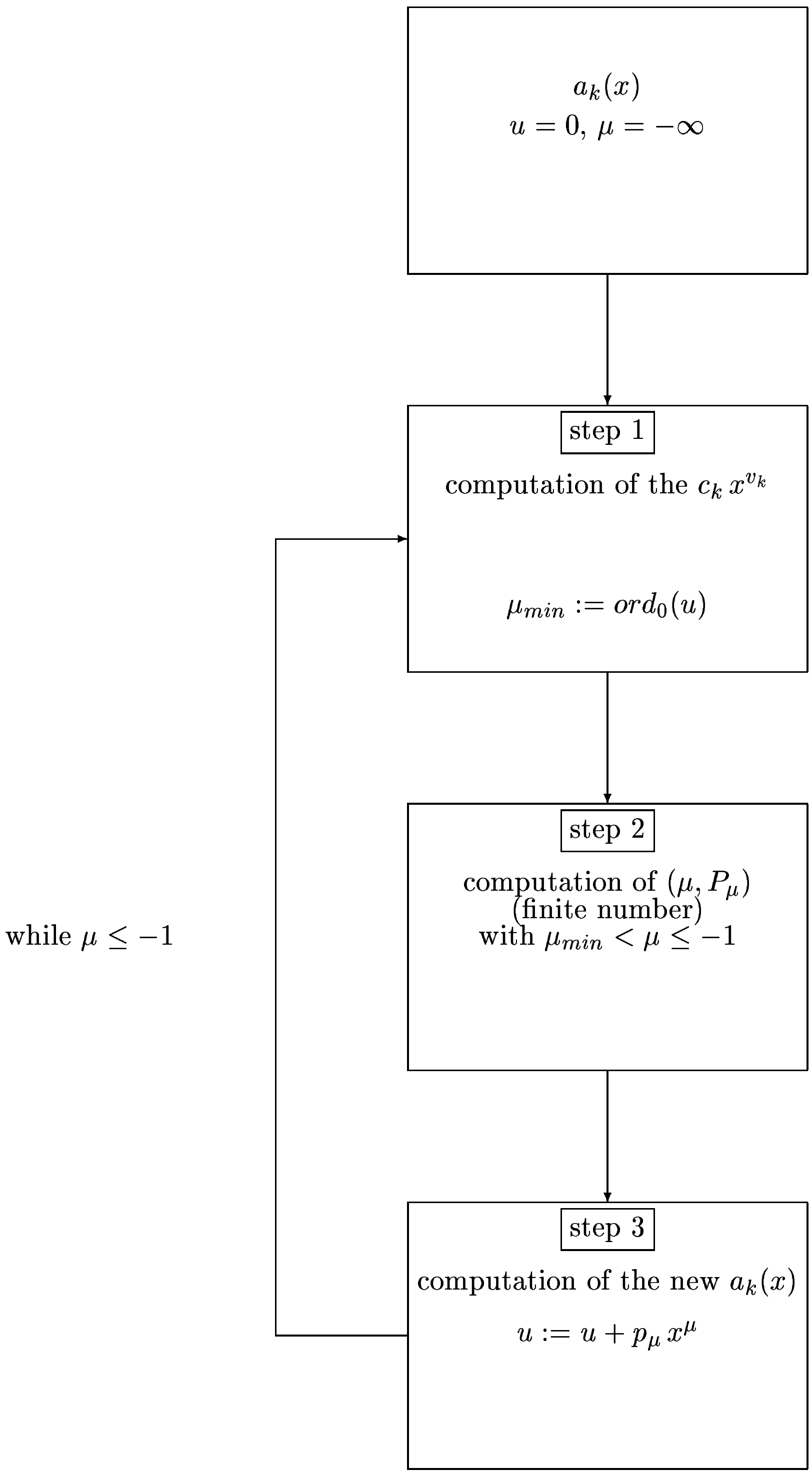
Now to compute the leadind term $p\,x^{\mu}$ of a solution to the Riccati equation, one first finds the exponent μ. Then the coefficient p is defined by the relation $P_{\mu}(p) = 0$. What is particularly interesting to notice here is that the exponent μ will depend only on the exponents v_k. This property will be preserved in the parameterized case and will enable us to state that the coefficient μ will not depend on the parameters (see step 2 in the section 3).
What is the method to compute the exponent μ ?
We propose here a geometrical interpretation inspired by D. Boularas.
Let us consider the graph $\mathcal{C}$ of the fonction $e(x) = min(v_k + k\,x, k = 0 \ldots n)$. It is the lower convex envelope of the set of lines of equation $y = v_k + k\,x$. It is analogous to the Newton Polygon which is generally represented by the lower convex envelope of the points $(k, v_k - k)$ ([Hoei96], [To87] for example).
One can extract two subsets from this graph:
the set $\mathcal{P}$ of the points with integer abscissa which are on the opened semi-lines and segments,
the set $\mathcal{I}$ of the points of intersection between these semi-lines and segments.
The abscissa of the points of the set $\mathcal{P}$ are the μ such that $card(I(\mu)) = 1$ (where $I(\mu)$ is defined in the proposition 1 page 4). For these μ, the leading term of $R(u(x))$ is of the type $c_k\,p^k$. It is equal to zero if, and only if, p is equal to zero, and then the only possibility for u is $u = 0$.
The abscissa of the points of the set $\mathcal{I}$ are the μ such that $card(I(\mu)) > 1$. For each of these μ there exists two distinct integers j and k such that $v_j + j\mu = v_k + k\mu$.
To conclude the sought exponents μ are the integer abscissa of the points of the part $\mathcal{I}$. Futhermore, as there are at most $n + 1$ lines in the graph $\mathcal{C}$, we can conclude that there are at most n possibilities for μ.

