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EXTENSIONS OF FRÉCHET $\epsilon-$SUBDIFFERENTIAL CALCULUS AND APPLICATIONS

HUYNH VAN NGAI, DINH THE LUC AND M. THERA

Abstract. In this paper, we establish some calculus rules for the limiting Fréchet $\epsilon-$subdiferentials of marginal functions and composite functions. Necessary conditions for approximate solutions of a constrained optimization problem are derived.

1. Preliminaries

Recently it has become clear that several useful subdifferentials of nonsmooth functions such as Clarke's and Kruger-Mordukhovich's subdifferentials, Ioffe's approximate subdifferential etc. can be expressed in terms of Fréchet subdifferential or its limiting version that belong to the so-called small subdifferentials (see [7-8], [15], [17-19]). In order to exploit this property of Fréchet subdifferential, the authors of [9-10] introduced the notion of Fréchet $\epsilon-$subdifferential by relaxing the original Fréchet subdifferential to a bit bigger one within a small positive error, and the notion of limiting Fréchet $\epsilon-$subdifferential by taking its sequential limits in the weak topology. These new kinds of subdifferentials enjoy rich calculus and turn out to be very useful in the study of approximate solutions of optimization problems and approximate convex functions (see [9-10], [16]).
The purpose of the present paper is to further develop calculus rules for the above mentioned subdifferentials. The main concern is with marginal functions and composite functions. The key technique we are going to use is a fuzzy sum rule established by Fabian [5] and extended by Jourani [10] using Fréchet $\epsilon-$subdifferential under a metric inequality condition. With the help of the new calculus rules we derive necessary conditions for approximate solutions of constrained optimization problems in terms of $\epsilon-$subdifferentials.
The paper is organized as follows. The remaining part of this section deals with notations and a fuzzy sum rule for Fréchet $\epsilon-$subdifferential that is needed in the sequel. In Section 2, a calculus formula for the limiting Fréchet $\epsilon-$subdifferential of marginal functions is presented. Section 3 is devoted to calculus rules for the Fréchet $\epsilon-$subdifferential and the limiting Fréchet $\epsilon-$subdifferential of composite functions. In the final section, we apply the calculus rules established in Section 3 to derive necessary conditions for approximate solutions of a general nonsmooth constrained optimization problem.

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Throughout this paper, let us denote by X a Banach space, X^* its topological dual, B_X the closed unit ball in X, $B(x,\delta)$ the closed ball centered at $x \in X$ with radius $\delta > 0$ and B^* the closed unit ball in X^*. We adopt the following notation: "$\stackrel{s}{\to}$" (respectively "$\stackrel{w^*}{\to}$") denotes the convergence with respect to the strong (respectively the weak*topology), while $x_n \stackrel{f}{\to} x$ (respectively $x_n \stackrel{C}{\to} x$) means that the sequence $\{x_n\}_{n\in N}$ converges to x while $\{f(x_n)\}_{n\in \mathbb{N}}$ converges to $f(x)$ (respectively $x_n \to x$ while $x_n \in C$). For each closed convex set $C \subset X$, d_C denotes the distance from x to C: $d_C(x) := \inf_{y\in C} \|x-y\|$. We use the symbol $F : X \rightrightarrows Y$ to denote a set-valued (multivalued) mapping F, that is a mapping which assigns to each $x \in X$ a subset (possibly empty) of Y. We note graph $F := \{(x,y) \in X \times Y : y \in F(x)\}$ le *graph* of F. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be given and ϵ be a fixed nonnegative number. We recall ([9]) that the Fréchet $\epsilon-$subdifferential of f at $x \in \mathrm{Dom} f$ is defined by

$$\partial^F_\epsilon f(x) = \left\{ x^* \in X^* : \liminf_{\|h\|\to 0} \frac{f(x+h) - f(x) - \langle x^*, h\rangle}{\|h\|} \geq -\epsilon \right\}. \tag{1.1}$$

