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Huynh Van Ngai and Michel Théra

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Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. 05 55 45 73 23 - Fax. 05 55 45 73 22 - laco@unilim.fr

http://www.unilim.fr/laco/

ON NECESSARY CONDITIONS FOR NONLIPSCHITZ OPTIMIZATION PROBLEMS

Huynh Van Ngai
Ecole Normale Supérieure de Quinhon, Vietnam
Michel Théra
LACO, UPRESSA 6090, Université de Limoges
123, avenue Albert Thomas, 87060 Limoges Cedex France

Abstract. In this paper, we establish a nessary condition for nonlipschitz constrained optimization problems in terms of Fréchet subdifferentials and normal cones in Asplund spaces.

Key words. Fuzzy calculus, Fréchet subdifferential, normal cone, optimality condition.

AMS subject classifications. 49J529, 90C30

1. Introduction

Throughout the paper, X denotes a Banach space, $C \subset X$ is a closed set of X and $f_i : X \to \mathrm{I\!R} \cup \{+\infty\}$, $i = 0, 1, ..., n.$ are extended-real valued functions. We consider constrained optimization problems with semicontinuous inequality and continuous equality data:

$$(\mathfrak{P}) \qquad \begin{aligned} \text{minimizer } & f_0(x) \quad \text{subjet to} \\ & f_i(x) \leq 0, \quad i = 1, ..., m; \\ & f_i(x) = 0, \quad i = m+1, ..., n; \\ & x \in C. \end{aligned}$$

In [1], was established a necessary optimality condition in reflexive Banach spaces in terms of (smooth) subderivatives and normal cones. In [17], we proved a necessary condition using limiting Fréchet subdifferentials and limiting normals in Asplund spaces with compactness assumptions. The purpose of this note is to prove a fuzzy multiplier rule for the above problem in terms of Fréchet subdifferentials and Fréchet normals in Asplund spaces. We establish a result stronger than in [1]. More precisely, multipliers corresponding to the inequality constraints are all positive and multipliers corresponding the equality constraints are all nonzero.

In Section 2, a "fuzzy calculus rule" for Fréchet subdifferentials of composite functions is established. Using this chain rule, we derive in Section 3, a fuzzy

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multiplier rule for problem $(\mathfrak{P})$. In the final section we present in the Asplund setting, a result on the relationship between the normal cone to a level set and the subdifferential of the corresponding function. Such a relationship for smooth subderivatives and smooth normal cones in reflexive spaces plays a key role in the proof of the main result in [1].

2. Fuzzy calculus for Fréchet subdifferentials

Let X be a Banach space with closed unit ball B_X and with dual space X^*. Let $f : X \to 1\mathrm{R} \cup \{+\infty\}$ be a lower semicontinuous function. We denote as usual by $\mathrm{dom} f := \{x \in X : f(x) < +\infty\}$, $\mathrm{epi} f := \{(x, \alpha) \in X \times 1\mathrm{R} : \alpha \geq f(x)\}$, and $\mathrm{gph} f := \{(x, \alpha) \in X \times 1\mathrm{R} : \alpha = f(x)\}$, the domain, the epigraph and the graph of f, respectively.

Recall that the Fréchet subdifferential of f at $x \in \mathrm{dom} f$ is defined by

$$\partial^F f(x) := \Big\{x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq 0\Big\}. \tag{2.1}$$

If $x \notin \mathrm{dom} f$, we set $\partial^F f(x) := \emptyset$.

For a closed subset C of X, the Fréchet normal cone to C at $x \in C$ is the set $N^F(C, x) := \partial^F \delta_C(x)$, where $\delta_C(.)$ is the indicator function of C given by $\delta_C(x) := 0$ if $x \in C$ and $+\infty$ if $x \notin C$.

Recall that a Banach space is said to be *Asplund* if every convex lower semicontinuous function is Fréchet differentiable on a dense $G_\delta-$subset of the interior of its domain. A important charaterization of Asplund spaces is the *fuzzy sum rule* for Fréchet subdifferentials proved by Fabian ([4],[5]) (see also another charaterization in Modukhovich-Shao [13] and Jourani [9]).

Proposition 2.1. (Fabian [5]) *Let X be an Asplund space, let $f_i : X \to 1\mathrm{R} \cup \{+\infty\}$, $i = 1, ..., n$ be lower semicontinuous functions. Let $\bar{x} \in \mathrm{dom} f_1 \cap ... \cap \mathrm{dom} f_n$. Then, for any $\epsilon > 0$ and any weak*neighbourhood V of 0 in X^*,*

$$\begin{aligned}\partial^F(f_1 + ... + f_n)(\bar{x}) \subseteq \bigcup \big\{&\partial^F f_1(x_1) + ... + \partial^F f_n(x_n) + V : \\ &(x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X\times 1\mathrm{R}},\ i = 1, ..., n\big\}.\end{aligned}$$

Moreover, in addition, if $f_2, ..., f_n$ are locally Lipschitz, then the following inclusion holds:

$$\begin{aligned}\partial^F(f_1 + ... + f_n)(\bar{x}) \subseteq \bigcup \big\{&\partial^F f_1(x_1) + ... + \partial^F f_n(x_n) + \epsilon B_{X^*} : \\ &(x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X\times 1\mathrm{R}},\ i = 1, ..., n\big\}.\end{aligned}$$

In the sequel we shall make use of the following pair of lemmata from [8], [14], [17].

