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ABSTRACT. In this paper, we establish characterizations of Asplund spaces in terms of conditions ensuring the metric inequality and intersection formulae. Then we establish chain rules for the limiting Fréchet subdifferentials. Necessary conditions for constrained optimization problems with non-Lipschitz data are derived.

1. Introduction

Let $f_1, f_2, ..., f_n$ be functions from a Banach space X to the extended real line $\mathbb{R} \cup \{+\infty\}$ and let g be a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$. We consider the following composite function:

$$g[f_1, ..., f_n] : X \to \mathbb{R} \cup \{+\infty\},$$

$$g[f_1, ..., f_n](x) := \begin{cases} g(f_1(x), ..., f_n(x)) & \text{if} \quad x \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n \\ +\infty & \text{otherwise.} \end{cases}$$

The purpose of the present paper is to establish calculus rules for the limiting Fréchet subdifferential of the above composite function in terms of subdifferentials of the functions entering in the composition. This is done when the f_i's are lower semicontinuous or continuous. Note that chain rules for composite functions subsume most of the basic calculus rules such as sum, product quotient, formulae for maxima, etc. A chain rule using the Mordukhovich coderivative has been given in [26], [29]. When the f_i's are Lipschitz, a chain rule for the Clarke subdifferential has been established in [3] and in [10] for the Ioffe geometric subdifferential. In this paper we use the metric inequality as a main tool to establish chain rules. This method has been previously used by Ioffe [10] for G-subdifferential and then by Jourani & Thibault [18],[19] and Jourani [14],$\cdots$. As an application of these new calculus rules we derive necessary conditions for constrained minimization problems with lower semicontinuous and continuous data in terms of limiting Fréchet subdifferentials.

The paper is organized as follows. Section 2 deals with notations and basic properties of Fréchet subdifferentials. In this sections we present a representation

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of the normal cone to the graph of a function in terms of the subdifferential of the corresponding function. In Section 3, we establish a condition ensuring the metric inequality. We show that this condition characterizes also the Asplund property. Section 4 is devoted to the limiting Fréchet subdifferential of composite function and to its corollaries. In the final section, we apply the calculus rules established in Section 4 to derive necessary optimality conditions for a non-Lipschitz constrained optimization problem.

2. Basic definitions and representations

Let X be a Banach space with closed unit ball B_X and with dual space X^*. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. We denote as usual by $\operatorname{dom} f := \{x \in X : f(x) < +\infty\}$, $\operatorname{epi} f := \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq f(x)\}$, $\operatorname{gph} f := \{(x, \alpha) \in X \times \mathbb{R} : \alpha = f(x)\}$, the *domain*, the *epigraph* and the *graph* of f, respectively.

Recall that the *Fréchet subdifferential* of f at $x \in \operatorname{dom} f$ is defined by

$$\partial^F f(x) := \left\{ x^* \in X^* : \liminf_{h \to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq 0 \right\}. \tag{2.1}$$

If $x \notin \operatorname{dom} f$, we set $\partial^F f(x) = \emptyset$.

The *limiting Fréchet subdifferential* and the *limiting singular subdifferential* of f at $x \in \operatorname{dom} f$ are the sets defined respectively by ([13],[29]):

$$\hat{\partial} f(x) := \left\{ x^* \in X^* : \exists x_n \xrightarrow{f} x,\, x_n^* \xrightarrow{w^*} x^* \quad \text{with} \quad x_n^* \in \partial^F f(x_n) \right\}; \tag{2.2}$$

(2.3)

$$\partial^\infty f(x) := \left\{ x^* \in X^* : \exists x_n \xrightarrow{f} x, t_n \downarrow 0^+, t_n x_n^* \xrightarrow{w^*} x^* \text{ with } x_n^* \in \partial^F f(x_n) \right\}.$$

Throughout the paper, the symbol $x_n \xrightarrow{f} x$ means that $(x_n, f(x_n)) \to (x, f(x))$, and " $\xrightarrow{w^*}$ " denotes the convergence with respect to the weak*-topology.

It is well-known that if f is Lipschitz at x, then $\partial^\infty f(x) = \{0\}$. Therefore, this construction makes sense only for non-Lipschitz functions.

For a closed subset C of X, the *Fréchet normal cone* and the *limiting Fréchet normal cone* to C at $x \in C$ are the sets defined respectively by:

$$N^F(C, x) := \partial^F \delta_C(x) = \left\{ x^* \in X^* : \limsup_{y \xrightarrow{C} x} \frac{\langle x^*, y - x\rangle}{\|y - x\|} \leq 0 \right\}; \tag{2.4}$$

$$\hat{N}(C, x) := \hat{\partial}\delta_C(x) = \delta_C^\infty(x). \tag{2.5}$$

Here, $\delta_C(.)$ stands for the *indicator function* of C defined by $\delta_C(x) = 0$ if $x \in C$ and $+\infty$, otherwise, while $y \xrightarrow{C} x$ means that $y \to x$ and $y \in C$.

As already observed by the authors of [13], [29] a nice framework to develop calculus rules for the limiting Fréchet subdifferential is the class of *Asplund spaces*, that is Banach spaces in which every convex lower semicontinuous function is Fréchet differentiable on a dense $G_\delta-$subset of the interior of its domain. An important charaterization of Asplund spaces is the *fuzzy sum rule* for Fréchet subdifferentials proved by Fabian ([6], [7]) (see also other charaterizations in Mordukhovich & Shao [28], Fabian & Modukhovich [8], [9], Mordukhovich & Wang [32] and Jourani [15]):

Assume that X be an Asplund space and $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., n$ are lower semicontinuous functions all but one of which is Lipschitz at $\bar{x} \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n$. Then for each $\epsilon > 0$, one has

$$\partial^F (f_1 + ... + f_n)(\bar{x}) \subseteq \bigcup \Big\{ \partial^F f_1(x_1) + ... + \partial^F f_n(x_n) + \epsilon B_{X^*} : (x_i, f_i(x_i)) \in (\bar{x}, f_i(\bar{x})) + \epsilon B_{X \times \mathbb{R}},\ i = 1, ..., n \Big\}.$$

The representations of the normal cones (2.4), (2.5) to epi$f$ as well as the related subdifferential constructions (2.1), (2.2) and (2.3) for lower semicontinuous functions were investigated in Kruger [22], Ioffe [11], Mordukhovich & Shao [29].

Proposition 2.1 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and let $(\bar{x}, \alpha) \in \operatorname{epi} f$.*

(i) Let $\lambda \neq 0$; the following equivalences are true in any Banach space X:

$$(2.6) \qquad (x^*, -\lambda) \in N^F(\operatorname{epi} f, (\bar{x}, \alpha)) \iff \lambda > 0,\ \alpha = f(\bar{x}),\ x^* \in \partial^F(\lambda f)(\bar{x});$$

$$(2.7) \qquad (x^*, -\lambda) \in \hat{N}(\operatorname{epi} f, (\bar{x}, f(\bar{x}))) \iff \lambda > 0,\ x^* \in \hat{\partial}(\lambda f)(\bar{x}).$$

(ii) Suppose that X is an Asplund space. If $(x^*, 0) \in N^F(\operatorname{epi} f, (\bar{x}, \alpha))$, then there exist sequences $\{x_n\}_{n \in \mathbb{N}}$, $\{x_n^*\}_{n \in \mathbb{N}}$, $\{\lambda_n\}_{n \in \mathbb{N}}$ such that

$$(2.8) \qquad x_n^* \in \lambda_n \partial^F f(x_n),\ x_n \xrightarrow{f} \bar{x},\ \lambda_n \downarrow 0^+,\ \|x_n^* - x^*\| \to 0.$$

As a result,

$$(2.9) \qquad (x^*, 0) \in \hat{N}(\operatorname{epi} f, (\bar{x}, f(\bar{x}))) \iff x^* \in \partial^\infty f(\bar{x}).$$

For a continuous function f, we have representations of the normal cones (2.4) and (2.5) to the graph of f in terms of subdifferentials (2.1), (2.2) and (2.3) as follows. Assertion (i) was established by Kruger [22]. We include its proof for sake of completness.

Proposition 2.2 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a continuous function and let $\bar{x} \in \operatorname{dom} f$.*

(i) Let $\lambda \neq 0$; the following equivalences are true in any Banach space X :

$$(2.10) \qquad (x^*, -\lambda) \in N^F(\operatorname{gph} f, (\bar{x}, f(\bar{x})) \iff x^* \in \partial^F(\lambda f)(\bar{x});$$

$$(x^*, -\lambda) \in \hat{N}(\operatorname{gph} f, (\bar{x}, f(\bar{x}))) \iff x^* \in \hat{\partial}(\lambda f)(\bar{x}). \tag{2.11}$$

(ii) Suppose that X is an Asplund space. If $(x^*, 0) \in N^F(\operatorname{gph} f, (\bar{x}, f(\bar{x}))$, then there exist sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{x_n^*\}_{n\in\mathbb{N}}$, $\{\lambda_n\}_{n\in\mathbb{N}}$ such that

$$x_n^* \in \partial^F(\lambda_n f)(x_n) \cup \partial^F(-\lambda_n f)(x_n),\ x_n \xrightarrow{f} \bar{x},\ \lambda_n \downarrow 0^+,\ \|x_n^* - x^*\| \to 0. \tag{2.12}$$

As a result,

$$(x^*, 0) \in \hat{N}(\operatorname{gph} f, (\bar{x}, f(\bar{x}))) \iff x^* \in \partial^\infty f(\bar{x}) \cup \partial^\infty(-f)(\bar{x}). \tag{2.13}$$

Proof. The proof of (i) follows from the definition, while proof of (ii) is inspired by Ioffe's techniques in [11], which are based on Ekeland's variational principle.

Proof of (i) We prove (2.10). Let $(x^*, -\lambda) \in N^F(\operatorname{gph} f, (\bar{x}, f(\bar{x}))$ with $\lambda \neq 0$. By definition, for each $\epsilon > 0$, there is $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle - \langle \lambda, f(x) - f(\bar{x})\rangle \le \epsilon(\|x - \bar{x}\| + |f(x) - f(\bar{x})| \text{ for all } x \in \bar{x} + \delta B_X.$$

Suppose that $\lambda > 0$, (the case of $\lambda < 0$ is similar). We may assume that $\epsilon < \lambda$. For any $x \in \bar{x} + \delta B_X$, $(x, \alpha) \in \operatorname{epi} f$, from the above inequality, we derive that

$$\begin{aligned}\langle x^*, x - \bar{x}\rangle - \langle \lambda, \alpha - f(\bar{x})\rangle &\le -\langle \lambda, \alpha - f(x)\rangle + \epsilon(\|x - \bar{x}\| + |f(x) - f(\bar{x})|)\\ &\le \epsilon(\|x - \bar{x}\| + |\alpha - f(\bar{x})|).\end{aligned}$$

This yields $(x^*, -\lambda) \in N^F(\operatorname{epi} f, (\bar{x}, f(\bar{x}))$. According to Proposition 2.2, we obtain $x^* \in \partial^F(\lambda f)(\bar{x})$. The converse of (2.10) is trivial, hence (2.10) is proved.