Clearly , $x^* \in \partial^F_\epsilon f(x)$ if and only if for each $\eta > 0$, there is $\delta > 0$ such that

$$\langle x^*, y - x\rangle \leq f(y) - f(x) + (\epsilon + \eta)\|y - x\| \quad \text{for all} \quad y \in x + \delta B. \tag{1.2}$$

The limiting Fréchet $\epsilon-$subdifferential at $x \in \mathrm{Dom} f$ is defined by

$$\hat{\partial}_\epsilon f(x) := \limsup_{y \stackrel{f}{\to} x} \partial^F_\epsilon f(y) \tag{1.3}$$

where "limsup" stands for the sequential Painlevé-Kuratowski upper limit of sets, i.e,

$$\limsup_{y \stackrel{f}{\to} x} \partial^F_\epsilon f(y) := \{x^* \in X^* : \exists\, x_n \stackrel{f}{\to} x, x^*_n \stackrel{w^*}{\to} x^* \quad \text{with} \quad x^*_n \in \partial^F_\epsilon f(x_n) \quad \forall n \in \mathbb{N}\}.$$

The limiting singular subdifferential of f at x is the set

$$\partial^\infty f(x) := \limsup_{y \stackrel{f}{\to} x, \lambda \downarrow 0^+} \lambda \partial^F f(y). \tag{1.4}$$

When $\epsilon = 0$, the sets (1.1) and (1.3) are denoted by $\partial^F f(x), \hat{\partial} f(x)$ respectively.

Note that if f is a lower semicontinuous convex function, then $\partial^F_\epsilon f(x) = \hat{\partial}_\epsilon f(x) = \partial f(x) + \epsilon B^*$ for all $\epsilon \geq 0$, where ∂f is the subdifferential in the sense of convex analysis. For nonconvex functions a similar formula is true (see [9], [19]). Namely, assume that X is an Apslund space, that is, a Banach space in which every convex lower semicontinuous function is Fréchet differentiable on a dense $G_\delta-$subset of the interior of its domain (see [18] for other characterizations of Asplund spaces). Then one has

$$\hat{\partial}_\epsilon f(x) = \hat{\partial} f(x) + \epsilon B^*.$$

Further, let $\delta_C(.)$ denote the indicator function of a set $C \subset X$, that is $\delta_C(x) = 0$ if $x \in C$ and $\delta_C(x) = +\infty$ otherwise. The set of Fréchet $\epsilon-$normals to C at x is given by

$$N^F_\epsilon(C, x) := \partial^F_\epsilon \delta_C(x).$$

Obviously we have

$$N^F_\epsilon(C,x) := \Big\{x^* \in X^* : \limsup_{y \to^C x} \frac{\langle x^*, y-x\rangle}{\|y-x\|} \le \epsilon \Big\}.$$

The set of limiting Fréchet $\epsilon-$normals to C at x is defined by

$$\hat{N}_\epsilon(C,x) := \hat{\partial}_\epsilon \delta_C(x) = \limsup_{y \to^C x} N^F_\epsilon(C,y).$$

In this paper we shall frequently make use of a "fuzzy sum rule", originally proven by Fabian [5] in the context of Asplund spaces for a sum of two functions, when one of which is locally Lipschitzian, and then extended by Jourani [10] to the case where both functions are lower semicontinuous. First let us introduce some notations.

For every $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$, we set

$$S_1 := \Big\{(x,\alpha,\beta) \in X \times \mathbb{R} \times \mathbb{R} : f_1(x) \le \alpha \Big\};$$

$$S_2 := \Big\{(x,\alpha,\beta) \in X \times \mathbb{R} \times \mathbb{R} : f_2(x) \le \beta \Big\}.$$