2. Computation of the coefficients of the Riccati equation satisfied by $u(x) - px^{\mu}$.

   Once μ and P_{μ} are computed, one can determine the differential equation satisfied by $y\,\exp(\int px^{\mu})$ under the condition $P_{\mu}(p) = 0$ and the Riccati equation satisfied by $u - px^{\mu}$. The new coefficients $a_k(x)$ still belong to $\overline{\mathbb{Q}}[x]$.

The following table sums up the steps of the computation of the generalized exponents. In the next section we study the adaptation of each of these steps to the parameterized case.

Non parameterized case



3 Parameterized case

Let

$$L_{\rm M}(y(x)) = a_n(x)y^{(n)}(x) + a_{n-1}({\rm M},x)y^{(n-1)}(x) + \cdots + a_1({\rm M},x)\,y'(x) + a_0({\rm M},x)\,y(x) = 0$$

where ${\rm M} = ({\rm M}_1,\ldots,{\rm M}_s)$, $a_k({\rm M},x) \in \overline{\mathbb{Q}}[{\rm M}][x]$, $a_n(x) \in \overline{\mathbb{Q}}[x]$.
We assume again for notational convenience that zero is a singular point for the equation.
The computation of the generalized exponents will require aspects of dynamic evaluation ([Du95]) and manipulation of *constructible sets* ([GD95]).

Definition 4 *[GD95] A* constructible set *of* $\overline{\mathbb{Q}}^s$ *is a set of* $(m_1,\ldots,m_s)$ *satisfying a finite number* k *of conditions:*

$$P_1(m_1,\ldots,m_s)\xi_1 0 \text{ and } \cdots \text{ and } P_k(m_1,\ldots,m_s)\xi_k 0$$

with $P_i \in \overline{\mathbb{Q}}[X_1,\ldots,X_s]$ *and* $\xi_i \in \{=,\neq\}$.