Lemma 2.2. (Ioffe ([8]), Modukhovich-Shao ([14])) *Let $f : X \to 1\mathrm{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $(\bar{x}, \alpha) \in \mathrm{epi} f$.*

(i) Let $\lambda \neq 0$; the equivalence

$$(x^*, -\lambda) \in N^F(\mathrm{epi} f, (\bar{x}, \alpha)) \iff \lambda > 0,\ \alpha = f(\bar{x}),\ x^* \in \partial^F(\lambda f)(\bar{x}),$$

is true in every Banach space X.

(ii) Suppose that X is an Asplund space. If $(x^*,0) \in N^F(\mathrm{epi}f,(\bar{x},\alpha))$, then there exist sequences $\{x_n\}_{n\in \mathbb{N}}$, $\{x_n^*\}_{n\in \mathbb{N}}$, $\{\lambda_n\}_{n\in \mathbb{N}}$ such that
$x_n^* \in \lambda_n \partial^F f(x_n)$, $(x_n, f(x_n)) \to (\bar{x}, f(\bar{x}))$, $\lambda_n \downarrow 0$ and $\|x_n^* - x^*\| \to 0$.

Lemma 2.3. ([17]) *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a continuous function and let $\bar{x} \in \mathrm{dom}f$.*

(i) Let $\lambda \neq 0$; the equivalence

$$(x^*, -\lambda) \in N^F(\mathrm{gph}f, (\bar{x}, f(\bar{x})) \iff x^* \in \partial^F(\lambda f)(\bar{x}),$$

is true in any Banach space X.

(ii) Suppose that X is an Asplund space. If $(x^*,0) \in N^F(\mathrm{gph}f,(\bar{x},f(\bar{x}))$, then there exist sequences $\{x_n\}_{n\in \mathbb{N}}$, $\{x_n^*\}_{n\in \mathbb{N}}$, $\{\lambda_n\}_{n\in \mathbb{N}}$ such that
$x_n^* \in \partial^F(\lambda_n f)(x_n) \cup \partial^F(-\lambda_n f)(x_n)$, $(x_n, f(x_n)) \to (\bar{x}, f(\bar{x}))$, $\lambda_n \downarrow 0$
and $\|x_n^* - x^*\| \to 0$.

Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., n$; $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Consider the composite function:

$$g[f_1, ..., f_n] : X \to \mathbb{R} \cup \{+\infty\},$$
$$g[f_1, ..., f_n](x) := \begin{cases} g(f_1(x), ..., f_n(x)) & \text{if} \quad x \in \mathrm{dom}f_1 \cap ... \cap \mathrm{dom}f_n \\ +\infty & \text{otherwise.} \end{cases}$$

We prove the following chain rule:

Theorem 2.4. *Let X be an Asplund space. Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous for $i = 1, ..., m$ and let $f_i : X \to \mathbb{R} \cup \{+\infty\}$ be continuous for $i = m+1, ..., n$. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. Suppose that g is nondecreasing for each of its first m variables. Let $\bar{x} \in \mathrm{dom}f_1 \cap ... \cap \mathrm{dom}f_n$ and $(f_1(\bar{x}), ..., f_n(\bar{x})) \in \mathrm{dom}g$. Then, for any $\epsilon > 0$ and any weak* neighbourhood V of 0 in X^*, we have*

$$\begin{aligned} \partial^F g[f_1, ..., f_n](\bar{x}) \subseteq \bigcup \Big\{ & \partial^F(\mu_1 f_1)(x_1) + ... + \partial^F(\mu_n f_n)(x_n) + V : \\ & (x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X\times \mathbb{R}},\ i = 1, ..., n; \\ & (\alpha_1, ..., \alpha_n) \in (f_1(\bar{x}), ..., f_n(\bar{x})) + \epsilon B_{\mathbb{R}^n}; \\ & (\mu_1, ..., \mu_n) \in \partial^F g(\alpha_1, ..., \alpha_n) + \epsilon B_{\mathbb{R}^n}; \\ & \mu_i > 0,\ i = 1, ..., m;\ \mu_i \neq 0,\ i = m+1, ..., n \Big\}. \end{aligned}$$