To prove (2.11), take $(x^*, -\lambda) \in \hat{N}(\operatorname{gph} f, (\bar{x}, f(\bar{x}))$ and select sequences $x_n \xrightarrow{f} \bar{x}$, $\lambda_n \to \lambda$, $(x_n^*, -\lambda_n) \in N^F(\operatorname{gph} f, (x_n, f(x_n))$ such that $(x_n^*, -\lambda_n) \xrightarrow{w^*} (x^*, -\lambda)$. Due to (2.10), $x_n^* \in \partial^F(\lambda_n f)(x_n)$ (when n is large). If $\lambda > 0$, then $\lambda_n > 0$ when n is large. Therefore, $x_n^*/\lambda_n \in \partial^F f(x_n)$; hence $x^*/\lambda \in \hat{\partial} f(\bar{x})$, equivalently, $x^* \in \hat{\partial}(\lambda f)(\bar{x})$. Otherwise, suppose that $\lambda < 0$. Then $x_n^*/(-\lambda_n) \in \partial^F(-f)(x_n)$. Hence $x^*/(-\lambda) \in \hat{\partial}(-f)(\bar{x})$, and we also have $x^* \in \hat{\partial}(\lambda f)(\bar{x})$. Therefore (2.11) is proved.

Proof of (ii) We prove (2.12). Without loss of generality, we may assume that $\bar{x} = 0$, $f(\bar{x}) = 0$. Let $(x^*, 0) \in N^F(\operatorname{gph} f, (\bar{x}, f(\bar{x}))$. For each $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that

$$\langle x^*, x\rangle < \epsilon(\|x\| + |f(x)|) \quad \text{for all} \quad x \in \delta_\epsilon B_X,\ x \neq 0; \tag{2.14}$$

$\delta_\epsilon \to 0$ as $\epsilon \to 0$ and f is bounded on $\delta_\epsilon B_X$.

Define $K_\epsilon := \{x \in X : \langle x^*, x\rangle \ge \epsilon\|x\|\}$. Observe that if $x \in K_\epsilon$ and $x \neq 0$, then $\lambda x \notin K_\epsilon$, $\forall \lambda < 0$, and either

$$(a)\ f(x) \ge 0 \quad \text{for all} \quad x \in K_\epsilon \cap \delta_\epsilon B_X,$$

or

$$(b)\ f(x) \le 0 \quad \text{for all} \quad x \in K_\epsilon \cap \delta_\epsilon B_X.$$

Indeed, if there are $x_1,\ x_2 \in K_\epsilon \cap \delta_\epsilon B_X$ such that $f(x_1) > 0$ and $f(x_2) < 0$, then $x_1,\ x_2 \neq 0$ and since f is continuous, there exists $z := \lambda x_1 + (1-\lambda)x_2$ with $\lambda \in (0,1)$ such that $f(z) = 0.$, Clearly, $z \in K_\epsilon \cap \delta_\epsilon B_X$ and $z \neq 0$. On the other hand, from (2.14), then $\langle x^*, z\rangle < \epsilon\|z\|$. That is, $z \notin K_\epsilon$, a contradiction.

For each $n \in \mathbb{N}$, if $d_{K_{2\epsilon}}(.)$ denotes the distance function to $K_{2\epsilon}$, consider the mapping $g_{n,\epsilon}$ defined by:

$$g_{n,\epsilon}(x) := \begin{cases} \epsilon f(x) + n d_{K_{2\epsilon}}(x) - \langle x^*, x\rangle + 2\epsilon\|x\| & \text{if} \quad (a) \\ -\epsilon f(x) + n d_{K_{2\epsilon}}(x) - \langle x^*, x\rangle + 2\epsilon\|x\| & \text{if} \quad (b). \end{cases}$$

Since f is continuous and is bounded on $\delta_\epsilon B_X$, then the functions $g_{n,\epsilon}$ are also continuous, and they are bounded on $\delta_\epsilon B_X$. By the Ekeland variational principle ([5]), for every $n = 1, 2, ...$, there exists a point $u_{n,\epsilon} \in \delta_\epsilon B_X$ minimizing the function $g_{n,\epsilon} + \frac{1}{n}\|. - u_{n,\epsilon}\|$ on $\delta_\epsilon B_X$. In particular, one has

$$g_{n,\epsilon}(u_{n,\epsilon}) \le g_{n,\epsilon}(0) = \frac{1}{n}\|u_{n,\epsilon}\|, \tag{2.15}$$

and therefore,

$$\lim_{n\to\infty} d_{K_{2\epsilon}}(u_{n,\epsilon}) = 0. \tag{2.16}$$

If $u_{n,\epsilon} \in K_\epsilon$, from (2.14), we derive that $g_{n,\epsilon}(u_{n,\epsilon}) \ge \epsilon\|u_{n,\epsilon}\|$. This together with (2.15) yield $u_{n,\epsilon} = 0$ when $n > \frac{1}{\epsilon}$; otherwise, suppose that $u_{n,\epsilon} \notin K_\epsilon$, i.e., $\langle x^*, u_{n,\epsilon}\rangle < \epsilon\|u_{n,\epsilon}\|$. Using Ioffe's argument developed in [11], we obtain

$$d_{K_{2\epsilon}}(u_{n,\epsilon}) \ge \frac{\epsilon}{1+2\epsilon}\|u_{n,\epsilon}\|.$$

Combining this inequality and (2.16), we derive that $\lim_{n\to\infty}\|u_{n,\epsilon}\| = 0$. Hence, for every $\epsilon > 0$, we can choose an index n_ϵ such that $\|u_{n_\epsilon}\| < \delta_\epsilon$ and $n_\epsilon \to \infty$ as $\epsilon \to 0$. Thus,

$$0 \in \partial^F\big(g_{n_\epsilon} + \frac{1}{n_\epsilon}\|. - u_{n_\epsilon}\|\big)(u_{n_\epsilon}).$$

According to the fuzzy sum rule, we can select $y_{n_\epsilon} \in \delta_\epsilon B_X$, $z_{n_\epsilon} \in \delta_\epsilon B_X$, $\zeta_{n_\epsilon} \in \partial^F f(y_{n_\epsilon}) \cup \partial^F(-f)(y_{n_\epsilon})$, $\xi_{n_\epsilon} \in \partial^F d_{K_{2\epsilon}}(z_{n_\epsilon})$ such that

$$\|\epsilon\zeta_{n_\epsilon} + n_\epsilon\xi_{n_\epsilon} - x^*\| \le 2(\epsilon + 1/n_\epsilon). \tag{2.17}$$

Note that $K_{2\epsilon}$ is a convex set and observe by a standard argument used in convex analysis that

$$\partial^F d_{K_{2\epsilon}}(.) \subseteq \{\alpha(-x^* + 2\epsilon B_{X^*}) : \alpha \ge 0\} \cap B_{X^*}.$$

Hence, there exists $\alpha_\epsilon \ge 0$ and $b^* \in B^*$ such that $\xi_{n_\epsilon} + \alpha_\epsilon x^* = 2\alpha_\epsilon\epsilon b^*$. Since $\|\xi_{n_\epsilon}\| \le 1$ (because $d_{K_{2\epsilon}}(.)$ is nonexpansive), $\{\alpha_\epsilon\}_\epsilon$ is bounded as $\epsilon \to 0$. From (2.17), we derive that

$$\|\epsilon\zeta_{n_\epsilon} - (n\alpha_\epsilon + 1)x^*\| \le \|\epsilon\zeta_{n_\epsilon} + n_\epsilon\xi_{n_\epsilon} - x^*\| + n\|\xi_{n_\epsilon} + \alpha_\epsilon x^*\| \le 2(\epsilon + 1/n_\epsilon) + 2n\alpha_\epsilon\epsilon.$$

Dividing this inequality by $n_\epsilon\alpha_\epsilon + 1$, and denoting $\lambda_\epsilon := \frac{\epsilon}{n_\epsilon\alpha_\epsilon+1}$ and $x^*_\epsilon := \lambda_\epsilon\zeta_{n_\epsilon}$, we obtain $x^*_\epsilon \in \partial^F(\lambda_\epsilon f)(y_{n_\epsilon}) \cup \partial^F(-\lambda_\epsilon f)(y_{n_\epsilon})$; $\|x^*_\epsilon - x^*\| \to 0$; $\lambda_\epsilon \downarrow 0^+$ as $\epsilon \to 0$. This completes the proof of (2.12).

Let us prove (2.13). The part " $\Leftarrow$ " is obvious. For the part " $\Rightarrow$ ", let $(x^*, 0) \in \hat{N}(\mathrm{gph}\, f, (\bar{x}, f(\bar{x}))$. There are sequences $x_n \xrightarrow{f} \bar{x}$; $(x^*_n, -\lambda_n) \in N^F(\mathrm{gph}\, f, (x_n, f(x_n))$ such that $(x^*_n, -\lambda_n) \xrightarrow{w^*} (x^*, 0)$.

If there exists a subsequence of $\{\lambda_n\}_{n\in\mathbb{N}}$ consisting all either of positive numbers or of negative numbers, we may assume that either $\lambda_n > 0$ for all n or $\lambda_n < 0$ for all n. By (2.11), we have $x^*_n \in \partial^F(\lambda_n f)(x_n)$. Hence $x^* \in \partial^\infty f(\bar{x}) \cup \partial^\infty(-f)(\bar{x})$. Otherwise, suppose that $\lambda_n = 0$ for all $n \in \mathbb{N}$. From (2.12), pick sequences $\gamma_n \downarrow 0^+$; $u_n \to \bar{x}$; $u^*_n \in \partial^F(\gamma_n f)(u_n) \cup \partial^F(-\gamma_n f)(u_n)$ such that $u^*_n \xrightarrow{w^*} x^*$. Again this implies that $x^* \in \partial^\infty f(\bar{x}) \cup \partial^\infty(-f)(\bar{x})$ establishing (2.13). The proof is complete. □

3. Metric inequality and intersection formulae

Let $S_i,\ i = 1, 2, ..., n$ be closed subsets of X. Recall ([10]) that the sets S_i satisfy the *metric inequality* ($\mathfrak{MI}$) for short) at $\bar{x} \in S_1 \cap ... \cap S_n$, if there are $a > 0$, $r > 0$ such that

$$(\mathfrak{MI}) \qquad d_{S_1\cap...\cap S_n}(x) \le a(d_{S_1}(x) + ... + d_{S_n}(x)) \quad \text{for all} \quad x \in \bar{x} + rB_X.$$

Here, $d_C(.)$ is the distance function to the set C. First, we establish the following result, which is the main tool in this section.