Theorem 1
Let

$$L_{\rm M}(y(x)) = a_n(x)\,y^{(n)}(x) + a_{n-1}({\rm M},x)\,y^{(n-1)}(x) + \cdots + a_0({\rm M},x)\,y(x) = 0$$

be a parameterized linear differential equation with $a_k({\rm M},x) \in \overline{\mathbb{Q}}[{\rm M}][x]$, $a_n(x) \in \overline{\mathbb{Q}}[x]$, $a_n(0) = 0$ *and* ${\rm M}=({\rm M}_1,\ldots,\ {\rm M}_s)$ *is a vector of parameters.*
One can construct a finite number of pairs $(\mathbf{C},(e_1({\rm M},x),\ldots,e_n({\rm M},x)))$ *where* $\mathbf{C}$ *is a constructible set,* $e_k({\rm M},x)$ *belongs to* $\overline{\mathbb{Q}}[{\rm M}][\frac{1}{x}]$ *and for all* m *in* $\mathbf{C}$ *the generalized exponents at zero for* $L_m(y(x)) = 0$ *are* $e_1(m,x),\ldots,e_n(m,x)$.
Proof
The proof is constructive and yiels an algorithm.
We use arguments coming from dynamic evaluation ([Du95]) and more precisely section 3 page 69 of [GD95].
To afford this we study the three steps of the table page 7.

step 1

The parameters m belong to the set $\mathbf{C}$ which is first equal to $\overline{\mathbb{Q}}^s$. Let us compute the leading terms at zero of the coefficients $a_k({\rm M},x)$.
We consider the following sets C_{k,j_k}:
$C_{k,j_k} = \{m;\ a_{k,0}(m) = \cdots = a_{k,j_k-1}(m) = 0,\ a_{k,j_k}(m) \neq 0\}$ where the coefficients $a_{k,j}(m)$ are the coefficients of the polynomials $a_k({\rm M},x)$: $a_k({\rm M},x) = \sum_{j=0}^{d} a_{k,j}({\rm M})\,x^j$.
For all m in C_{k,j_k}, the leading term at zero of $a_k(m,x)$ is $a_{k,j_k}(m)x^{j_k}$.
Let us consider the sets $\mathbf{C} := \mathbf{C} \cap C_{0,j_0} \cap \cdots \cap C_{n-1,j_{n-1}}$, to each of them one can associate the leading terms at zero $a_{k,j_k}(m)x^{j_k}$ of the polynomials $a_k(m,x)$. We denote them $c_k(m)\,x^{v_k}$.

Remark 3 *We can restrict ourselves to the* j_k *such that* $j_k - k \leq v_n - n$.
Indeed if $j_k - k > v_n - n$, *then the line of equation* $y = j_k + kx$ *does not belong to* $\mathcal{C}$.

step 2

According to the definition of the set $\mathbf{C}$, for each m in $\mathbf{C}$, the coefficients $c_k(m)$ are non zero. Furthermore the exponents v_k do not depend on the parameters and belong to $\mathbb{Z}$.
So like in the non parameterized case, the exponent μ can be computed thanks to the polygon made of the lines of equations $y = k\,x + v_k$ (independant of the parameters because v_k is one of the exponents of the polynomial a_k and does not depend on the coefficients of the polynomial a_k). So we still get at most n possibilities for μ.
For each of these μ, we add to the set $\mathbf{C}$ the conditions $P_\mu(p_\mu) = 0$ and $p_\mu \neq 0$. The polynomial $P_\mu(p)$ is defined in the same way as in proposition 1 page 4:

$$P_\mu(p) = \begin{cases} \sum\limits_{j \in I(\mu)} c_j(m) p^j & \text{if } \mu < -1 \\ \sum\limits_{j \in I(\mu)} c_j(m) p(p-1)\cdots(p-j+1) & \text{if } \mu = -1 \end{cases}$$

$$e(\mu) = min\{v_k + k\mu, k = 0 \ldots n\}$$
$$\text{and } I(\mu) = \{k;\ e(\mu) = v_k + k\mu\}.$$

So we have added a new parameter (p_μ) and two conditions ($p_\mu \neq 0$ and $P_\mu(p_\mu) = 0$).

step 3

For each $(\mu, \mathbf{C})$ computed in the step 2, one computes the new coefficients $a_k(m, x)$. They are the coefficients of the equation $L_m(\exp(\int \frac{p_\mu}{x^\mu})\, z) = 0$. As μ is independant of the parameters, these new coefficients are still polynomials depending on m and the new parameter p_μ.