Proof. First, we set

$$S_i := \Big\{ (x, \alpha_1, ..., \alpha_n) \in X \times \mathbb{R}^n : \alpha_i \geq f_i(x) \Big\},\ i = 1, ..., m;$$

$$S_i := \Big\{ (x, \alpha_1, ..., \alpha_n) \in X \times \mathbb{R}^n : \alpha_i = f_i(x) \Big\},\ i = m+1, ..., n.$$

Clearly,

$$\begin{aligned} N^F(S_i, (x, \alpha_1, ..., \alpha_n)) = \Big\{ (x^*, \lambda_1, ..., \lambda_n) \in X^* \times \mathbb{R}^n : \lambda_j = 0 \quad \text{if} \quad j \neq i; \\ (x^*, \lambda_i) \in N^F(\mathrm{epi}f_i, (x, \alpha_i)) \Big\}, \\ \text{for} \quad i = 1, ..., m; \end{aligned}$$

$$N^F(S_i,(x,\alpha_1,...,\alpha_n)) = \Big\{(x^*,\lambda_1,...,\lambda_n) \in X^* \times \mathrm{I\!R}^n : \lambda_j = 0 \quad \text{if} \quad j \neq i; \\ (x^*,\lambda_i) \in N^F(\mathrm{gph} f_i,(x,\alpha_i))\Big\}, \\ \text{for} \quad i = m+1,...,n.$$

Fix $x^* \in \partial^F g[f_1,...,f_n](\bar{x})$. Observe that

$$g[f_1,...,f_n](x) = \min\Big\{g(\alpha_1,...,\alpha_n)+\sum_{i=1}^{n}\delta_{S_i}(x,\alpha_1,...,\alpha_n) : (x,\alpha_1,...,\alpha_n) \in X\times\mathrm{I\!R}^n\Big\}.$$

Hence,

$$(x^*,0,...,0) \in \partial^F\big[g(.)+\sum_{i=1}^{n}\delta_{S_i}(.)\big](\bar{x},f_1(\bar{x}),...,f_n(\bar{x})). \tag{2.2}$$

For any $\epsilon > 0$, any weak* neighbourhood V of 0 in X^*, let U be a weak* neighbourhood of 0 in X^* such that $U + n\epsilon B_{X^*} \subseteq V$ (B_{X^*} is the closed unit ball in X^*).

Since the functions f_i are lower semicontinuous, there exists $\eta \in (0,\frac{\epsilon}{2})$ such that

$$f_i(x) > f_i(\bar{x}) - \frac{\epsilon}{2} \text{ for all } x \in \bar{x}+\eta B_X,\ i = 1,...n. \tag{2.3}$$

Using the fuzzy sum rule for (2.2), there exist

$$(x_i,r_i) \in \big((\bar{x},f_i(\bar{x}))+\eta B_{X\times\mathrm{I\!R}}\big)\cap \mathrm{epi} f_i,\ i=1,...,m; \tag{2.4}$$

$$(x_i,f_i(x_i)) \in (\bar{x},f_i(\bar{x}))+\eta B_{X\times\mathrm{I\!R}},\ i=m+1,...,n; \tag{2.5}$$

$$(\alpha_1,...,\alpha_n) \in (f_1(\bar{x}),...,f_n(\bar{x}))+\eta B_{\mathrm{I\!R}^n}; \tag{2.6}$$

$$(\lambda_1,...,\lambda_n) \in \partial^F g(\alpha_1,...,\alpha_n);$$

$$(\zeta_i,-\gamma_i) \in N^F(\mathrm{epi} f_i,(x_i,r_i)),\ i=1,...,m; \tag{2.7}$$

$$(\zeta_i,-\gamma_i) \in N^F(\mathrm{gph} f_i,(x_i,f_i(x_i)),\ i=m+1,...,n \tag{2.8}$$

such that

$$(x^*,0,...,0) \in (0,\lambda_1,...,\lambda_n)+(\zeta_1+...+\zeta_n,-\gamma_1,...,-\gamma_n)+U\times\eta B_{\mathrm{I\!R}^n}.$$

Thus,

$$x^* \in \zeta_1+...+\zeta_n+U; \tag{2.9}$$

$$(\gamma_1, ..., \gamma_n) \in \partial^F g(\alpha_1, ..., \alpha_n) + \eta B_{R^n}. \tag{2.10}$$

For every $i = 1, ..., m$, by (2.7), using Lemma 2.2, we obtain:

If $\gamma_i \neq 0$, then $r_i = f_i(x_i)$, $\gamma_i > 0$ and $\zeta_i \in \gamma_i \partial^F f_i(x_i)$, we set $\mu_i := \gamma_i$, $z_i := x_i$, and $\xi_i := \zeta_i$. Else, $\gamma_i = 0$, there exist $\mu_i \in (0, \eta)$, $(z_i, f_i(z_i)) \in (x_i, f_i(x_i)) + \eta B_{X \times \mathrm{I\!R}}$, $\xi_i \in \mu_i \partial^F f_i(z_i)$ such that $\|\xi_i - \zeta_i\| < \epsilon$.