Theorem 3.1 *Let X be an Asplund space and let $S_1, ..., S_n$ be non-empty closed subsets of X with $S_1 \cap ... \cap S_n$ non-empty. Let $\bar{x} \notin S_1 \cap ... \cap S_n$. Then one has*

$$(3.1) \qquad m d_{S_1\cap...\cap S_n}(\bar{x}) \le d_{S_1}(\bar{x}) + ... + d_{S_n}(\bar{x}) \le M d_{S_1\cap...\cap S_n}(\bar{x}),$$

where

$$m = \inf\left\{ \|x^*\| : \, x^* \in \partial^F\big(d_{S_1} + ... + d_{S_n}\big)(x); \|x - \bar{x}\| < d_{s_1\cap...\cap S_n}(\bar{x}) \right\}$$

and

$$M = \sup\left\{ \|x^*\| : \, x^* \in \partial^F\big(d_{S_1} + ... + d_{S_n}\big)(x); \|x - \bar{x}\| < d_{s_1\cap...\cap S_n}(\bar{x}) \right\}.$$

Proof. We prove the first inequality of (3.1). Consider the function $f : X \to \mathbb{R}$ defined by $f(x) = d_{S_1}(x) + ... + d_{S_n}(x)$. Obviously,

$$f(\bar{x}) = \min_{x\in X} f(x) + d_{S_1\cap...\cap S_n}(\bar{x}) \frac{d_{S_1}(\bar{x}) + ... + d_{S_1}(\bar{x})}{d_{S_1\cap...\cap S_n}(\bar{x})}.$$

By the Ekeland variational principle, there exists $z \in X$ with $\|z - \bar{x}\| < d_{S_1\cap...\cap S_n}(\bar{x})$ such that the function

$$f(.) + \frac{d_{S_1}(\bar{x}) + ... + d_{S_1}(\bar{x})}{d_{S_1\cap...\cap S_n}(\bar{x})} \|. - z\|$$

attains a minimum at z. Consequently

$$0 \in \partial^F\Big(f(.) + \frac{d_{S_1}(\bar{x}) + ... + d_{S_1}(\bar{x})}{d_{S_1\cap...\cap S_n}(\bar{x})}\|. - z\|\Big)(z).$$

For each $\epsilon > 0$, apply the fuzzy sum rule to derive the existence of $u \in X$ and $u^* \in X^*$ such that $\|u - z\| < d_{S_1\cap...\cap S_n}(\bar{x}) - \|z - \bar{x}\|$; $u^* \in \partial^F(d_{S_1} + ... + d_{S_n})(u)$ and

$$-u^* \in \left(\frac{d_{S_1}(\bar{x}) + ... + d_{S_1}(\bar{x})}{d_{S_1\cap...\cap S_n}(\bar{x})} + \epsilon\right) B^*.$$

Therefore,

$$\|u - \bar{x}\| < d_{S_1\cap...\cap S_n}(\bar{x})$$

and

$$\|u^*\| d_{S_1\cap...\cap S_n}(\bar{x}) \le d_{S_1}(\bar{x}) + ... + d_{S_n}(\bar{x}) + \epsilon d_{S_1\cap...\cap S_n}(\bar{x}).$$

Thus

$$m d_{S_1\cap...\cap S_n}(\bar{x}) \le d_{S_1}(\bar{x}) + ... + d_{S_n}(\bar{x}) + \epsilon d_{S_1\cap...\cap S_n}(\bar{x}).$$

As ϵ is arbitrary, we obtain

$$m d_{S_1\cap...\cap S_n}(\bar{x}) \le d_{S_1}(\bar{x}) + ... + d_{S_n}(\bar{x}),$$

which completes the proof of the first part.
To prove the second inequality, let $\{x_i\}_{i\in\mathbb{N}}$ be a sequence of elements of $S_1\cap...\cap S_n$ such that $\|x_i - \bar{x}\| < \frac{i+1}{i} d_{S_1\cap...\cap S_n}(\bar{x})$. We set $y_i = \bar{x} + \frac{i}{i+1}(x_i - \bar{x})$. Then,

$$\|y_i - \bar{x}\| < d_{S_1\cap...\cap S_n}(\bar{x}),\ \|y_i - \bar{x}\| \to d_{S_1\cap...\cap S_n}(\bar{x}) \text{ and } d_{S_1}(y_i) + ... + d_{S_n}(y_i) \to 0$$

as $i \to \infty$. Let us apply the mean value theorem to the function $(d_{S_1} + ...d_{S_n})(.)$ on the segment $[y_i, \bar{x}]$, there exist sequences $\{c_i^k\}_{k\in\mathbb{N}}$ converging to $c_i \in [y_i, \bar{x})$ and $\xi_i^k \in \partial^F\big(d_{S_1} + ... + d_{S_n}(c_i^k)$ such that

$$\liminf_{k\to\infty}\langle \xi_i^k, \bar{x} - y_n\rangle \ge (d_{S_1} + ... + d_{S_n})(\bar{x}) - (d_{S_1} + ... + d_{S_n})(y_i).$$

Consequently, for each $i \ge 1$, there exist $u_i \in X$ with $\|u_i - \bar{x}\| < d_{S_1\cap...\cap S_n}(\bar{x})$ and $\xi_i \in \partial^F\big(d_{S_1} + ... + d_{s_n}\big)(u_i)$ such that

$$\langle \xi_i, \bar{x} - y_i\rangle \ge (d_{S_1} + ... + d_{S_n})(\bar{x}) - (d_{S_1} + ... + d_{S_n})(y_i) - \frac{1}{i}.$$

Therefore $M\|\bar{x} - y_i\| \ge (d_{S_1} + ... + d_{S_n})(\bar{x}) - (d_{S_1} + ... + d_{S_n})(y_i) - \frac{1}{i}$. Passing to limit in the latter inequality as $i \to \infty$, one obtains the second inequality of (3.1) and the proof is complete. □

The following corollary gives necessary or sufficient conditions ensuring the metric inequality.

Corollary 3.2 *Let X be an Asplund space and let $S_1, ..., S_n$ be non-empty closed subsets of X. Then the following assertions hold:*

1) If x_0 is either a interior point of $S_1\cap...\cap S_n$, or a boundary point of $S_1\cap...\cap S_n$ and there exist $\gamma > 0$ and $\delta > 0$ such that $\|x^*\| \ge \gamma$ for all $x \in B(x_0, \delta)$, $x \notin$

$S_1 \cap ... \cap S_n$ and $x^* \in \partial^F(d_{S_1} + ... + d_{S_n})(x)$, then the metric inequality holds with $a = 1/\gamma$.

2) If x_0 is a boundary point of $S_1 \cap ... \cap S_n$, for every $\gamma > 0$, there exists $\delta > 0$ such that $\|x^*\| < \gamma$ for all $x \in B(x_0, \delta)$; $x \notin S_1 \cap ... \cap S_n$ and $x^* \in \partial^F(d_{S_1} + ... + d_{S_n})(x)$, then the $(\mathfrak{MI})$ is not satisfied at x_0.

Proof. The proof immediately follows from Theorem 3.1. □

Note that the converses of assertions 1) and 2) do not hold in general. To see this, consider the following examples:

Example 3.3 Let $S_1 = \{0\}$ and $S_2 = \{0, 1, 1/2, ..., 1/n, ...\}$ be closed subsets of $\mathbb{R}$. Obviously, S_1 and S_2 satisfy the $(\mathfrak{MI})$ at 0. Despite this, for all $x \in (1/n - 1/2n(n+1), 1/n), d_{S_1}(x) + d_{S_2}(x) = 1/n - x + x = 0$. Consequenly, $\partial^F(d_{S_1} + d_{S_2})(x) = 0$. Hence, for any $\delta > 0$, we can always find $x \in \mathbb{R}$ with $0 < x < \delta$ such that $\partial^F(d_{S_1} + d_{S_2})(x) = 0$.

Example 3.4 Consider the two following closed subsets of $\mathbb{R}$:

$$S_1 = \{\, 0, 1/2, ..., 1/2n, ... \,\} \text{ and } S_2 = \{\, 0, 1/3, ..., 1/(2n+1), ... \,\}.$$

Then, $S_1 \cap S_2 = \{0\}$ and obviously $d_{S_1 \cap S_2}(1/2n) = 1/2n$ and $d_{S_1}(1/2n) + d_{S_2}(1/2n) \leq 1/(2n(2n+1))$. Hence S_1, S_2 do not satisfy the (MI) at 0. However, for all $x \in \big(1/(2n), 1/(2n+1) + 1/[(2n-1)(2n+1)]\big)$, $d_{S_1}(x) + d_{S_2}(x) = x - 1/(2n) + x - 1/(2n+1)$ and therefore $2 \in \partial^F(d_{S_1} + d_{S_2})(x)$.

In order to derive a condition ensuring the metric inequality in terms of limiting Fréchet normal cones, recall ([8]) that a closed subset C of X is said to be *sequentially normally compact* at $\bar{x} \in C$, if for any sequences $x_n \xrightarrow{C} \bar{x}$, $x_n^* \in N^F(C, x_n)$, one has

$$x_n^* \xrightarrow{w^*} 0 \quad \Longleftrightarrow \quad \|x_n^*\| \to 0.$$

Obviously, any closed in finite dimensions is sequentially normally compact. A compact epi-Lipschitz set in the sense of Borwein-Strojwas (see [2]) is also sequentially normally compact. But the converse does not hold in general. For more details on compact epi-Lipschitz sets, the reader is refered to [2], [12].

The following properties of the Fréchet subdifferential of distance functions will be needed:

Lemma 3.5 (Jourani & Thibault [19]) *Let $C \subset X$ be a closed set and let $\bar{x} \notin C$. Then*

$$x^* \in \partial^F d_C(\bar{x}) \quad \Longrightarrow \quad \|x^*\| = 1.$$

The next lemma follows an idea by Thibault [36].

Lemma 3.6 *Let $C \subset X$ be a nonempty closed set.*

a). Let $\bar{x} \in C$ and let $x^* \in N^F(C, \bar{x})$. Then $x^* \in \lambda \partial^F d_C(\bar{x})$ holds for any $\lambda \geq \|x^*\| + 1$, and any Banach space X.

b) Let X be an Asplund space and let $x \in X$. If $x^* \in \partial^F d_C(\bar{x})$, then for any $\epsilon \in (0, 1)$, there exist $x_\epsilon \in C$ and $x_\epsilon^* \in N^F(C, x_\epsilon)$ such that

$$\|x_\epsilon - \bar{x}\| < d_C(\bar{x}) + \epsilon \quad \text{and} \quad \|x_\epsilon^* - x^*\| \leq \epsilon. \tag{3.2}$$

As a result, for all $x \in C$, one has

$$\hat{N}(C,x) = \bigcup_{\lambda > 0} \lambda \hat{\partial} d_C(x). \tag{3.3}$$

Proof. To prove a), let $\bar{x} \in C$ and let $x^* \in N^F(C,\bar{x})$. By definition, for each $\epsilon \in (0,1)$, there is $\delta > 0$ such that

$$\langle x^*, x - \bar{x}\rangle \le \epsilon\|x - \bar{x}\| \quad \forall x \in (\bar{x} + \delta B_X) \cap C.$$

Fix $x \in \bar{x} + \frac{\delta}{2}B_X$. Take a sequence of points $c_n \in C$ such that $\lim_{n\to\infty}\|c_n - x\| = d_C(x)$. Since for n large enough, $c_n \in \bar{x} + \delta B_X$, we have

$$\begin{aligned}\langle x^*, x - \bar{x}\rangle = \langle x^*, x - c_n\rangle + \langle x^*, c_n - \bar{x}\rangle &\le \|x^*\|\|c_n - x\| + \epsilon\|c_n - \bar{x}\| \\ &\le (\|x^*\| + 1)\|c_n - x\| + \epsilon\|x - \bar{x}\|.\end{aligned}$$

Letting $n \to \infty$ in the previous inequality, we obtain

$$\langle x^*, x - \bar{x}\rangle \le (\|x^*\| + 1)d_C(x) + \epsilon\|x - \bar{x}\|,$$

establishing a).