□

Example 1 *Let us consider the linear differential equation*

$$L_{\mathrm{M}_1}(y(x)) = a_0(\mathrm{M}_1, x)\, y(x) + a_1(\mathrm{M}_1, x)\, y'(x) + a_2(\mathrm{M}_1, x)\, y^{(2)}(x) = 0$$

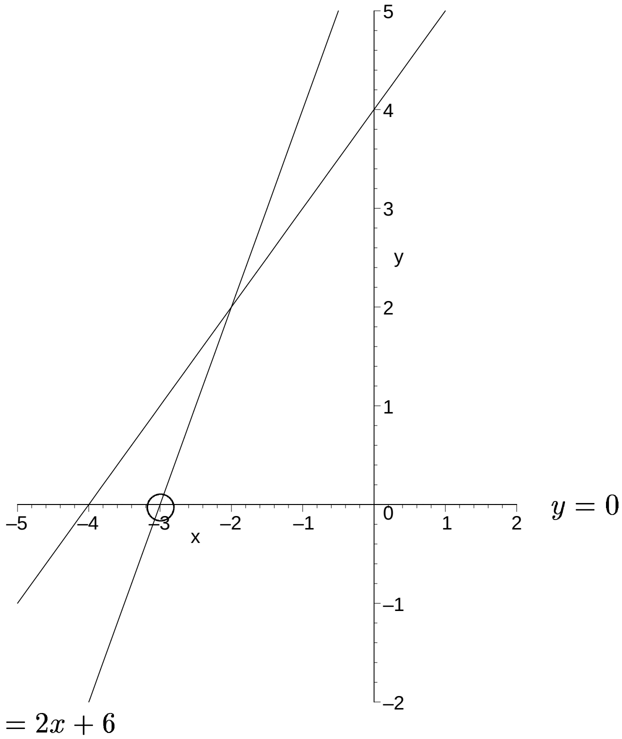
with
$a_0(\mathrm{M}_1, x) = -x^3 - \mathrm{M}_1^2$, $a_1(\mathrm{M}_1, x) = x^4\,(x + \mathrm{M}_1)$ *and* $a_2(\mathrm{M}_1, x) = x^6$.
We get the two following pairs $(\mathbf{C}, (e_1(\mathrm{M}_1, x), e_2(\mathrm{M}_1, x)))$.

1. $\mathbf{C} = \{m_1;\ m_1 \neq 0\}$,
   $e_1(\mathrm{M}_1, x) = -\frac{\mathrm{M}_1}{x^2} - \frac{\mathrm{M}_1}{2}\frac{1}{x} + 1 - \frac{\mathrm{M}_1}{8}$,
   $e_2(\mathrm{M}_1, x) = \frac{\mathrm{M}_1}{x^2} - \frac{\mathrm{M}_1}{2}\frac{1}{x} + 1 + \frac{m_1}{8}$.

2. $\mathbf{C} = \{m_1;\ m_1 = 0\}$,
   $e_1(\mathrm{M}_1, x) = \frac{1}{\sqrt{x}} + \frac{1}{4}$,
   $e_2(\mathrm{M}_1, x) = \frac{-1}{\sqrt{x}} + \frac{1}{4}$.

Let us first of all study the leading terms at zero of the polynomials $a_k(\mathrm{M}_1, x)$.

1. *first case:* $\mathbf{C} = \{m_1;\ m_1 \neq 0\}$.
   The leading coefficient at zero of the polynomial $a_0(m_1, x)$ *is* $-m_1^2$*; the one of* $a_1(m_1, x)$ *is* $m_1\, x^4$.