Similarly, for every $i = m + 1, ..., n$, applying Lemma 2.3 to (2.8), one has:

If $\gamma_i \neq 0$, then $\zeta_i \in \partial^F(\gamma_i f_i)(x_i)$, denoting $\mu_i := \gamma_i$, $z_i := x_i$, and $\xi_i := \zeta_i$. Else, $\gamma_i = 0$, there exist $\mu_i \in (-\eta, \eta)$, $\mu_i \neq 0$, $(z_i, f_i(z_i)) \in (x_i, f_i(x_i)) + \eta B_{X \times \mathrm{I\!R}}$, $\xi_i \in \partial^F(\mu_i f_i)(z_i)$ such that $\|\xi_i - \zeta_i\| < \epsilon$. Hence, by (2.10) we have

$$(\mu_1, ..., \mu_n) \in \partial^F g(\alpha_1, ..., \alpha_n) + \epsilon B_{R^n}$$

and moreover, from (2.9) we derive that

$$x^* \in \zeta_1 + ... + \zeta_n + U \subseteq \xi_1 + ... + \xi_n + n\epsilon B_{X^*} + U \subseteq \xi_1 + ... + \xi_n + V.$$

on the other hand, Combining (2.3), (2.4), (2.5), yields

$$(z_i, f_i(z_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X \times \mathrm{I\!R}} \text{ for all } i = 1, ..., n.$$

The proof is complete. □

3. A NECESSARY OPTIMALITY CONDITION

Let us consider the constrained optimization problem:

$$(\mathfrak{P}) \qquad \begin{aligned} \min f_0(x) &\quad \text{s.t.} \\ f_i(x) \leq 0, &\quad i = 1, ..., m; \\ f_i(x) = 0, &\quad i = m + 1, ..., n; \\ x \in C. & \end{aligned}$$

Suppose that the $f_i : X \to \mathrm{I\!R} \cup \{+\infty\}$ are lower semicontinuous for $i = 1, ..., m$; the $f_i : X \to \mathrm{I\!R} \cup \{+\infty\}$ are continuous for $i = m + 1, ..., n$ and $C \subseteq X$ is a nonempty closed set.

Now, we use the chain rule established in Section 2 to obtain a multiplier rule for problem $(\mathfrak{P})$. In the proof, following Treiman ([21]), we use the function $g : \mathrm{I\!R}^{n+1} \to \mathrm{I\!R}$ given by

$$g(\alpha_0, ..., \alpha_n) := \begin{cases} \max\{\alpha_0, ..., \alpha_n\} & \text{if} \quad \alpha_{m+1} = ... = \alpha_n = 0 \\ \max\{\alpha_0, ..., \alpha_m, |\alpha_{m+1}|, ..., |\alpha_n|\} & \text{otherwise.} \end{cases}$$

We first establish the simple lemma:

Lemma 3.1. *For any* $(\alpha_0, ..., \alpha_n) \in \mathrm{I\!R}^{n+1}$, *then*

$$(\mu_0, ..., \mu_n) \in \partial^F g(\alpha_0, ..., \alpha_n) \Rightarrow \mu_i \geq 0, \ i = 0, ..., m \text{ and } \sum_{i=0}^{n} |\mu_i| \geq 1.$$

Proof. Clearly, g is nondecreasing for each of its first $m+1$ variables. Hence, immediately, $\mu_i \geq 0$ for all $i = 0, ..., m$.

By definition, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sum_{i=0}^{n} \mu_i h_i \leq g(\alpha_0 + h_0, ..., \alpha_n + h_n) - g(\alpha_0, ..., \alpha_n) + \epsilon \max\{|h_0|, ..., |h_n|\} \tag{3.1}$$

$$\text{for all } (h_0, ..., h_n) \in \delta B_{R^{n+1}}.$$

Let $h \in (0, \delta)$. Take $(h_0, ..., h_n)$ in (3.1) such that $h_0 = ... = h_m = -h$, and for $i = m+1, ..., n$, $h_i = -h$ if $\alpha_i > 0$; $h_i = h$ if $\alpha_i < 0$; $h_i = 0$ if $\alpha_i = 0$. Then, when δ is small, we have

$$g(\alpha_0 + h_0, ..., \alpha_n + h_n) - g(\alpha_0, ..., \alpha_n) = -h.$$

Hence, from (3.1),

$$\sum_{i=0}^{n} |\mu_i| h \geq - \sum_{i=0}^{n} \mu_i h_i \geq h(1 - \epsilon).$$