We next prove b). Let X be an Asplund space. Let $x^* \in \partial^F d_C(\bar{x})$. For each $\epsilon \in (0,1)$, we can find a number $\delta \in (0, \epsilon/8)$ such that

$$\langle x^*, x - \bar{x}\rangle \le d_C(x) - d_C(\bar{x}) + \frac{\epsilon}{2}\|x - \bar{x}\| \quad \forall x \in \bar{x} + \delta B_X.$$

Take $z_\epsilon \in C$ such that

$$\|z_\epsilon - \bar{x}\| \le d_C(\bar{x}) + \delta^2.$$

Fix $x \in (z_\epsilon + \delta B_X) \cap C$. Then $d_C(\bar{x} + x - z_\epsilon) \le \|\bar{x} - z_\epsilon\|$. This together with the two latter inequalities yield

$$\langle x^*, x - z_\epsilon\rangle \le \delta^2 + \frac{\epsilon}{2}\|z_\epsilon - \bar{x}\|.$$

By the Ekeland variational principle, there exist $\bar{z}_\epsilon \in (z_\epsilon + \frac{\delta}{2}B_X) \cap C$ such that

$$-\langle x^*, \bar{z}_\epsilon - z_\epsilon\rangle + \frac{\epsilon}{2}\|\bar{z}_\epsilon - z_\epsilon\| - 2\delta\|x - \bar{z}_\epsilon\| \le -\langle x^*, x - z_\epsilon\rangle + \frac{\epsilon}{2}\|z_\epsilon - x\| \quad \forall x \in (z_\epsilon + \delta B_X) \cap C.$$

Since $\delta < \epsilon/8$ and $\bar{z}_\epsilon + \frac{\delta}{2}B_X \subseteq z_\epsilon + \delta B_X$, we have

$$\langle x^*, x - \bar{z}_\epsilon\rangle \le \frac{3\epsilon}{4}\|x - \bar{z}_\epsilon\| \quad \forall x \in (\bar{z}_\epsilon + \frac{\delta}{2}B_X) \cap C.$$

This shows that $x^* \in \partial^F(\delta_C(.) + \frac{3\epsilon}{4}\|. - \bar{z}_\epsilon\|)(\bar{z}_\epsilon)$, (recall $\delta_C(.)$ is the indicator function of C). By the fuzzy sum rule, there exist $x_\epsilon \in (\bar{z}_\epsilon + \frac{\delta}{2}B) \cap C$ and $x^*_\epsilon \in N^F(C, x_\epsilon)$ such that $\|x^*_\epsilon - x^*\| \le \epsilon$. Combining (3.7), $\bar{z}_\epsilon \in z_\epsilon + \frac{\delta}{2}B_X$ and $x_\epsilon \in (\bar{z}_\epsilon + \frac{\delta}{2}B_X) \cap C$, we derive that

$$\|x_\epsilon - \bar{x}\| \le d_C(\bar{x}) + \delta^2 + \delta < d_C(\bar{x}) + \epsilon,$$

which proves (3.2). Use a) and (3.2), and take limits to obtain (3.3) and complete the proof. □

Proposition 3.7 *Let X be an Asplund space. Let $S_1, ..., S_n$ be closed subsets of X and let $\bar{x} \in S_1 \cap ... \cap S_n$. If the following condition are satisfied :*
i) $S_2, ..., S_n$ are sequentially normally compact and

$$x_i^* \in \hat{N}(S_i, \bar{x}),\ i = 1, ..., n \quad \& \quad x_1^* + ... + x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0,$$

then either $\bar{x}$ is a interior point of $S_1 \cap ... \cap S_n$ or a boundary point of $S_1 \cap ... \cap S_n$ and there exist $\gamma > 0$ and $\delta > 0$ such that $\|x^*\| \geq \gamma$ for all $x \in B(\bar{x}, \delta)$; $x \notin S_1 \cap ... \cap S_n$ and $x^* \in \partial^F(d_{S_1} + ... + d_{S_n})(x)$. As a result, the condition i) is sufficient for the metric inequality to be satisfied at $\bar{x}$.

Proof. Assume that for each $k \geq 1$, there exist $x_k \in B(\bar{x}, 1/k)$; $x_k \notin S_1 \cap ... \cap S_n$ and $x_k^* \in \partial^F(d_{S_1} + ... + d_{S_n})(x_k)$ such that $\|x_k^*\| < 1/k$. By the fuzzy sum rule, there exist $u_k^i \in x_k + \delta_k B_X$; $\zeta_k^i \in \partial^F d_{S_i}(u_k^i)$, $i = 1, ..., n$ such that

$$x_k^* \in \sum_{i=1}^n \zeta_k^i + \frac{2}{k} B_{X^*},\ k = 1, 2, ...,\tag{3.4}$$

where $\delta_k = d_{S_1 \cap ... \cap S_n}(x_k)/2$. Therefore, $u_k^i \notin S_1 \cap ... \cap S_n$ for all $i = 1, ..., n$ and $k \geq 1$. Without loss of generality, we may assume that $u_k^1 \notin S_1$, $k = 1, 2, ...$ Hence by Lemma 3.5, then $\|\zeta_k^1\| = 1$, $k = 1, 2, ...$ On the other hand, the sequences $\{\zeta_k^i\}_{k \in \mathbb{N}}$, $i = 1, 2, ..., n$ are bounded ($\|\zeta_k^i\| \leq 1$). Since X is an Asplund space, we may assume that $\zeta_k^i \xrightarrow{w^*} \zeta^i$, $i = 1, ..., n$. According to (3.10), we have

$$\sum_{i=1}^n \zeta^i = 0, \quad \zeta^i \in \hat{\partial} d_{S_i}(\bar{x}) \subset \hat{N}(S_i, \bar{x}).$$

Using the assumption, we obtain $\zeta^1 = ... = \zeta^n = 0$.

Now, from $\zeta_k^i \in \partial^F d_{S_i}(u_k^i)$, using Lemma 3.6, there exist $v_k^i \in S_i$, $\xi_k^i \in N^F(S_i, v_k^i)$ such that

$$\|v_k^i - u_k^i\| < d_{S_i}(u_k^i) + \delta_k \quad \text{and} \quad \|\xi_k^i - \zeta_k^i\| \leq \delta_k,\ i = 1, ..., n, k = 1, 2, ...$$

Therefore, for every $i = 1, ..., n$, then $v_k^i \to \bar{x}$ and $\xi_k^i \xrightarrow{w^*} 0$. Due to the assumption that all but one of sets S_i are sequentially normal compact, all but one of the sequences $\{\xi_k^i\}_{k \in \mathbb{N}}$ must strongly converge to zero. Consequently, all but one of the sequences $\{\zeta_k^i\}_{k \in \mathbb{N}}$ must strongly converge to zero. This together with (3.4) imply that $\|\zeta_k^i\| \to 0$ for all $i = 1, ..., n$. In particular, $\|\zeta_k^1\| \to 0$, which contradits $\|\zeta_k^1\| = 1$ for all $k = 1, 2....$ The second part immediately follows Corollary 3.2. The proof is complete □

Similar results of the second part of the above proposition for $G-$subdifferential (or abstract subdifferentials verifying certain properties) was obtained in [10], [12], [14], [17] and [20]. Note that the converse of the first part of Proposition 3.7 does not hold in general. For instance, take the two subsets of $\mathbb{R}$: $S_1 = (-\infty, 0]$ and $S_2 = [0, +\infty)$. Then S_1, S_2 do not satisfy i), but for every $x \in \mathbb{R}$ with $x \neq 0$, one has $\partial^F(d_{S_1} + d_{S_2})(x) = \{1\}$ if $x > 0$ and $\{-1\}$ if $x < 0$.

The next theorem gives characterizations of Asplund spaces in terms of conditions ensuring the metric inequality and the intersection formulae.

Theorem 3.8 *Let X be a Banach space. The following assertions are equivalent:*

1) X is an Asplund space;

2) For every non-empty closed subsets $S_1, S_2, ..., S_n$ of X, all but one of which are sequentially normally compact at $\bar{x} \in S_1 \cap ... \cap S_n$ and satisfying the following qualification condition:

$$x_i^* \in \hat{N}(S_i, \bar{x}),\ i = 1, ..., n \quad \& \quad x_1^* + ... + x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0,$$

then $S_1, S_2, ..., S_n$ satisfy the metric inequality at $\bar{x}$;

3) For every non-empty closed subsets $S_1, S_2, ..., S_n$ satisfying the $(\mathfrak{MI})$ at $\bar{x} \in S_1 \cap ... \cap S_n$, one has

$$\hat{N}(S_1 \cap ... \cap S_n, \bar{x}) \subseteq \hat{N}(S_1, \bar{x}) + ... + \hat{N}(S_n, \bar{x}).$$

Proof. The implication 1) $\Rightarrow$ 2) was established in Proposition 3.7. To prove 1) $\Rightarrow$ 3), fix $x^* \in \hat{N}(S_1 \cap ... \cap S_n, \bar{x})$ and select sequences $x_k \overset{S_1 \cap ... \cap S_n}{\longrightarrow} \bar{x}$, and $x_k^* \in N^F(S_1 \cap ... \cap S_n, x_k)$ such that $x_k^* \overset{w^*}{\longrightarrow} x^*$. Hence, the sequence $\{x_k^*\}_{k \in \mathbb{N}}$ is bounded. Assume that $\|x_k^*\| \leq \lambda - 1$, $k = 1, 2....$ By Lemma 3.6, then

$$x_k^* \in \lambda \partial^F d_{S_1 \cap ... \cap S_n}(x_k),\ k = 1, 2, ...$$

Since the $(\mathfrak{MI})$ is satisfied at $\bar{x}$, it is also satisfied at x_k for k is large enough, that is, there is $r_k > 0$ such that

$$d_{S_1 \cap ... \cap S_n}(x) \leq a(d_{S_1}(x) + ... + d_{S_n}(x)) \quad \text{for all} \quad x \in \bar{x}_k + r_k B_X,$$

for all k large. Therefore,

$$x_k^* \in \lambda \partial^F d_{S_1 \cap ... \cap S_n}(x_k) \subseteq a\lambda \partial^F \big(d_{S_1}(.) + ... + d_{S_n}(.)\big)(x_k).$$

Apply the fuzzy sum rule to derive the existence of sequences $x_k^i \in x_k + \frac{1}{k} B_X$, $\zeta_k^i \in a\lambda \partial^F d_{S_i}(x_k^i)$, $i = 1, ..., n$ such that

$$\Big\| x_k^* - \sum_{i=1}^n \zeta_k^i \Big\| \leq \frac{1}{k}.$$

Clearly, the sequences $\{\zeta_k^i\}_{k \in \mathbb{N}}$ are bounded. Hence assume that

$$\zeta_k^i \overset{w^*}{\longrightarrow} \zeta^i \in a\lambda \hat{\partial} d_{S_i}(\bar{x}) \subseteq \hat{N}(S_i, \bar{x}), \quad i = 1, ..., n.$$

Therefore,

$$\sum_{i=1}^n \zeta_k^i \overset{w^*}{\longrightarrow} x^* = \sum_{i=1}^n \zeta^i \in \sum_{i=1}^n \hat{N}(S_i, \bar{x}),$$

establishing the proof of 1)$\Rightarrow$ 3).

To prove 2)⇒1) and 3)⇒1), suppose that X is not Asplund. We prove that 2) and 3) are not true for some sets $S_1, ..., S_n \subset X$. Following [8], [32], we represent X in the form $X = Y \times \mathbb{R}$ with norm $\|(y,\alpha)\| = \|y\| + |\alpha|$ for $x = (y,\alpha) \in Y \times \mathbb{R} = X$. Note that X is not Asplund if and only if Y is not Asplund. Due to Theorem 1.5.3 in [4] (also Theorem 2.1 in [8]), there exist an equivalent norm $|.|$ on Y and $\gamma > 0$ such that $\frac{1}{2}\|.\| \le |.| \le \|.\|$ and

$$\limsup_{h\to 0} \frac{|y+h+|y-h|-2|y|}{\|h\|} > \gamma \quad \text{for all} \quad y \in Y.$$

For 2)⇒ 1), let $f : Y \to \mathbb{R}$ be defined by $f(y) := -\sqrt{|y|}$ and let $S_1 := \{0\} \times (-\infty, 0]$ and $S_2 := \operatorname{epi} f$ be closed subsets of X. Then $S_1 \cap S_2 = \{(0,0)\}$ and S_1, S_2 do not satisfy the $(\mathfrak{MI})$ at (0,0). Indeed, for each $n \ge 1$, take $y_n \in Y$ such that $0 < \|y_n\| < 1/2(n-1)^2$. One has

$$(y_n, -\sqrt{|y_n|}) \in B((0,0), 1/(n-1));$$

$$d_{S_1}(y_n, -\sqrt{|y_n|}) = \|y_n\|; \quad d_{S_2}(y_n, -\sqrt{|y_n|}) = 0.$$

Hence

$$d_{S_1\cap S_2}(y_n, -\sqrt{|y_n|}) = \|y_n\| + \sqrt{|y_n|} > n\big[d_{S_1}(y_n, -\sqrt{|y_n|}) + d_{S_2}(y_n, -\sqrt{|y_n|})\big].$$

Despite this, we show that S_2 is sequentially normally compact and

$$\hat{N}(S_1, (0,0)) \cap (-\hat{N}(S_2, (0,0)) = \{(0,0)\}.$$

Indeed, it is sufficient to prove that for all $(y,\alpha) \in S_2$, $N^F(S_2, (y,\alpha)) = \{(0,0)\}$. Let $(y_0, \alpha_0) \in S_2$ and $(y^*, -\lambda) \in N^F(S_2, (y_0, \alpha_0))$.