We say that the pair (f_1, f_2) satisfies the *metric inequality* (MI) at $x_0 \in \mathrm{Dom} f_1 \bigcap \mathrm{Dom} f_2$, if there are $a > 0, r > 0$ such that

$$d_{S_1 \bigcap S_2}(x,\alpha,\beta) \le a[d_{S_1}(x,\alpha,\beta) + d_{S_2}(x,\alpha,\beta)] \qquad (MI)$$

for all $(x,\alpha,\beta) \in B(x_0,r) \times B(f_1(x_0),r) \times B(f_2(x_0),r)$. Note that if one of the functions f_1 and f_2 is locally Lipschitzian at x_0, then (MI) holds. Moreover, if X is an Asplund space, then (MI) also holds provided there are a cone K^* locally compact in the weak* topology and $r > 0$ such that

$$\partial^F d((x,\alpha), \mathrm{epi} f_1) \subset K^* \times \mathbb{R}$$

for all $(x,\alpha) \in \big(B(x_0,r) \times B(f_1(x_0),r)\big) \bigcap \mathrm{epi} f_1$ and

$$\partial^\infty f_1(x_0) \bigcap (-\partial^\infty f_2(x_0)) = \{0\}.$$

Actually the first part of the above condition can be substituted by the following weaker one: for every sequences $\{(x_n,\lambda_n)\}_{n\in\mathbb{N}}$ and $\{(x_n^*,\lambda_n^*)\}_{n\in\mathbb{N}}$ satisfying the following relations:

$$(x_n^*,\lambda_n^*) \in N^F(\mathrm{epi} f_1,(x_n,\lambda_n);\ (x_n,\lambda_n) \to (x_0,f_1(x_0));\ x_n^* \overset{w^*}{\to} 0 \ \text{ and } \lambda_n^* \to 0,$$

one has $\|x_n^*\| \to 0$ as $n \to \infty$.
In [20] a function satisfying this latter condition is called *sequentially normally epi-compact.*

We finally recall the promised extended fuzzy sum rule (see [10]): Assume that X is an Asplund space and $f_1, f_2 : X \to \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous and satisfy (MI) at $x_0 \in \mathrm{Dom} f_1 \cap \mathrm{Dom} f_2$. Then for each $x^* \in \partial^F_\epsilon(f_1+f_2)(x_0)$, for each $\gamma > 0, \delta > 0, b_1 > a\|x^*\|+3$ and $b_2 > a\|x^*\|+3$, there exist $x_i \in x_0 + \gamma B_X$, $f_i(x_i) \in f_i(x_0) + \gamma B_{\mathbb{R}}$, and $x_i^* \in \partial^F f_i(x_i)$, $\|x_i^*\| \le 2b_i, i = 1,2$ such that

$$\|x^* - x_1^* - x_2^*\| \le \epsilon + 2\delta(1 + b_1 + b_2).$$

In the sequel, the following sum rule for the limiting Fréchet $\epsilon-$subdifferential (see [10]) will be needed: Let f_1 and f_2 be lower semicontinous and satisfy (MI) at $x_0 \in \mathrm{Dom} f_1 \cap \mathrm{Dom} f_2$. Then for every $\epsilon \geq 0$, we have

$$\hat{\partial}_\epsilon (f_1 + f_2)(x_0) \subset \bigcap_{\alpha_1+\alpha_2=\epsilon} \Big(\hat{\partial}_{\alpha_1} f_1(x_0) + \hat{\partial}_{\alpha_2} f_2(x_0) \Big).$$

2. Limiting Fréchet $\epsilon-$subdifferential of marginal functions

Let us consider a general parametrized constrained optimization problem:

$$(P_u) : p(u) = \min_{x \in F(u)} \varphi(u, x)$$

where $\varphi : U \times X \to \mathrm{I\!R} \bigcup \{+\infty\}$ is a lower semicontinuous function defined on the product of two Banach spaces U and X, and F is a set-valued map from U to X. In general, p is nonsmooth, even if φ is differentiable and $F(u) = X$ for all $u \in U$. In this section we wish to establish a calculus rule for the limiting Fréchet $\epsilon-$subdifferential of p in terms of the limiting Fréchet $\epsilon-$subdifferential of φ and the normal cone to the graph of F. For this purpose, let us derive a formula for the $\epsilon-$normal set $\hat{N}_\epsilon(\mathrm{graph} F, .)$ by using the distance function $d(F, .)(u, x) := d_{F(u)}(x)$.