Dividing the latter inequality by h and letting ϵ go to zero, we obtain $\sum_{i=0}^{n} |\mu_i| \geq 1$. The proof is complete. □

We prove the main result:

Theorem 3.2. *Let X be an Asplund space, let C be a closed subset of X. Let functions f_i be lower semicontinuous for $i = 0, ..., m$ and let f_i be continuous for $i = m+1, ..., n$. Assume that $\bar{x}$ is a local solution of ($\mathfrak{P}$). Then for any $\epsilon > 0$, any weak* neighborhood V of 0 in X^*, there exist $(x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X \times \mathbb{R}}$, $i = 0, ..., n$; $x_{n+1} \in \bar{x} + \epsilon B_X$, and $\mu_i > 0$ for $i = 0, ..., m$; $\mu_i \neq 0$ for $i = m+1, ..., n$ such that*

$$|\mu_0| + ... + |\mu_n| = 1,$$

$$0 \in \partial^F(\mu_0 f_0)(x_0) + ... + \partial^F(\mu_n f_n)(x_n) + N^F(C, x_{n+1}) + V.$$

Proof. Observe that if $\bar{x}$ is a local solution of ($\mathfrak{P}$), then $\bar{x}$ is a local minimum point of function $g \circ F + \delta_C$, where $g : \mathbb{R}^{n+1} \to \mathbb{R}$,

$$g(\alpha_0, ..., \alpha_n) := \begin{cases} \max\{\alpha_0, ..., \alpha_n\} & \text{if} \quad \alpha_{m+1} = ... = \alpha_n = 0 \\ \max\{\alpha_0, ..., \alpha_m, |\alpha_{m+1}|, ..., |\alpha_n|\} & \text{otherwise,} \end{cases}$$

and

$$F(x) := (f_0(x) - f_0(\bar{x}), ..., f_m(x) - f_m(\bar{x}), f_{m+1}(x), ..., f_n(x)).$$

Therefore,

$$0 \in \partial^F(g \circ F + \delta_C)(\bar{x}). \tag{3.2}$$

Of course, we only need to consider $\epsilon \in (0, 1)$. Since the functions f_i are lower semicontinuous, there exists $\eta \in (0, \frac{\epsilon}{2})$ such that

$$f_i(x) > f_i(\bar{x}) - \frac{\epsilon}{2} \quad \text{for all} \quad x \in \bar{x} + \eta B_X, \ i = 0, ..., n. \tag{3.3}$$

Let U be a convex weak* neighbourhood of 0 in X^* such that $2U \subseteq V$. By the fuzzy sum rule, applied to (3.2), there exist

$$(y, g \circ F(y)) \in (\bar{x}, g \circ F(\bar{x})) + \eta B_{X \times \mathrm{I\!R}}, \tag{3.4}$$

$x_{n+1} \in \bar{x} + \eta B_X$, $\zeta \in \partial^F(g \circ F)(y)$, $\zeta_{n+1} \in N^F(C, x_{n+1})$ such that

$$0 \in \zeta + \zeta_{n+1} + \frac{U}{2}. \tag{3.5}$$

Combining (3.3) and (3.4), we derive that

$$(y, f_i(y)) \in (\bar{x}, f_i(\bar{x})) + \frac{\epsilon}{2} B_{X \times \mathrm{I\!R}}, \quad i = 0, ..., n. \tag{3.6}$$

On the other hand, since $\zeta \in \partial^F(g \circ F)(y)$, by virtue of Theorem 2.4, there exist

$$(x_i, f_i(x_i)) \in (y, f_i(y)) + \frac{\epsilon}{2} B_{X \times \mathrm{I\!R}}, \quad i = 0, ..., n; \tag{3.7}$$

$$(\alpha_0, ..., \alpha_n) \in (f_0(y), ..., f_n(y)) + \frac{\epsilon}{2} B_{\mathrm{I\!R}^{n+1}}; \tag{3.8}$$

$$(\lambda_0, ..., \lambda_n) \in \partial^F g(\alpha_0, ..., \alpha_n) + \frac{\epsilon}{2} B_{\mathrm{I\!R}^{n+1}}; \tag{3.9}$$

and $\zeta_i \in \partial^F(\lambda_i f_i)(x_i)$, $i = 0, ..., n$ such that $\lambda_i > 0$ for $i = 0, ..., m$; $\lambda_i \neq 0$ for $i = m+1, ..., n$ and

$$\zeta \in \zeta_0 + ... + \zeta_n + \frac{U}{2}. \tag{3.10}$$

¿From (3.5), (3.10), we have

$$0 \in \zeta_0 + ... + \zeta_n + \zeta_{n+1} + U. \tag{3.11}$$

By (3.6), (3.7), we have

$$(x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X \times \mathrm{I\!R}}. \tag{3.12}$$