Assume that $\lambda \neq 0$, by Lemma 2.1, $\lambda > 0$ and $y^* \in \lambda\partial^F f(y_0)$. Therefore

$$\liminf_{\|h\|\to 0} \frac{-\sqrt{|y_0+h|} + \sqrt{|y_0|} - \langle y^*, h\rangle}{\|h\|} \ge 0.$$

Thus

$$\limsup_{\|h\|\to 0} \frac{\sqrt{|y_0+h|} + \sqrt{|y_0-h|} - 2\sqrt{|y_0|}}{\|h\|} \le 0.$$

On the other hand, set $q(h) = \big(\sqrt{|y_0+h|} + \sqrt{|y_0-h|} - 2\sqrt{|y_0|}\big) \cdot \|h\|^{-1}$. Then

$$\begin{aligned} q(h) &= \frac{1}{\|h\|}\left[\frac{|y_0+h|-|y_0|}{\sqrt{|y_0+h|}+\sqrt{|y_0|}} + \frac{|y_0-h|-|y_0|}{\sqrt{|y_0-h|}+\sqrt{|y_0|}}\right] \\ &= \frac{|y_0+h|+|y_0-h|-2|y_0|}{\|h\|(\sqrt{|y_0+h|}+\sqrt{|y_0|})} + \frac{(|y_0-h|-|y_0|)(\sqrt{|y_0+h|}-\sqrt{|y_0-h|})}{\|h\|(\sqrt{|y_0+h|}+\sqrt{|y_0|})(\sqrt{|y_0-h|}-\sqrt{|y_0|})} \end{aligned}$$

If $y_0 = 0$, then

$$q(h) = \frac{2|h|}{\|h\|\sqrt{|h|}} \ge \frac{\sqrt{2}}{\sqrt{\|h\|}}.$$

Thus $q(h) \to +\infty$ as $\|h\| \to 0$.

If $y_0 \neq 0$, then

$$\limsup_{\|h\|\to 0} q(h) = \frac{1}{2\sqrt{|y_0|}} \limsup_{\|h\|\to 0} \frac{|y_0+h| + |y_0-h| - 2|y_0|}{\|h\|} > \frac{\gamma}{2\sqrt{|y_0|}},$$

a contradiction. Hence $\lambda = 0$. By definition, for each $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle y^*, y - y_0\rangle \le \epsilon(\|y-y_0\| + |\alpha - \alpha_0|) \text{ for all } (y,\alpha) \in \big((y_0,\alpha_0) + \delta B_{Y\times \mathbb{R}}\big) \cap S_2.$$

Note that

$$(y,\alpha) \in \big((y_0, -\sqrt{|y_0|}) + \delta B_{Y\times \mathbb{R}}\big) \cap S_2 \Longrightarrow (y, \alpha + \alpha_0 + \sqrt{|y_0|}) \in \big((y_0,\alpha_0) + \delta B_{Y\times \mathbb{R}}\big) \cap S_2.$$

Therefore,

$$\langle y^*, y - y_0\rangle \le \epsilon(\|y-y_0\| + |\alpha + \sqrt{|y_0|}|) \text{ for all } (y,\alpha) \in \big((y_0,-\sqrt{|y_0|}) + \delta B_{Y\times \mathbb{R}}\big) \cap S_2.$$

Thus, if $y_0 = 0$, then $\langle y^*, y\rangle \le \epsilon\|y\|$ for all $y \in \delta B_Y$ and as ϵ is arbitrary, one obtains $y^* = 0$. If $y_0 \neq 0$, for every $y \in y_0 + \min\{\delta, \delta\|y_0\|, \|y_0\|/2\}B_Y$, one has $(y, -\sqrt{|y|}) \in (y_0, -\sqrt{|y_0|}) + \delta B_{Y\times\mathbb{R}}$. Consequently,

$$\langle y^*, y - y_0\rangle \le \epsilon(\|y-y_0\| + |\sqrt{|y|} - \sqrt{|y_0|}|).$$

Hence

$$\left\langle y^*, \frac{y-y_0}{\|y-y_0\|}\right\rangle \le \epsilon\left(1 + \frac{1}{\sqrt{|y|}+\sqrt{|y_0|}}\right) \le \epsilon\left(1 + \frac{2}{3\sqrt{|y_0|}}\right)$$

and we also obtain $y^* = 0$. So $N^F(S_2,(y,\alpha)) = \{(0,0)\}$ for all $(y,\alpha) \in S_2$ and establish the proof of the implication 2)$\Rightarrow$ 1).

To prove 3)$\Rightarrow$ 1), take $g : Y \to \mathbb{R}$ defined by $g(y) := -|y|$ and consider the two subsets S_1, S_2 of X as in [8]: $S_1 = \{0\} \times (-\infty, 0]$ and $S_2 = \text{epi}g$. Observe that $S_1 \cap S_2 = \{(0,0)\}$; $\hat{N}(S_1,(0,0)) = Y^* \times [0,+\infty)$; $\hat{N}(S_2,(0,0)) = \{(0,0)\}$; $\hat{N}(S_1\cap S_2,(0,0)) = Y^* \times \mathbb{R}$. Therefore,

$$\hat{N}(S_1\cap S_2,(0,0)) \not\subseteq \hat{N}(S_1,(0,0)) + \hat{N}(S_2,(0,0)).$$

However, S_1, S_2 satisfy the $(\mathfrak{MI})$ at (0,0). Indeed, let $(y,\alpha) \in Y \times \mathbb{R}$. We have $d_{S_1\cap S_2}(y,\alpha) = \|y\| + |\alpha|$ and $d_{S_1}(y,\alpha) = \|y\|$ if $\alpha \le 0$ and $\|y\| + |\alpha|$ if $\alpha > 0$. If $\alpha \ge 0$, then

$$d_{S_1\cap S_2}(y,\alpha) = d_{S_1}(y,\alpha) \le d_{S_1}(y,\alpha) + d_{S_2}(y,\alpha);$$

if $-|y| \le \alpha < 0$, then

$$d_{S_1\cap S_2}(y,\alpha) = \|y\| + |\alpha| \le 2\|y\| \le 2(d_{S_1}(y,\alpha) + d_{S_2}(y,\alpha));$$

if $\alpha < -|y|$, observe that

$$\begin{aligned} d_{S_2}(y,\alpha) &= \min\left\{\|y-z\| + |\alpha - \mu| :\ \mu \ge -|z|\right\} \\ &\ge \min\left\{|y-z| + |\alpha-\mu| :\ \mu \ge -|z|\right\} \\ &\ge \min\left\{||y| - |z|| + |\alpha - \mu| :\ \mu \ge -|z|\right\} \ge (||y| + \alpha|)/\sqrt{2}. \end{aligned}$$

Consequently,

$$d_{S_1\cap S_2}(y,\alpha) = \|y\| + |\alpha| \le 2|\alpha| \le 2\sqrt{2}(d_{S_1}(y,\alpha) + d_{S_2}(y,\alpha)).$$

Hence S_1, S_2 satisfy the $(\mathfrak{MI})$ at (0,0). The proof is complete. $\square$

4. Subdifferential calculus

Let $f_i : X \to 1\mathrm{R} \cup \{+\infty\}$, $i = 1, ..., m$ be semicontinuous functions and let $f_i : X \to 1\mathrm{R} \cup \{+\infty\}$, $i = m+1, ..., n$ be continuous functions. Let $g : 1\mathrm{R}^n \to 1\mathrm{R} \cup \{+\infty\}$. Consider the composite function:

$$g[f_1, ..., f_n] : X \to 1\mathrm{R} \cup \{+\infty\},$$

$$g[f_1, ..., f_n](x) := \begin{cases} g(f_1(x), ..., f_n(x)) & \text{if} \quad x \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n \\ +\infty & \text{otherwise.} \end{cases}$$

In this section, we establish chain rules for subdifferential of the above composite function where the components f_i are semicontinuous or continuous. This chain rules are used to obtain a mutiplier rule for optimization problems with non-Lipschitz data in the sequel. To simplify notations, we will use the quatities τ_i as in [1]: $\tau_i = 1$ for $i = 1, ..., m$ and τ_j equal to either 1 or -1 for $j = m+1, ...n$.

We first prove a chain rule when g is Lipschitz:

Theorem 4.1. *Let X be an Asplund space. Let $f_i : X \to 1\mathrm{R} \cup \{+\infty\}$, $i = 1, ..., m$ be semicontinuous functions and let $f_i : X \to 1\mathrm{R} \cup \{+\infty\}$, $i = m+1, ..., n$ be continuous functions. Let $g : 1\mathrm{R}^n \to 1\mathrm{R}$ be a locally Lipschitz function and nondereasing for each of its first m variables. Let $\bar{x} \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n$. Suppose that the following asumptions are fulfilled:*

i) All but one of sets $\operatorname{epi} f_i$, $i = 1, ..., m$; $\operatorname{gph} f_i$, $i = m+1, ..., n$ are sequentially normally compact respectively at $(\bar{x}, f_i(\bar{x}))$ and

$$x_i^* \in \partial^\infty(\tau_i f_i)(\bar{x}),\ i = 1, ..., n\ \&\ \sum_{i=1}^n x_i^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0.$$

$$ii) \quad x_k \overset{g[f_1,\ldots,f_n]}{\longrightarrow} \bar{x} \quad \Longrightarrow \quad f_i(x_k) \to f_i(\bar{x}),\ i = 1, ..., m.$$

Then one has the inclusions:

$$\hat{\partial} g[f_1, ..., f_n](\bar{x}) \subseteq \bigcup \Bigg\{ \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f)(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty(\tau_i f_i)(\bar{x}) : (\lambda_1, ..., \lambda_n) \in \hat{\partial} g\big(f_1(\bar{x}), ..., f_n(\bar{x})\big) \Bigg\} \tag{4.1}$$

and

$$\partial^\infty g[f_1, ..., f_n](\bar{x}) \subseteq \sum_{i=1}^n \partial^\infty(\tau_i f_i)(\bar{x}). \tag{4.2}$$

To prove the theorem, we need the following results established by Mordukhovich & Shao [29] and Jofré-Luc-Théra [13]:

Proposition 4.2. ([13], [29]) *Let X, Y be Banach spaces. Suppose that φ is a function on $X \times Y$, and*

$$p(x) = \inf_{y \in Y} \varphi(x, y).$$

Assume that the following holds:

$p(\bar{x}) = \varphi(\bar{x}, \bar{y})$ *and for every sequence* $x_n \xrightarrow{p} \bar{x}$*, there exists a sequence* $y_n \to \bar{y}$ *such that* $p(x_n) = \varphi(x_n, y_n)$*, then*

$$\hat{\partial} p(\bar{x}) \times \{0\} \subseteq \hat{\partial}\varphi(\bar{x}, \bar{y}); \quad \partial^{\infty} p(\bar{x}) \times \{0\} \subseteq \partial^{\infty}\varphi(\bar{x}, \bar{y}).$$

Proposition 4.3.([13],[29]) *Let* X *be an Asplund space. Let* $f_1, f_2 : X \to \mathrm{I\!R} \cup \{+\infty\}$ *be lower semicontinuous functions. Let* $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$*. Suppose that* f_1 *is Lipschitz at* $\bar{x}$*, then*

$$\hat{\partial}(f_1 + f_2)(\bar{x}) \subseteq \hat{\partial} f_1(\bar{x}) + \hat{\partial} f_2(\bar{x});$$

$$\partial^{\infty}(f_1 + f_2)(\bar{x}) \subseteq \partial^{\infty} f_2(\bar{x}).$$

Proof of Theorem 4.1. We consider the following sets:

$$S_i := \Big\{ (x, \alpha_1, ..., \alpha_n) \in X \times \mathrm{I\!R}^n : \alpha_i \geq f_i(x) \Big\},\ i = 1, ..., m;$$

$$S_i := \Big\{ (x, \alpha_1, ..., \alpha_n) \in X \times \mathrm{I\!R}^n : \alpha_i = f_i(x) \Big\},\ i = m+1, ..., n.$$

Clearly,

$$N^F(S_i, (x, \alpha_1, ..., \alpha_n)) = \Big\{ (x^*, \lambda_1, ..., \lambda_n) \in X^* \times \mathrm{I\!R}^n : \lambda_j = 0 \quad \text{if} \quad j \neq i; \quad (x^*, \lambda_i) \in N^F(\operatorname{epi} f_i, (x, \alpha_i)) \Big\},$$

$$\text{for} \quad i = 1, ..., m;$$

$$N^F(S_i, (x, \alpha_1, ..., \alpha_n)) = \Big\{ (x^*, \lambda_1, ..., \lambda_n) \in X^* \times \mathrm{I\!R}^n : \lambda_j = 0 \quad \text{if} \quad j \neq i; \quad (x^*, \lambda_i) \in N^F(\operatorname{gph} f_i, (x, \alpha_i)) \Big\},$$

$$\text{for} \quad i = m+1, ..., n.$$

and

$$\hat{N}(S_i, (x, \alpha_1, ..., \alpha_n)) = \Big\{ (x^*, \lambda_1, ..., \lambda_n) \in X^* \times \mathrm{I\!R}^n : \lambda_j = 0 \quad \text{if} \quad j \neq i; \quad (x^*, \lambda_i) \in \hat{N}(\operatorname{epi} f_i, (x, \alpha_i)) \Big\},$$

$$\text{for} \quad i = 1, ..., m;$$

$$\hat{N}(S_i,(x,\alpha_1,...,\alpha_n)) = \Big\{(x^*,\lambda_1,...,\lambda_n)\in X^*\times 1\mathrm{R}^n : \lambda_j = 0 \quad \text{if} \quad j\neq i; \\ (x^*,\lambda_i)\in \hat{N}(\operatorname{gph} f_i,(x,\alpha_i))\Big\}, \\ \text{for} \quad i = m+1,...,n.$$

Therefore, from assumption (i), we derive that all but one of sets S_i, $i=1,...,n$ are sequentially normal compact at $(\bar{x}, f_1(\bar{x}),...,f_n(\bar{x}))$ and that

$$(x_i^*,\lambda_1^i,...,\lambda_n^i)\in \hat{N}(S_i,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x})), i=1,...,n \quad \& \quad \sum_{i=1}^n (x_i^*,\lambda_1^i,...,\lambda_n^i)=0 \\ \Longrightarrow \quad (x_i^*,\lambda_1^i,...,\lambda_n^i)=0,\ i=1,...,n.$$

Using Theorem 3.8, we obtain

$$\hat{N}(S_1\cap ...\cap S_n,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x}))\subseteq \sum_{i=1}^n \hat{N}(S_i,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x})). \tag{4.3}$$

On the other hand, observe that

$$g[f_1,...,f_n](x) = \min_{(\lambda_i)\in 1\mathrm{R}^n}\Big(g(\lambda_1,...,\lambda_n)+\delta_{S_1\cap...\cap S_n}(x,\lambda_1,...,\lambda_n)\Big).$$

Asumptions (i) and (ii) ensure that the conditions of Proposition 4.2 and 4.3 are satisfied. Hence use Proposition 4.2 and 4.3 to observe that

$$\hat{\partial} g[f_1,...,f_n](\bar{x})\times\{0\}\times ...\times\{0\}\subseteq \{0\}\times\hat{\partial} g\big(f_1(\bar{x}),...,f_n(\bar{x})\big) \\ +\hat{N}(S_1\cap ...\cap S_n,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x})); \tag{4.4}$$

$$\partial^\infty g[f_1,...,f_n](\bar{x})\times\{0\}\times ...\times\{0\}\subseteq \hat{N}(S_1\cap ...\cap S_n,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x})). \tag{4.5}$$

Now, fix $x^*\in\hat{\partial} g[f_1,...,f_n](\bar{x})$. By (4.3) and (4.4), there are $(\lambda_1,...,\lambda_n)\in \hat{\partial} g\big(f_1(\bar{x}),...,f_n(\bar{x})\big)$ and $x_i^*\in X^*$, $i=1,...,n$ such that

$$(x_i^*,-\lambda_i)\in\hat{N}(\operatorname{epi} f_i,(\bar{x},f_i(\bar{x})),\quad i=1,...,m;$$

$$(x_i^*,-\lambda_i)\in\hat{N}(\operatorname{gph} f_i,(\bar{x},f_i(\bar{x})),\quad i=m+1,...,n;$$

and

$$x^* = x_1^*+...+x_n^*.$$

Since g is nondecreasing for each of its first m variables, then $\lambda_i\geq 0$ for $i=1,...,m$. By virtue of Proposition 2.1 and 2.2, for every $i=1,...,n$, $x_i^*\in\hat{\partial}(\lambda_i f)(\bar{x})$ for $\lambda_i\neq 0$ and $x_i^*\in\partial^\infty(\tau_i f_i)(\bar{x})$ for $\lambda_i=0$. Hence (4.1) is proved. Similarly, let $x^*\in\partial^\infty g[f_1,...,f_n](\bar{x})$. Using (4.3) and (4.5), then there are $x_i^*\in X^*$, $i=1,...,n$ such that

$$(x_i^*,0)\in\hat{N}(\operatorname{epi} f_i,(\bar{x},f_i(\bar{x})),\quad i=1,...,m;$$

$$(x_i^*, 0) \in \hat{N}(\operatorname{gph} f_i, (\bar{x}, f_i(\bar{x})), \quad i = m+1, ..., n$$

and $x^* = x_1^* + ... + x_n^*$. Therefore, $x_i^* \in \partial^\infty(\tau_i f_i)(\bar{x})$, $i = 1, ..., n$ and

$$x^* \in \partial^\infty(\tau_1 f_1)(\bar{x}) + ... + \partial^\infty(\tau_n f_n)(\bar{x}).$$

This proves (4.2) and completes the proof. □

Remark 4.4 When either the $f_i's$, $i = 1, ..., m$ also are continuous, or g is strictly creasing for each of its first m variables, then assumption (ii) of Theorem 4.1 is satisfied. If all but one of the $f_i's$, $i = 1, ..., n$ are Lipschitz, then (i) is satisfied.

Now we derive several corollaries of Theorem 4.1. The first one is the sum rule given under the standard qualification condition:

Corollary 4.5 *Let X be an Asplund. Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., n$ be lower semicontiuous functions. Let $\bar{x} \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n$. Assume that all but of the sets* epi f_i *are sequentially normally compact respestively at $(\bar{x}, f_i(\bar{x}))$, $i = 1, ..., n$ and*

$$x_i^* \in \partial^\infty(f_i)(\bar{x}),\ i = 1, ..., n\ \&\ \sum_{i=1}^n x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0.$$

Then

$$\hat{\partial}(f_1 + ... + f_n)(\bar{x}) \subseteq \hat{\partial} f_1(\bar{x}) + ... + \hat{\partial} f_n(\bar{x});$$
$$\partial^\infty(f_1 + ... + f_n)(\bar{x}) \subseteq \hat{\partial}^\infty f_1(\bar{x}) + ... + \hat{\partial}^\infty f_n(\bar{x}).$$

Proof. The proof immediately follows from Theorem 4.1 by applying to the function $g(\lambda_1, ..., \lambda_n) = \lambda_1 + ... + \lambda_n$. □

Corollary 4.6 *Let X be an Asplund space. Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., n$ be lower semicontiuous functions. Let $\bar{x} \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n$. Assume that*
$f_1(\bar{x}) = ... = f_n(\bar{x})$ and define $f(x) := \max_i f_i(x)$. Suppose also that all but of the sets epi f_i *are sequentially normally compact respestively at $(\bar{x}, f_i(\bar{x}))$, $i = 1, ..., n$ and the qualification condition holds:*

$$x_i^* \in \partial^\infty(f_i)(\bar{x}),\ i = 1, ..., n\ \&\ \sum_{i=1}^n x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0.$$

Then

$$\hat{\partial} f(\bar{x}) \subseteq \left\{ \sum_{\lambda_i > 0} \lambda_i \hat{\partial} f_i(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty f_i(\bar{x}) : \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1 \right\};$$
$$\partial^\infty f(\bar{x}) \subseteq \partial^\infty f_1(\bar{x}) + .. + \partial^\infty f_n(\bar{x}).$$

Proof. The result follows immediately from Theorem 4.1 using the function g defined by $g(\lambda_1, ..., \lambda_n) = \max_i \lambda_i$. Note that g is convex, Lipschitz and nondecreasing for each of its variables and satisfies:

$$\hat{\partial} g(f_1(\bar{x}), ..., f_n(\bar{x})) = \left\{ \lambda_1, ..., \lambda_n) \in \mathbb{R}^n : \lambda_i \geq 0,\ \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The condition $f_1(\bar{x}) = ... = f_n(\bar{x})$ ensures that assumption (ii) of Theorem 4.1 is satisfied. □

Corollary 4.5 and 4.6 have been established in Mordukhovich & Shao [29] under the compact epi-Lipschitz condition and then, e.g., in [31] under the sequentially normal compactness presented above. Their method is based on the Extremal principle. The above proof is different from that in [29], [31].