Necessarly, $\mu = -3$ *and* $p_{-3}^2 - m_1^2 = 0$.
The coefficients of the equation $L(\exp(\int \frac{p_{-3}}{x^3})\, z(x)) = 0$ *are:*

$$a_0(m_1, x) = m_1\, p_{-3}\, x - 2p_{-3}x^2 - x^3$$

$$a_1(m_1, x) = 2p_{-3}x^3 + m_1\, x^4 + x^5.$$

Their leading coefficients at zero are respectively $m_1\, p_{-3}\, x$ *and* $2p_{-3}x^3$, *for all* m_1 *in* $\mathbf{C}$. *The new value for* μ *is* -2 *and* p_{-2} *satisfies* $m_1\, p_{-3} + p_{-2}\, 2\, p_{-3} = 0$ *that is to say* $p_{-2} = -\frac{m_1}{2}$.
The coefficients of the equation $L(\exp(\int \frac{p_{-3}}{x^3} + \frac{p_{-2}}{x^2})\, z(x)) = 0$ *are:*

$$a_0(m_1, x) = (-2\, p_{-3} - \frac{m_1^2}{4})x^2 + (\frac{m_1}{2} - 1)x^3$$

$$a_1(m_1, x) = 2p_{-3}x^3 + x^5.$$

The new value for μ *is* -1 *and* p_{-1} *satisfies* $(-2\, p_{-3} - \frac{m_1^2}{4}) + p_{-1}\, 2\, p_{-3} = 0$ *that is to say* $p_{-1} = 1 + \frac{m_1^2}{8p_{-3}}$.

2. *second case:* $\mathbf{C} = \{m_1;\ m_1 = 0\}$.
   The equation is

$$y(x) + x^2\, y'(x) + x^3\, y''(x) = 0.$$

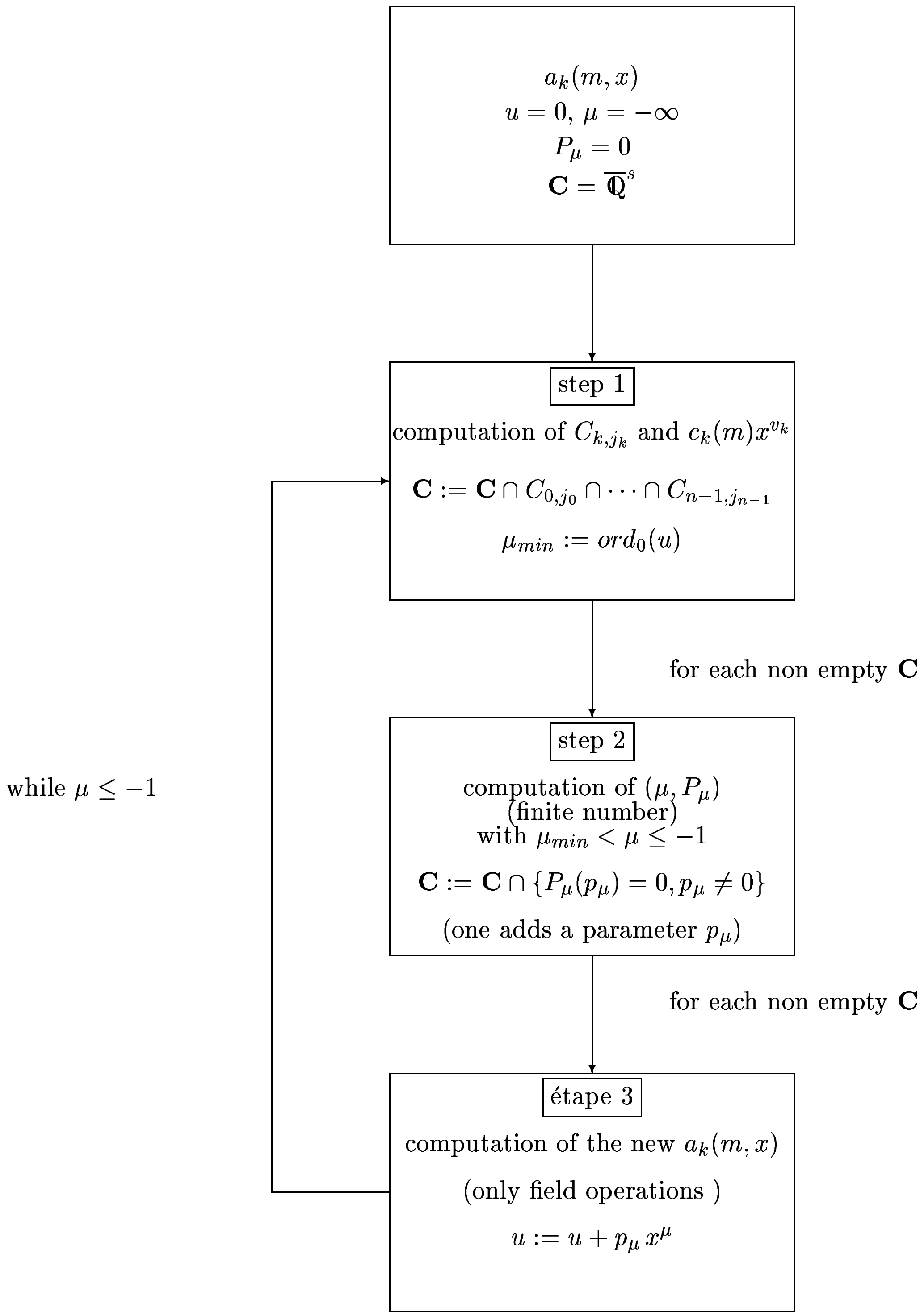
   The generalized exponents are:
   $e_1(m_1, x) = e_1(0, x) = \frac{1}{\sqrt{x}} + \frac{1}{4}$
   $e_2(m_1, x) = e_2(0, x) = \frac{-1}{\sqrt{x}} + \frac{1}{4}$.