By Lemma 3.1 and (3.9), we derive that $|\lambda_0| + ... + |\lambda_n| \geq 1 - \frac{\epsilon}{2} > \frac{1}{2}$. Dividing the inclusion (3.11) by $\lambda := \sum_{i=0}^n |\lambda_i| > \frac{1}{2}$, and setting $\mu_i := \lambda_i/\lambda$; $\xi_i := \zeta_i/\lambda \in \partial^F(\mu_i f_i)(x_i)$, $i = 0, ..., n$; $\xi_{n+1} = \zeta_{n+1}/\lambda$, we obtain $\mu_i > 0$ for $i = 0, ..., m$; $\mu_i \neq 0$ for $i = m+1, ..., n$; $\sum_{i=0}^n |\mu_i| = 1$, and

$$0 \in \sum_{i=0}^{n} \xi_i + \xi_{n+1} + U/\lambda \subseteq \sum_{i=0}^{n} \partial^F(\mu_i f_i)(x_i) + N^F(C, x_{n+1}) + V.$$

The proof is complete. □

4. The normal cone to a level set

In [1], was established in the reflexive setting,the relationship between the normal cone to a level set and the subderivative of the corresponding function. This relation was the key ingredient of the proof of the main result in [1]. This relationship is of some independent interest, however in the final section, we prove that it also holds true for Fréchet subdifferentials and Fréchet normals in an Asplund setting.

Theorem 4.1. *Let X be an Asplund space, let $f : X \to \mathrm{I\!R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $S := \{x \in X : f(x) \leq 0\}$, and let $\bar{x} \in S$. Then, either*

(A1) for any $\epsilon, \eta > 0$, there exists $(x, f(x)) \in (\bar{x}, f(\bar{x})) + \eta B_{X\times \mathrm{I\!R}}$ such that $\partial^F f(x) \cap \epsilon B_{X^*} \neq \emptyset$,

or

(A2) for any $\xi \in N^F(S, \bar{x})$, any $\epsilon > 0$, there exist $(x, f(x)) \in (\bar{x}, f(\bar{x})) + \epsilon B_{X\times \mathrm{I\!R}}$; $\zeta \in \partial^F f(x)$ and $\lambda > 0$ such that $\|\lambda\zeta - \xi\| < \epsilon$.

We first recall the following two lemmata from [10], [17]. Denote by $d_C(.)$ the distance function to a set C.

Lemma 4.2. (Jourani-Thibault [10]) *Let C be a nonempty closed subset of X and let $x_0 \notin C$. Then the implication*

$$x^* \in \partial^F d_C(x_0) \Rightarrow \|x^*\| = 1,$$

holds for any Banach space X. The following lemma is proved in [17]. It follows Thibault [20].

Lemma 4.3. *Let $C \subset X$ be a nonempty closed set.*

(i). Let $\bar{x} \in C$ and let $x^* \in N^F(C, \bar{x})$. Then $x^* \in \lambda\partial^F d_C(\bar{x})$ holds for any $\lambda \geq \|x^*\| + 1$, and any Banach space X.

(ii) Let X be an Asplund space and let $x \in X$. If $x^* \in \partial^F d_C(\bar{x})$, then for any $\epsilon \in (0, 1)$, there exist $x_\epsilon \in C$ and $x^*_\epsilon \in N^F(C, x_\epsilon)$ such that

$$\|x_\epsilon - \bar{x}\| < d_C(\bar{x}) + \epsilon \quad \text{and} \quad \|x^*_\epsilon - x^*\| \leq \epsilon.$$

Proof of Theorem 4.1. We set: $S_1 := \big\{(x, \alpha) \in X \times \mathrm{I\!R} : \alpha \leq 0\big\}$; $S_2 := \mathrm{epi} f$. Clearly,

$$\partial^F d_{S_1}(x, \alpha) = \begin{cases} \{(0,0)\} & \text{if} \quad \alpha < 0 \\ \{0\} \times [0,1] & \text{if} \quad \alpha = 0 \\ \{(0,1)\} & \text{if} \quad \alpha > 0. \end{cases}$$

We consider the following two cases:

Case 1. There exists a sequence $(x_n, \alpha_n) \to (\bar{x}, f(\bar{x}))$ such that

$$r_n := d_{S_1\cap S_2}(x_n, \alpha_n) > n\big(d_{S_1}(x_n, \alpha_n) + d_{S_2}(x_n, \alpha_n)\big),\ n = 1, 2, ...$$

Thus,

$$d_{S_1}(x_n, \alpha_n) + d_{S_2}(x_n, \alpha_n) < \frac{r_n}{n} \leq \min_{X\times \mathrm{I\!R}} \big(d_{S_1}(.) + d_{S_2}(.)\big) + \frac{r_n}{n}.$$