Proposition 2.1 *Let U and X be Banach spaces and let F be a set-valued map from U to X with closed graph. Let $\bar{x} \in F(\bar{u})$. Then one has*

$$\hat{N}_\epsilon(\mathrm{graph} F, .)(\bar{u}, \bar{x}) = \bigcup_{\lambda > 0} \hat{\partial}_\epsilon (\lambda d(F, .))(\bar{u}, \bar{x}). \tag{2.1}$$

Proof. We follow the proof of [21]. For the inclusion "$\subseteq$", let $(u^*, x^*) \in \hat{N}_\epsilon(\mathrm{graph} F, .)(\bar{u}, \bar{x})$. There exist sequences $\{(u_n, x_n)\}_{n \in \mathrm{I\!N}} \subset \mathrm{graph} F$, $(u_n, x_n) \to (\bar{u}, \bar{x})$, $\{(u_n^*, x_n^*)\}_{n \in \mathrm{I\!N}} \subset N_\epsilon^F(\mathrm{graph} F, .)(u_n, x_n)$ such that $(u_n^*, x_n^*) \stackrel{w^*}{\to} (u^*, x^*)$. Therefore, the sequence $\{(u_n^*, x_n^*)\}_{n \geq 1}$ is bounded and there is $\lambda_0 > 0$ such that

$$\|u_n^*\| + \|x_n^*\| \leq \lambda_0, \quad \forall n \in \mathrm{I\!N}. \tag{2.2}$$

By (1.2), for each $1 > \eta > 0$, there is $\delta_n > 0$, such that

$$\langle u_n^*, u - u_n \rangle + \langle x_n^*, x - x_n \rangle \leq (\eta + \epsilon)(\|u - u_n\| + \|x - x_n\|), \tag{2.3}$$

for all $(u, x) \in \big((u_n, x_n) + \delta_n B_X \times B_Y\big) \cap \mathrm{graph} F$. We claim that there is some $\lambda > 0$ such that $(u_n^*, x_n^*) \in \partial^F \lambda d(F, .)(u_n, x_n)$. Indeed, let $(u, x) \in \big((u_n, x_n) + \frac{\delta_n}{2} B_X \times B_Y\big)$ be arbitrarily fixed. Choose $z \in F(u)$ with $\|z - x\| \leq 2 d_{F(u)}(x)$. If $d(F(u), x) \geq \delta_n/2$, then

$$\langle u_n^*, u - u_n \rangle + \langle x_n^*, x - x_n \rangle \leq \lambda_0(\delta_n/2 + \delta_n/2) \leq 2\lambda_0 d(F(u), x).$$

If $d(F(u), x) < \delta_n/2$, then $\|z - x_n\| < \delta_n$. By (2.2) and (2.3) we obtain

$$\begin{aligned} \langle u_n^*, u - u_n \rangle + \langle x^*, x - x_n \rangle &= \langle u_n^*, u - u_n \rangle + \langle x_n^*, x - z \rangle + \langle x_n^*, z - x_n \rangle \\ &\leq \lambda_0 \|x - z\| + (\eta + \epsilon)(\|u - u_n\| + \|z - x_n\|) \\ &\leq \lambda d_{F(u)}(x) + (\eta + \epsilon)(\|u - u_n\| + \|x - x_n\|) \end{aligned}$$

with $\lambda = 2(\lambda_0 + \epsilon + \eta)$. Consequently, $(u_n^*, x_n^*) \in \partial^F \lambda d(F, .)(u_n, x_n)$ with $\lambda = 2(\lambda_0 + \epsilon + 1)$.