Remark 4 *In the previous example the exponents associated to the set* $\mathbf{C} = \{m_1; m_1 \neq 0\}$ *are well defined when* $m_1 = 0$ *(*$-\frac{m_1}{x^2} - \frac{m_1}{2}\,\frac{1}{x} + 1 - \frac{m_1}{8} = 1$ *and* $\frac{m_1}{x^2} - \frac{m_1}{2}\,\frac{1}{x} + 1 + \frac{m_1}{8} = 1$ *when* $m_1 = 0$*).*

However they are not equal to the exponents associated to the set $\mathbf{C} = \{m_1; m_1 = 0\}$ *(i.e.* $\frac{1}{\sqrt{x}} + \frac{1}{4}$ *and* $\frac{-1}{\sqrt{x}} + \frac{1}{4}$*).*
So this example shows that our process cannot be bypassed even when the exponents associated to a set $\mathbf{C}$ *are defined at the forbidden points of* $\mathbf{C}$.

The following table sums up the computation of the generalized exponents at zero in the parameterized case.

Parameterized case



4 Conclusion

The problem of computing the generalized exponents at a singularity (independant of the parameters) of a parameterized linear differential equation can be achieved thanks to arguments developped in [GD95] (we implemented a maple program for this computation [1]). Indeed one can construct a *finite* number of *finite* sets of *algebraic* or *semi-algebraic* conditions on the parameters to each of which one can associate the generalized exponents.
The difficulty then encountered during the computation of the exponential solutions comes from the possible presence of *arithmetic* conditions on the parameters. These conditions rely on the degree of the polynomial solutions. More precisely the study of the polynomial solutions is indecidable (J.A. Weil, [B99]). To determine these solutions one first constructs a finite number of arithmetic conditions on the parameters coming from the valuations and the degree. One can prove that it is not possible to get all the necessary and sufficient conditions on the parameters if one allows oneself to add only a finite number of algebraic conditions to these arithmetic conditions ([B00-2]). At the moment we can only provide partial tools to help this study ([B99], [B00-2]).
Furthermore one can hope to simplify the detection of logarithmic terms in the formal solutions and the computation of exponential solutions when one deals with equations coming from physical questions such as the study of non integrability conditions on hamiltonian systems ([B00-1], [BW01]). Indeed we get strong constraints on the parameters both coming from physical properties and from the symplectic structure of the equation. This last point will be further developped in a forthcoming work with J.-A. Weil.

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[1] For details concerning the implementation please contact the author at dboucher@univ-lr.fr

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