By the Ekeland variational principle, for every $n = 1, 2, ...$, there exists a point $(z_n, \beta_n) \in X \times 1\mathrm{R}$ such that $\|(z_n, \beta_n) - (x_n, \alpha_n)\| < r_n$ and (z_n, β_n) is a minimizer of the function

$$g_n(x, \alpha) := d_{S_1}(x, \alpha) + d_{S_2}(x, \alpha) + \frac{1}{n}\|(x, \alpha) - (z_n, \beta_n)\|.$$

Therefore,

$$(0, 0) \in \partial^F g_n(z_n, \beta_n), \quad n = 1, 2, ... \tag{4.1}$$

Observe that $(z_n, \beta_n) \notin S_1 \cap S_2$ (if $(z_n, \beta_n) \in S_1 \cap S_2$, then $\|(x_n, \alpha_n) - (z_n, \beta_n)\| \geq d_{S_1 \cap S_2}(x_n, \alpha_n) = r_n$, a contradiction). Since S_1 and S_2 are closed, there exists $\delta_n > 0$ such that either $\big((z_n, \beta_n) + \delta_n B_{X \times 1\mathrm{R}}\big) \cap S_1 = \emptyset$ or $\big((z_n, \beta_n) + \delta_n B_{X \times 1\mathrm{R}}\big) \cap S_2 = \emptyset$, and $\delta_n \to 0$ as $n \to \infty$.

Now, using the fuzzy sum rule in (4.1) (note that d_{S_1} and d_{S_2} are Lipschitz), there exist $(u_n, a_n) \in (z_n, \beta_n) + \delta_n B_{X \times 1\mathrm{R}}$; $(v_n, b_n) \in (z_n, \beta_n) + \delta_n B_{X \times 1\mathrm{R}}$; $(0, a_n^*) \in \partial^F d_{S_1}(u_n, a_n)$; $(\zeta_n, -b_n^*) \in \partial^F d_{S_2}(v_n, b_n)$ such that

$$(0, 0) \in (0, a_n^*) + (\zeta_n, -b_n^*) + \frac{2}{n} B_{X^* \times 1\mathrm{R}}. \tag{4.2}$$

Thus, either $(u_n, a_n) \notin S_1$ or $(v_n, b_n) \notin S_2$. By Lemma 4.2, then either $|a_n^*| = 1$ (since $a_n^* \geq 0$, this means that $a_n^* = 1$) or $\|(\zeta_n, -b_n^*)\| = 1$. Combining this observation and (4.2), we derive that

$$\zeta_n \to 0 \quad \text{and} \quad b_n^* \to 1 \text{ as } n \to \infty. \tag{4.3}$$

On the other hand, $(\zeta_n, -b_n^*) \in \partial^F d_{S_2}(v_n, b_n)$. Using Lemma 4.3 we derive the existence of $(y_n, \lambda_n) \in S_2$; $(\xi_n, -\mu_n) \in N^F(S_2, (y_n, \lambda_n))$ such that

$$\|(y_n, \lambda_n) - (v_n, b_n)\| < d_{S_2}(v_n, b_n) + \frac{1}{n}; \ \|(\xi_n, -\mu_n) - (\zeta_n, -b_n^*)\| < \frac{2}{n}.$$

Hence, $(y_n, f(y_n)) \to (\bar{x}, f(\bar{x}))$, and by (4.3), $\mu_n \to 1$; $\xi_n \to 0$. When n is large, then $\mu_n > 0$. Since $(\xi_n, -\mu_n) \in N^F(\mathrm{epi} f, (y_n, \lambda_n))$, using Lemma 2.2, we obtain

$$\xi_n / \mu_n \in \partial^F f(y_n).$$

¿From the above, we derive that, for any $\epsilon > 0$, $\eta > 0$, when n is large enough, we have

$$(y_n, f(y_n)) \in (\bar{x}, f(\bar{x})) + \eta B_{X \times 1\mathrm{R}} \quad \text{and} \quad \xi_n / \mu_n \in \partial^F f(y_n) \cap \epsilon B_{X^*}.$$

We obtain (A1).