For the inclusion " $\supseteq$", let $(u^*, x^*) \in \hat{\partial}_\epsilon(\lambda d(F, .)(\bar{u}, \bar{x}))$, for some $\lambda > 0$. We claim that $(u^*, x^*) \in \hat{N}_\epsilon(\mathrm{graph} F)(\bar{u}, \bar{x})$. Indeed, by the definition, there are sequences

$\{(u_n, x_n)\}_{n\in\mathbb{N}}$ with $(u_n, x_n) \overset{d(F,\cdot)}{\to} (\bar{u}, \bar{x})$ and $\{(u_n^*, x_n^*)\}_{n\in\mathbb{N}}$ with $(u_n^*, x_n^*) \in \partial_\epsilon^F \lambda d(F, .)$ (u_n, x_n) and $(u_n^*, x_n^*) \overset{w^*}{\to} (u^*, x^*)$. Fix a sequence $\{\eta_n\}_{n\in\mathbb{N}}$ such that $\eta_n \downarrow 0$. Then pick a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ with $0 < \delta_n < \frac{\eta_n}{4}$ such that for all $(u, x) \in (u_n, x_n) + \delta_n B_U \times B_X$. One has

$$\langle u_n^*, u-u_n\rangle+\langle x_n^*, x-x_n\rangle \leq \lambda\big(d(F(u), x)-d(F(u_n), x_n)\big)+(\frac{\eta_n}{2}+\epsilon)(\|(u, x)-(u_n, x_n)\|). \tag{2.4}$$

We now construct sequences $\{(\bar{u}_n, \bar{x}_n)\}_{n\geq 1} \subset \text{graph}F$ converging to $(\bar{u}, \bar{x})$ and $\{(\bar{u}_n^*, \bar{x}_n^*)\}_{n\geq 1}$ with $(\bar{u}_n^*, \bar{x}_n^*) \in N_\epsilon^F(\text{graph}F)(\bar{u}_n, \bar{x}_n)$ and $(u_n^*, x_n^*) \overset{w^*}{\to} (u^*, x^*)$. For this purpose, choose $x_n' \in F(u_n)$ with

$$\|x_n' - x_n\| \leq d(F(u_n), x_n) + \frac{\delta_n^2}{\lambda}. \tag{2.5}$$

Then, $(u_n, x_n') \overset{\text{graphF}}{\to} (\bar{u}, \bar{x})$. For $(u, x) \in \big((u_n, x_n') + \delta_n B_U \times B_X)\big) \cap \text{graph}F$, one has $d(F(u), x_n + x - x_n') \leq \|x_n + x - x_n' - x\| = \|x_n - x_n'\|$ and hence by (2.4) and (2.5) one obtains

$$\langle u_n^*, u - u_n\rangle + \langle x_n^*, x - x_n'\rangle \leq \delta_n^2 + (\frac{\eta_n}{2} + \epsilon)(\|u - u_n\| + \|x_n - x_n'\|.$$

By the Ekeland variational principle [2], there exists $(\bar{u}_n, \bar{x}_n) \in \big((u_n, x_n') + \frac{\delta_n}{2} B_U \times B_X\big) \cap \text{graph}F$ such that

$$-\langle u_n^*, \bar{u}_n-u_n\rangle-\langle x_n^*, \bar{x}_n-x_n'\rangle+(\frac{\eta_n}{2}+\epsilon)(\|(\bar{u}_n, \bar{x}_n)-(u_n, x_n')\|)-2\delta_n(\|(u, x)-(\bar{u}_n, \bar{x}_n)\|)$$
$$\leq -\langle u_n^*, u - u_n\rangle - \langle x_n^*, x - x_n'\rangle + (\frac{\eta_n}{2} + \epsilon)(\|(u, x) - (u_n, x_n')\|)$$

for all $(u, x) \in \big((u_n, x_n') + \delta_n B_U \times B_X\big) \cap \text{graph}F$. Since $\delta_n < \frac{\eta_n}{4}$ one has

$$\langle u_n^*, u - \bar{u}_n\rangle + \langle x_n^*, x - \bar{x}_n\rangle \leq (\eta_n + \epsilon)(\|u - \bar{u}_n\| + \|x - \bar{x}_n\|)$$

or equivalently

$$\frac{\epsilon}{\eta_n + \epsilon}\big(\langle u_n^*, u - \bar{u}_n\rangle + \langle x_n^*, x - \bar{x}_n\rangle\big) \leq \epsilon(\|u - \bar{u}_n\| + \|x - \bar{x}_n\|).$$