Another consequence of Theorem 4.1 is the chain rule for subdifferential of the norm function as follows:

Corollary 4.7 *Let X be an Asplund space. Let $f_i : X \to \mathbb{R}$, $i = 1, ..., n$ be continuous functions and set*

$$f(x) := \|(f_1(x), ..., f(x))\|_{\mathbb{R}^n} = |f_1(x)| + ... + |f_n(x)|.$$

Assume that all but one of sets $\operatorname{gph} f_i$ *are sequentialy normally compact respestively at $(\bar{x}, f_i(\bar{x}))$ and*

$$x_i^* \in \partial^\infty(\tau_i f_i)(\bar{x}),\ i = 1, ..., n\ \&\ \sum_{i=1}^n x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0.$$

Here, $\tau_i \in \{-1, 1\}$, $i = 1, ..., n$. Then

$$\begin{aligned} \hat{\partial} f(\bar{x}) \subseteq \Big\{ & \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f_i)(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty(\tau_i f_i)(\bar{x}) : \\ & \sum_{i=1}^n |\lambda_i| \leq 1, \sum_{i=1}^n \lambda_i f_i(\bar{x}) = \sum_{i=1}^n |f_i(\bar{x})| \Big\}; \\ \partial^\infty f(\bar{x}) \subseteq\ & \partial^\infty(\tau_1 f_1)(\bar{x}) + .. + \partial^\infty(\tau_n f_n)(\bar{x}). \end{aligned}$$

Proof. Apply Theorem 4.1 to the function $g(\lambda_1, ..., \lambda_n) := |\lambda_1| + ... + |\lambda_n|$. It well known from the convex analysis that:

$$\partial g(\lambda_1, ..., \lambda_n) = \left\{ (\gamma_1, ..., \gamma_n) \in \mathbb{R}^n : \sum_{i=1}^n |\gamma_i| \leq 1, \sum_{i=1}^n \lambda_i \gamma_i = \sum_{i=1}^n |\lambda_i| \right\}.$$

Hence, we can complete the proof. □

The following corollaries are chain rules for subdifferential of the product and the quotient of lower semicontinuous (or continuous) functions. The case of Lipschitz functions has been investigated in [29].

Corollary 4.8 *Let X be an Asplund space. Let f_1, $f_2 : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous functions. Let $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$.*

(i) If $f_1(\bar{x}) > 0$ and $f_2(\bar{x}) > 0$, and suppose that $\operatorname{epi} f_1$ *(or* $\operatorname{epi} f_2$*) is sequentially normally compact at $(\bar{x}, f_1(\bar{x}))$ $((\bar{x}, f_2(\bar{x}))$,respestively) and*

$$\partial^\infty f_1(\bar{x}) \cap (-\partial^\infty f_2(\bar{x})) = \{0\}.$$

Then

$$\hat{\partial}(f_1f_2)(\bar{x}) \subseteq f_2(\bar{x})\hat{\partial}f_1(\bar{x}) + f_1(\bar{x})\hat{\partial}f_2(\bar{x});$$

$$\partial^\infty(f_1f_2)(\bar{x}) \subseteq \partial^\infty f_1(\bar{x}) + \partial^\infty f_2(\bar{x}).$$

(ii) Fix $\tau_1, \tau_2 \in \{-1, 1\}$. Suppose that f_1, f_2 are continuous and gph f_1 *(or* gph f_2*) is sequentially normally compact at $(\bar{x}, f_1(\bar{x}))$ ($(\bar{x}, f_2(\bar{x}))$,respestively) and satisfy:*

$$\partial^\infty(\tau_1f_1)(\bar{x}) \cap (-\partial^\infty(\tau_2f_2)(\bar{x})) = \{0\}.$$

Then if we define $\hat{\partial}(\lambda \circ f_i)(\bar{x})$ by $\hat{\partial}(\lambda f_i)(\bar{x})$ for $\lambda \neq 0$ and by $\partial^\infty(\tau_if_i)(\bar{x})$ for $\lambda = 0$, we obtain

$$\hat{\partial}(f_1f_2)(\bar{x}) \subseteq \hat{\partial}(f_2(\bar{x}) \circ f_1)(\bar{x}) + \hat{\partial}(f_1(\bar{x}) \circ f_2)(\bar{x});$$

$$\partial^\infty(f_1f_2)(\bar{x}) \subseteq \partial^\infty(\tau_1f_1)(\bar{x}) + \partial^\infty(\tau_2f_2)(\bar{x}).$$

Proof. Consider the function $g : 1\mathrm{R}^2 \to 1\mathrm{R}$, given by: $g(\lambda_1, \lambda_2) = \lambda_1\lambda_2$. Then $(f_1f_2)(x) = g[f_1, f_2](x)$. To prove (i), note that when $f_1(\bar{x}) > 0$ and $f_2(\bar{x}) > 0$, then g is nondecreasing for each of its variables on some neighborhood of $(f_1(\bar{x}), f_2(\bar{x}))$. The results follows immediately from Theorem 4.1. □

Similarly, apply Theorem 4.1 to the function $g : 1\mathrm{R}^2 \to 1\mathrm{R}$ given by $g(\lambda_1, \lambda_2) = \frac{\lambda_1}{\lambda_2}$. We obtain chain rules for the quotient of two functions as follows:

Corollary 4.9 *Let X be an Asplund space. Let f_1, $f_2 : X \to 1\mathrm{R} \cup \{+\infty\}$. Let f_1 be a lower semicontinuous function and let f_2 be continuous. Let $\bar{x} \in$* dom $f_1 \cap$dom f_2.*, and let $f_2(\bar{x}) \neq 0$.*

(i). If $f_2(\bar{x}) > 0$, epi f_1 *(or* gph f_2*) is sequentially normally compact at $(\bar{x}, f_1(\bar{x}))$ ($(\bar{x}, f_2(\bar{x}))$,respestively) and,*

$$\partial^\infty f_1(\bar{x}) \cap (-\partial^\infty(\tau_2f_2)(\bar{x})) = \{0\},\ (\tau_2 \in \{-1, 1\}).$$

Then

$$\hat{\partial}\Big(\frac{f_1}{f_2}\Big)(\bar{x}) \subseteq \frac{\hat{\partial}(f_2(\bar{x})f_1)(\bar{x}) - \hat{\partial}(f_1(\bar{x}) \circ f_2)(\bar{x})}{(f_2(\bar{x}))^2};$$

$$\partial^\infty\Big(\frac{f_1}{f_2}\Big)(\bar{x}) \subseteq \partial^\infty f_1(\bar{x}) + \partial^\infty(\tau_2f_2)(\bar{x}).$$

(ii). Suppose that f_1 is continuous, and gph f_1 *(or* gph f_2*) is sequentially normally compact at $(\bar{x}, f_1(\bar{x}))$ ($(\bar{x}, f_2(\bar{x}))$,respestively) and*

$$\partial^\infty(\tau_1f_1)(\bar{x}) \cap (-\partial^\infty(\tau_2f_2)(\bar{x})) = \{0\},\ (\tau_1,\ \tau_2 \in \{-1, 1\}).$$

Then

$$\hat{\partial}\Big(\frac{f_1}{f_2}\Big)(\bar{x}) \subseteq \frac{\hat{\partial}(f_2(\bar{x})f_1)(\bar{x}) - \hat{\partial}(f_1(\bar{x}) \circ f_2)(\bar{x})}{(f_2(\bar{x}))^2};$$

$$\partial^\infty\Big(\frac{f_1}{f_2}\Big)(\bar{x}) \subseteq \partial^\infty(\tau_1f_1)(\bar{x}) + \partial^\infty(\tau_2f_2)(\bar{x}).$$

Finally, we establish the following chain rule, which is a generalization of Theorem 4.1 in the case where no Lipschitz assumption is assumed.

Theorem 4.10 *Let X be an Asplund space. Let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = 1, ..., m$ be semicontinuous functions and let $f_i : X \to \mathbb{R} \cup \{+\infty\}$, $i = m+1, ..., n$ be continuous functions. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and nondereasing for each of its first m variables. Let $\bar{x} \in \operatorname{dom} f_1 \cap ... \cap \operatorname{dom} f_n$, and let $(f_1(\bar{x}), ..., f_n(\bar{x})) \in \operatorname{dom} g$. Suppose that the following asumptions are satisfied:*

i) All but one of sets $\operatorname{epi} f_i$, $i = 1, ..., m$; $\operatorname{gph} f_j$, $j = m+1, ..., n$ are sequentially normally compact respestively at $(\bar{x}, f_i(\bar{x}))$ and

$$(\lambda_1, ..., \lambda_n) \in \partial^\infty g(f_1(\bar{x}), ..., f_n(\bar{x})),\ x_i^* \in \hat{\partial}(\lambda_i f_i)(\bar{x}), \text{ if } \lambda_i \neq 0;\ x_i^* \in \partial^\infty(\tau_i f_i)(\bar{x}),$$

$$\text{if } \lambda_i = 0, i = 1, ..., n \ \& \sum_{i=1}^n x_n^* = 0 \implies x_1^* = ... = x_n^* = 0; \lambda_1 = ... = \lambda_n = 0.$$

$$ii) \quad x_k \overset{g[f_1,\ldots,f_n]}{\longrightarrow} \bar{x} \quad \implies \quad f_i(x_k) \to f_i(\bar{x}),\ i = 1, ..., m.$$

Then one has the chain of inclusions:

$$\hat{\partial} g[f_1, \cdots, f_n](\bar{x}) \subseteq \bigcup \Big\{ \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f)(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty(\tau_i f_i)(\bar{x}) : (\lambda_1, \cdots, \lambda_n) \in \hat{\partial} g\big(f_1(\bar{x}), \cdots, f_n(\bar{x})\big) \Big\}; \tag{4.6}$$

$$\partial^\infty g[f_1, \cdots, f_n](\bar{x}) \subseteq \bigcup \Big\{ \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f)(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty(\tau_i f_i)(\bar{x}) : (\lambda_1, \cdots, \lambda_n) \in \partial^\infty g\big(f_1(\bar{x}), \cdots, f_n(\bar{x})\big) \Big\}. \tag{4.7}$$

Proof. Similarly to the proof of Theorem 4.1, we have

$$g[f_1, ..., f_n](x) = \min_{(\lambda_i) \in \mathbb{R}^n} \left\{ (g(\lambda_1, ..., \lambda_n) + \sum_{i=1}^n \delta_{S_i}(x, \lambda_1, ..., \lambda_n) \right\}.$$

Hence, by Proposition 4.2, we obtain

$$\hat{\partial} g[f_1, ..., f_n](\bar{x}) \times \{0\} \times ... \times \{0\} \subseteq \hat{\partial}\big(g(.) + \delta_{S_1}(.)\big)(\bar{x}, f_1(\bar{x}), ..., f_n(\bar{x}));$$

$$\partial^\infty g[f_1, ..., f_n](\bar{x}) \times \{0\} \times ... \times \{0\} \subseteq \partial^\infty\big(g(.) + \delta_{S_1}(.)\big)(\bar{x}, f_1(\bar{x}), ..., f_n(\bar{x})).$$

It is easy to verify that all conditions of Corollary 4.4 are satisfied. Therefore, we derive that

$$\hat{\partial} g[f_1, ..., f_n](\bar{x}) \times \{0\} \times ... \times \{0\} \subseteq \{0\} \times \hat{\partial} g\big(f_1(\bar{x}), ..., f_n(\bar{x})\big) + \sum_{i=1}^n \hat{N}(S_i, (\bar{x}, f_1(\bar{x}), ..., f_n(\bar{x}));$$

$$\partial^\infty g[f_1,...,f_n](\bar{x})\times\{0\}\times...\times\{0\}\subseteq\{0\}\times\partial^\infty g\big(f_1(\bar{x}),...,f_n(\bar{x})\big)$$
$$+\sum_{i=1}^{n}\hat{N}(S_i,(\bar{x},f_1(\bar{x}),...,f_n(\bar{x})).$$

We use the argument similar to the one used in the proof of Theorem 4.1 to obtain (4.6), (4.7). The proof is complete. ☐

5. Optimality necessary conditions

Consider the standard problem of mathematical programming

$$(\mathfrak{P})\qquad \begin{aligned}&\text{minimizer } f_0(x)\quad\text{subject to}\\ &f_i(x)\le 0,\quad i=1,...,m;\\ &f_i(x)=0,\quad i=m+1,...,n;\\ &x\in C.\end{aligned}$$