Case 2. There are $a > 0$, $r > 0$ such that

$$d_{S_1 \cap S_2}(x, \alpha) \leq a\big(d_{S_1}(x, \alpha) + d_{S_2}(x, \alpha)\big) \text{ for all } (x, \alpha) \in (\bar{x}, f(\bar{x})) + r B_{X \times 1\mathrm{R}}.$$

Fix $\xi \in N^F(S, \bar{x})$. Clearly, $(\xi, 0) \in N^F(S_1 \cap S_2, (\bar{x}, f(\bar{x})))$. By Lemma 4.3,

$$(\xi, 0) \in b\partial^F d_{S_1 \cap S_2}(\bar{x}, f(\bar{x})) \quad \text{for some} \quad b > \|\xi\| + 1.$$

Therefore,

$$(\xi,0)\in\kappa\partial^F\big(d_{S_1}(.)+d_{S_2}(.)\big)(\bar{x},f(\bar{x})),\quad \text{where,}\quad \kappa:=ab. \tag{4.4}$$

Since f is lower semicontinuous, there exists $\eta\in(0,\frac{\epsilon}{2})$ such that

$$f(x)>f(\bar{x})-\frac{\epsilon}{2}\quad\text{for all}\quad x\in\bar{x}+\eta B_X. \tag{4.5}$$

Now, apply the fuzzy sum rule to (4.4), to obtain the existence of (u,α), $(v,\beta)\in(\bar{x},f(\bar{x}))+\frac{\eta}{4}B_{X\times\mathrm{I\!R}}$; $(0,\gamma)\in\partial^F d_{S_1}(u,\alpha)$; $(v^*,\lambda)\in\partial^F d_{S_2}(v,\beta)$ such that

$$(\xi,0)\in\kappa\big((0,\gamma)+(v^*,\lambda)\big)+\frac{\epsilon}{4}B_{X^*\times\mathrm{I\!R}}.$$

Thus,

$$\xi\in\kappa v^*+\frac{\epsilon}{4}B_{X^*}. \tag{4.6}$$

Since $(v^*,\lambda)\in\partial^F d_{S_2}(v,\beta)$, by Lemma 4.3, there exist $(z,\nu)\in S_2:=\mathrm{epi}f$; $(z^*,-\nu^*)\in N^F(S_2,(z,\nu))$ such that

$$\|(z,\nu)-(v,\beta)\|\le d_{S_2}(v,\beta)+\frac{\eta}{2}; \tag{4.7}$$

$$\|(v^*,\lambda)-(z^*,-\nu^*)\|<\frac{\epsilon}{4\kappa}. \tag{4.8}$$

Combining (4.5) and (4.7), we obtain $(z,f(z))\in(\bar{x},f(\bar{x}))+\frac{\epsilon}{2}B_{X\times\mathrm{I\!R}}$. ¿From (4.6) and (4.8) we derive that $\|\kappa z^*-\xi\|<\frac{\epsilon}{2}$. Next, we apply Lemma 2.2 to $(z^*,-\nu^*)\in N^F(S_2,(z,\nu))$, to obtain:

If $\nu^*\neq0$, then $\zeta:=z^*/\nu^*\in\partial^F f(z)$, and $\|\lambda\zeta-\xi\|<\epsilon$. Here, $\lambda=\kappa\nu^*$. We obtain (A2). Else, there exist

$$(y,f(y))\in(z,f(z))+\frac{\epsilon}{2}B_{X\times\mathrm{I\!R}}\subseteq(\bar{x},f(\bar{x}))+\epsilon B_{X\times\mathrm{I\!R}};$$

and $t>0$; $\zeta:=y^*/t\in\partial^F f(y)$ such that $\|y^*-z^*\|<\frac{\epsilon}{2\kappa}$. Therefore, $\|\lambda\zeta-\xi\|<\epsilon$, where $\lambda=\kappa t$. We again obtain (A2). The proof is complete. □

Theorem 4.4. *Let X be an Asplund space, let $f:X\to\mathrm{I\!R}\cup\{+\infty\}$ be a continuous function. Let $S:=\{x\in X: f(x)=0\}$, and let $\bar{x}\in S$. Then, either*

(B1). for any ϵ, $\eta>0$, there exists $(x,f(x))\in(\bar{x},f(\bar{x}))+\eta B_{X\times\mathrm{I\!R}}$ such that $\big[\partial^F f(x)\cup\partial^F(-f)(x)\big]\cap\epsilon B_{X^*}\neq\emptyset$,

or

(B2). for any $\xi\in N^F(S,\bar{x})$, any $\epsilon>0$, there exist $(x,f(x))\in(\bar{x},f(\bar{x}))+\epsilon B_{X\times\mathrm{I\!R}}$; $\zeta\in\big[\partial^F f(x)\cup\partial^F(-f)(x)\big]$ and $\lambda>0$ such that $\|\lambda\zeta-\xi\|<\epsilon$.

Proof. Just mimic the proof of Theorem 4.1, but with

$$S_1:=\big\{(x,\alpha)\in X\times\mathrm{I\!R}:\alpha=0\big\}\quad\text{and}\quad S_2:=\mathrm{gph}f,$$

and apply Lemma 2.3 instead of Lemma 2.2. □

We can use Theorem 4.1 and Theorem 4.4 and the method in [1] to establish a fuzzy multiplier rule for problem $(\mathfrak{P})$ in an Asplund space setting. However, this method gives a result which is weaker than the one established in Theorem 3.2.

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