This shows that $\frac{\epsilon}{\eta_n+\epsilon}(u_n^*, x_n^*) \in N_\epsilon^F(\text{graph}F, .)(\bar{u}_n, \bar{x}_n)$. Passing to limit when n tends to ∞ we obtain the required inclusion $(u^*, x^*) \in \hat{N}_\epsilon(\text{graph}F, .)(\bar{u}, \bar{x})$. △

Corollary 2.2 *Let X be a Banach space and $C \subset X$ a nonempty closed subset of X. Then for every $\bar{x} \in C$ one has*

$$\hat{N}_\epsilon(C, .)(\bar{x}) = \bigcup_{\lambda>0} \hat{\partial}_\epsilon(\lambda d(C, .)(\bar{x}).$$

Proof. This is derived from Proposition 2.1, by using the set-valued mapping $F : X \Longrightarrow X$ defined by $F(x) := C$ for all $x \in X$. △

In [9], was given a formula to compute the $\epsilon-$subdifferential of p when $\varphi(., .)$ is locally Lipschitzian. A similar formula for the limiting Fréchet $\epsilon-$subdifferential of p can be established under a compactness assumption and a qualification condition.

Theorem 2.3 *Assume that U and X are Asplund spaces, F has a closed graph and φ is lower semicontinuous and sequentially normally epi-compact at $(\bar{u}, \bar{x})$ where $\bar{u} \in U$ and $\bar{x} \in F(\bar{u})$ with $p(\bar{u}) = \varphi(\bar{u}, \bar{x})$. Assume further the following conditions:*
i) $(-\partial^\infty\varphi(\bar{u}, \bar{x}))\bigcap \hat{N}(\text{graph}F, .)(\bar{u}, \bar{x}) = \{(0, 0)\}$;

ii) For every sequence $\{u_n\}_{n\in \mathbb{N}}$ such that $u_n \xrightarrow{p} \bar{u}$, there exists a subsequence $\{u_{n_m}\}$ such that there exists a sequence $x_{n_m} \in F(u_{n_m})$ with limit $\bar{x}$ and $p(u_{n_m}) = \varphi(u_{n_m}, x_{n_m})$.
Then one has

$$\hat{\partial}_\epsilon p(\bar{u}) \times \{0\} \subset \bigcap_{\alpha_1+\alpha_2=\epsilon} \left(\hat{\partial}_{\alpha_1}\varphi(\bar{u},\bar{x}) + \hat{N}_{\alpha_2}(\text{graph}F,.)(\bar{u},\bar{x})\right).$$

Proof. Invoke the proof of Theorem 2.18 in [9] by using the fuzzy sum rule (see Section 1) instead of Theorem 2.17 of [9]. △

In order to proceed to another rule, let us recall [23] that a set-valued map F from U to X is said to be Lipschitzian at $\bar{u} \in U$ if it has nonempty closed values on U and if there exist $\kappa > 0$ and a neighbourhood V of $\bar{u}$ such that

$$F(u') \subseteq F(u) + \kappa\|u' - u\|B_X,$$

for all $u', u \in V$. Observe that F is *Lipschitzian* at $\bar{u}$ iff it has nonempty closed values and the function $d(F,.)(u,x) := d_{F(u)}(x)$ is Lipschitzian at $(\bar{u}, x)$ for all $x \in X$.

The following result of [3] will be also needed.

Lemma 2.4 *Suppose that $f : X \to \mathbb{R}$ attains a minimum over $C \subset X$ at $x \in C$ and f is Lipschitzian in $B(x,\delta)$ with Lipschitz constant $\kappa_0 > 0$. Then for any $\kappa \geq \kappa_0$ the function $g(y) := f(y) + \kappa d_C(y)$ attains a local minimum over $B(x, \frac{\delta}{2})$ at x.*

We now are able to provide a rule to compute the limiting Fréchet $\epsilon-$subdifferential of p when $\varphi(u,.)$ is uniformly Lipschitzian with respect to the second variable. The compactness asumption and the qualification condition i) of the previous theorem are no more needed.