Here the $f_i's$, $i=1,...,n$ are extended real-valued functions on X and C is a closed subset of X. We use the quantities τ_i, $i=0,...,n$: $\tau_i=1$, $i=0,...,m$ and $\tau_i\in\{-1,1\}$, $i=m+1,...,n$. We use the chain rules established in the previous section to obtain a Lagrange multiplier as follows:

Theorem 5.1 *Let X be an Asplund space. Let $f_i: X\to \mathbb{R}\cup\{+\infty\}$ be lower semicontinuous functions for $i=0,...,m$ and let $f_i: X\to\mathbb{R}\cup\{+\infty\}$ be continuous functions for $i=m+1,...,n$. Let $\bar{x}$ is local minimizer of $(\mathfrak{P})$. Suppose that C is sequentially normally compact at $\bar{x}$ and all but one of sets* $\operatorname{epi} f_i$, $i=0....m$; $\operatorname{gph} f_i$, $i=m+1....n$ *are sequentially normally compact respectively at $(\bar{x},f_i(\bar{x}))$. Then either*

*(**M1**). There exist $x_i^*\in\partial^\infty(\tau_i f_i)(\bar{x})$, $i=0,...,n$; $x_{n+1}^*\in\hat{N}(C,\bar{x})$ such that*

$$\sum_{i=1}^{n+1}x_i^*=0\quad\text{and}\quad\sum_{i=1}^{n+1}\|x_i^*\|=1,\quad \textit{or}$$

*(**M2**). There exist λ_i, $i=0,...,n$ not all zero such that $\lambda_i\ge 0$, $i=0,...,m$ and*

$$0\in\sum_{\lambda_i\neq 0}\hat{\partial}(\lambda_i f_i)(\bar{x})+\sum_{\lambda_i=0}\partial^\infty(\tau_i f_i)(\bar{x})+\hat{N}(C,\bar{x}).$$

Proof. Let us consider the following functions: $g:\mathbb{R}^{n+1}\to\mathbb{R}$,

$$g(\alpha_0,...,\alpha_n):=\begin{cases}\max\{\alpha_0,...,\alpha_n\} & \text{if}\quad \alpha_{m+1}=...=\alpha_n=0\\ \max\{\alpha_0,...,\alpha_m,|\alpha_{m+1}|,...,|\alpha_n|\} & \text{otherwise,}\end{cases}$$

and

$$F(x):=(f_0(x)-f_0(\bar{x}),...,f_m(x)-f_m(\bar{x}),f_{m+1}(x),...,f_n(x)).$$

If $\bar{x}$ is a local mimimum of $(\mathfrak{P})$, then $\bar{x}$ is a local mimimum of the function:

$$g\circ F(.)+\delta_C(.).$$

Therefore,

$$0 \in \hat{\partial}(g \circ F + \delta_C)(\bar{x}). \tag{5.1}$$

Observe that $(0,...,0) \notin \hat{\partial} g(0,...,0)$ and the first $m+1$ components of any vector in $\partial^{\infty} g(0) \cup \hat{\partial} g(0)$ must be nonnegative. Consider the following three cases:

Case 1. There exist $(\lambda_1, ..., \lambda_n) \in \partial^{\infty} g(F(\bar{x}))$ and $x_i^* \in \hat{\partial}(\lambda_i f_i)(\bar{x})$ for $\lambda_i \neq 0$; $x_i^* \in \hat{\partial}(\tau_i f_i)(\bar{x})$ for $\lambda_i = 0$ such that

$$\sum_{i=1}^{n} x_i^* = 0; \quad \sum_{i=1}^{n} \|x_i^*\| = 1.$$

In this case, if $\lambda_1 = ... = \lambda_n = 0$, then we obtain (**M1**), else, we obtain (**M2**), with $x_{n+1}^* = 0$.

Case 2. There exist $(\lambda_1, ..., \lambda_n) \in \partial^{\infty} g(F(\bar{x}))$ such that $(\lambda_1, ..., \lambda_n) \neq (0, ...0)$ and

$$0 \in \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f_i)(\bar{x}) + \sum_{\lambda_i = 0} \partial^{\infty}(\tau_i f_i)(\bar{x}).$$

Then, (**M2**) is satisfied with $x_{n+1}^* = 0$.

Case 3.

$$(\lambda_1, ..., \lambda_n) \in \partial^{\infty} g(F(\bar{x})),\ x_i^* \in \hat{\partial}(\lambda_i f_i)(\bar{x}), \text{ if } \lambda_i \neq 0;\ x_i^* \in \partial^{\infty}(\tau_i f_i)(\bar{x}), \text{ if } \lambda_i = 0,$$
$$i = 1, ..., n \ \&\ \sum_{i=1}^{n} x_n^* = 0 \quad \Longrightarrow \quad x_1^* = ... = x_n^* = 0 \quad \text{and} \quad \lambda_1 = ... = \lambda_n = 0.$$

Observe that all conditions of Theorem 4.10 are satisfied in this case. By virtue of the sum rule (Corollary 4.4), then either

(3.a) there are $u^* \in \partial^{\infty}(g \circ F)(\bar{x})$ and $x_{n+1}^* \in \hat{N}(C, \bar{x})$ such that

$$u^* + x_{n+1}^* = 0 \quad \text{and} \quad \|u^*\| + \|x_{n+1}^*\| = 1, \quad \text{or}$$

(3.b)

$$0 \in \hat{\partial}(g \circ F)(\bar{x}) + \hat{N}(C, \bar{x}).$$

In the case (3.a), it follows from Theorem 4.10 that there exist $(\lambda_1, ..., \lambda_n) \in \partial^{\infty} g(F(\bar{x}))$ and $x_i^* \in \hat{\partial}(\lambda_i f_i)(\bar{x})$ for $\lambda_i \neq 0$; $x_i^* \in \hat{\partial}(\tau_i f_i)(\bar{x})$ for $\lambda_i = 0$ such that

$$u^* = \sum_{i=1}^{n} x_i^*.$$

Hence

$$\sum_{i=1}^{n+1} x_i^* = 0 \quad \text{and} \quad \sum_{i=1}^{n+1} \|x_i^*\| = 1.$$

If $\lambda_1, .., \lambda_n$ are all zero, then one has (**M1**); otherwise, one has (**M2**).

In the case (3.b), From Theorem 4.10, we derive that

$$0 \in \bigcup_{(\lambda_i)_i \in \hat{\partial} g(F(\bar{x}))} \left\{ \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f_i)(\bar{x}) + \sum_{\lambda_i = 0} \partial^\infty(\tau_i f_i)(\bar{x}) \right\} + \hat{N}(C, \bar{x}).$$

Hence, we obtain (**M2**). The proof is complete. □

Remark 5.2 Function g used in proof of Theorem 5.1 follows Treiman ([37]). When all functions $f_i's$, $i = 0, ..., n$ are locally Lipschitz, we obtain in the framework of Asplund spaces a result which is stronger than Clarke's necessary condition ([3]). When X is finite dimensional, this result was established by Borwein-Treiman-Zhu [1]. Using another procedure, based on the extremal principle, Mordukhovich ([26]) has proved the necessary condition for problem $(\mathfrak{P})$ using normal cones to epigraphs or graphs of the f_i. The Lipschitz version for Fréchet smooth spaces was also obtained by Kruger & Mordukhovich ([21]). After the paper was completed, we did receive a preprint by Mordukhovich in which was proved necessary conditions in the framework of Asplund spaces. Observe from Propositions 2.1 and 2.2 that if $f_1(\bar{x}) = \cdots = f_m(\bar{x}) = 0$, then Theorem 5.1 (ii) in [31] is equivalent to the above result. Let us also remark that this equivalence is no longer true if $f_i(\bar{x}) < 0$ for some index $i \in \{1, \cdots, m\}$. For instance, it suffices to consider the problem:

$$\min f_0(x) \quad \text{subject to} \quad f_1(x) \leq 0$$

where $f_0(x) = x$ and

$$f_1(x) := \begin{cases} x - 1 & \text{if} \quad x \leq 0 \\ -x & \text{if} \quad x > 0. \end{cases}$$

Obviously, $x = 0$ is not a solution of $(\mathfrak{P})$ and the conclusion of the theorem above is not satisfied for $x = 0$. Despite this, the conclusion of Theorem 5.1 (ii) in [31] holds. However, if we replace epi f_i, $i \in \{1, \cdots, m\}$ by epi$f_i - f_i(\bar{x})$ in Mordukhovich's proof and if we make use of Propositions 2.1 and 2.2, we derive the result above.

In general, assertion (**M1**) of Theorem 5.1 cannot be avoided. For instance, consider $(\mathfrak{P})$: $\min f_0(x)$ s.t. $f_1(x) \leq 0$, where the functions $f_0, f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_0(x) = x^{1/3}$ and $f_1(x) = -x^{1/3}$. Obviously 0 is a solution of $(\mathfrak{P})$. However, $\hat{\partial} f_0(0) = \hat{\partial} f_1(0) = \emptyset$.

Now, we set

$$I(\bar{x}) := \{i \in 1, ..., m : f_i(\bar{x}) = 0\},$$

the set of *active inequality constraints*. In addition to assumptions in Theorem 4.1, assume that f_i is continuous at $\bar{x}$ for $i \notin I(\bar{x})$. We then obtain a Fritz Jhon type necessary condition as follows:

Theorem 5.3 *Under the assumptions in Theorem 5.1, in addition, suppose that f_i are continuous at $\bar{x}$ for all $i \notin I(\bar{x})$. If $\bar{x}$ is a local solution of $(\mathfrak{P})$, then either*

*(**M1**). There exist $x_i^* \in \partial^\infty(\tau_i f_i)(\bar{x})$, $i = \{0\} \cup I(x) \cup \{m+1, ..., n\}$; $x_{n+1}^* \in \hat{N}(C, \bar{x})$ such that*

$$x_0^* + \sum_{i \in I(\bar{x})} x_i^* + \sum_{i=m+1}^{n+1} x_i^* = 0 \quad \text{and} \quad \|x_0^*\| + \sum_{i \in I(\bar{x})} \|x_i^*\| + \sum_{i=m+1}^{n+1} \|x_i^*\| = 1, \qquad \textit{or}$$

(**M2**). *There exist* λ_i, $i \in \{0\} \cup I(\bar{x}) \cup \{m+1, ..., n\}$ *not all zero such that* $\lambda_i \geq 0$, $i \in \{0\} \cup I(\bar{x})$ *and*

$$0 \in \sum_{\lambda_i \neq 0} \hat{\partial}(\lambda_i f_i)(\bar{x}) + \sum_{\lambda_i = 0} \partial^{\infty}(\tau_i f_i)(\bar{x}) + \hat{N}(C, \bar{x}).$$

Proof. By Assumption, the $f_i's$ are continuous at $\bar{x}$, for $i \notin I(\bar{x})$, $i = 1, ..., m$, hence if $\bar{x}$ is a local solution of $(\mathfrak{P})$, then $\bar{x}$ is a local minimum of function:

$$f(x) + \delta_C(x).$$

Here,

$$f(x) := \begin{cases} \max\{f_0(x) - f_0(\bar{x}), \max_{i \in I(\bar{x})} f_i(x)\} & \text{if} \quad f_{m+1}(x) = ... = f_n(x) = 0 \\ \max\{f_0(x) - f_0(\bar{x}), \max_{i \in I(\bar{x})} f_i(x), |f_{m+1}(x)|, ..., |f_n(x)|\} & \text{otherwise.} \end{cases}$$

Without loss of generality, we can assume that $I(\bar{x}) := \{1, ..., m\}$. Then $\bar{x}$ is a local mimimum of function $g \circ F + \delta_C$. Here, g and F as in proof of Theorem 5.1. Hence, we derive the result. □

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