Theorem 2.5 *Assume that U and X are Asplund spaces, F is Lipschitzian at $\bar{u}$, where $(\bar{u},\bar{x}) \in U \times X$ with $p(\bar{u}) = \varphi(\bar{u},\bar{x})$ and φ is lower semicontinuous in both variables and uniformly Lipschitzian in the second variable at $\bar{x}$ for u sufficiently close to $\bar{u}$ with a common Lipschitz constant κ. If condition ii) of the previous theorem is satisfied, then one has*

$$\hat{\partial}_\epsilon p(\bar{u}) \times \{0\} \subset \hat{\partial}_\epsilon\varphi(\bar{u},\bar{x}) + \kappa\hat{\partial}d(F,.)(\bar{u},\bar{x}). \tag{2.6}$$

As a result one obtains

$$\hat{\partial}_\epsilon p(\bar{u}) \times \{0\} \subset \hat{\partial}_\epsilon\varphi(\bar{u},\bar{x}) + \hat{N}(\text{graph}F,.)(\bar{u},\bar{x}). \tag{2.7}$$

Proof. Assume that $\varphi(u,.)$ is Lipschitzian in a ball $B(\bar{x}, 3\delta_1)$ with a common Lipschitz constant κ for every u near $\bar{u}$. Obviously, $\varphi(u,.)$ is also uniformly Lipschitzian in the ball $B(x, 2\delta_1)$ for all $x \in B(\bar{x},\delta_1)$. According to condition ii), there exists $\delta_0 > 0$ such that for each $u \in B(\bar{u},\delta_0)$ with $|p(u) - p(\bar{u})| \leq \delta_0$, there is $x_u \in B(\bar{x}, \frac{\delta_1}{4}) \cap F(u)$ such that $p(u) = \min_{x\in F(u)} \varphi(u,x) = \varphi(u,x_u)$. Choose a positive number $\delta_2 < \delta_0$ and take $u^* \in \hat{\partial}_\epsilon p(\bar{u})$. Due to the definition, there are sequences $u_n \in U$, $x_n \in F(u_n)$, $u_n^* \in \partial_\epsilon^F p(u_n)$, such that $u_n \xrightarrow{p} \bar{u}$, $u_n^* \xrightarrow{w^*} u^*$. Moreover, by condition ii), without loss of generality, we may assume that there is a sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ with $x_n \to \bar{x}$ and $p(u_n) = \varphi(u_n, x_n)$. Then, the sequence $\{u_n^*\}_{n\geq 1}$ is bounded, that is there exists $\lambda_0 > 0$ such that $\|u_n^*\| \leq \lambda_0$ for all $n \in \mathbb{N}$. By (1.2), for each $\eta > 0$, there exists $\delta_{n,\eta} > 0$ such that

$$p(u_n + h) - p(u_n) - \langle u_n^*, h\rangle \geq -(\eta + \epsilon)\|h\| \quad \forall h \in \delta_{n,\eta} B_U. \tag{2.8}$$

Thus, for every $n \geq 1$, there exists a positive number $\gamma_{1,n}$ such that $\gamma_{1,n} < \min\{\delta_{n,\eta}, \frac{\delta_0}{4\lambda_0}\}$, and

$$p(u_n + h) - p(u_n) > -\frac{\delta_0}{2} \quad \forall h \in \gamma_{1,n} B_U. \tag{2.9}$$

When n is large, say $n \geq n_0$, one has $\|x_n - \bar{x}\| < \frac{\delta_1}{8}$, $\|u_n - \bar{u}\| < \delta_2$ and $|p(u_n) - p(\bar{u})| < \frac{\delta_0}{2}$. Set $f(u, x) := \varphi(u, x) + \kappa d(F, .)(u, x)$. For $n \geq n_0$ and for $h \in U$ and $k \in X$ small enough, since F is Lipchizt at $\bar{u}$, there exists a constant $M > 0$ such that $F(u_n) \subseteq F(u_n + h) + M\|h\| B_X$. Consequently, there are $z_{n,h} \in F(u_n + h)$ and $b \in B_X$ such that $x_n = z_{n,h} + M\|h\| b$. Moreover, since $\varphi(u, .)$ is Lipschitzian with constant κ, one deduces that

$$\varphi(u_n + h, x_n + k) \geq \varphi(u_n + h, z_{n,h}) - \kappa\|k + M\|h\| b\|$$

and therefore

$$f(u_n + h, x_n + k) \geq p(u_n + h) - \kappa\|k + M\|h\| b\|.$$

It follows that for $n \geq n_0$, there exists $\gamma_{2,n} > 0$ such that

$$f(u_n + h, x_n + k) - f(u_n, x_n) > p(u_n + h) - p(u_n) - \frac{\delta_0}{4} \tag{2.10}$$

for all $h \in \gamma_{2,n} B_U$, $k \in \gamma_{2,n} B_X$. Let $h \in \min\{\gamma_{1,n}, \delta_0 - \delta_2, \gamma_{2,n}\} B_U$ and $k \in \min\{\gamma_{1,n}, \frac{\delta_1}{8}, \gamma_{2,n}\} B_X$. One has

$$\|u_n + h - \bar{u}\| \leq \|u_n - \bar{u}\| + \|h\| < \delta_0.$$

We distinguish two cases: $|p(u_n + h) - p(\bar{u})| \leq \delta_0$ and $|p(u_n + h) - p(\bar{u})| > \delta_0$. In the first case, as noticed before, there exists $x_{n,h} \in B(\bar{x}, \frac{\delta_1}{4})$ such that

$$p(u_n + h) = \min_{x \in F(u)} \varphi(u_n + h, x) = \varphi(u_n + h, x_{n,h}).$$

By Lemma 2.4, one has $p(u_n + h) = \min_{x \in B(x_{n,h}, \frac{\delta_1}{2})} f(u_n + h, x)$. On the other hand, $\|x_n + k - x_{n,h}\| < \frac{\delta_1}{2}$, hence $p(u_n + h) \leq f(u_n + h, x_n + k)$. Thus, by (2.8), we get

$$f(u_n + h, x_n + k) - f(u_n, x_n) - \langle (u_n^*, 0), (h, k)\rangle \geq -(\eta + \epsilon)(\|h\| + \|k\|). \tag{2.11}$$

In the second case, since $|p(u_n) - p(\bar{u})| < \delta_0/2$, one derives $|p(u_n + h) - p(u_n)| > \delta_0/2$. Moreover, (2.9) implies $p(u_n + h) - p(u_n) > \delta_0/2$. Hence by (2.10), we obtain

$$f(u_n + h, x_n + k) - f(u_n, x_n) > \frac{\delta_0}{4} > \langle (u_n^*, 0), (h, k)\rangle - (\eta + \epsilon)(\|h\| + \|k\|).$$

Thus, (2.11) also holds. Therefore, $(u_n^*, 0) \in \partial^F f(u_n, x_n)$ and $(u^*, 0) \in \hat{\partial}_\epsilon f(\bar{u}, \bar{x})$ as well. By the assumption, we can apply the sum rule to obtain

$$\hat{\partial}_\epsilon f(\bar{u}, \bar{x}) \subset \hat{\partial}_\epsilon \varphi(\bar{u}, \bar{x}) + \kappa \hat{\partial} d(F, .)(\bar{u}, \bar{x})$$

and to derive (2.6). Using Proposition 2.1, we deduce (2.7) and the proof is complete. △

It is worthwhile noticing that if $\varphi(.,.)$ is locally Lipschitzian, then $\varphi(u, .)$ is uniformly Lipschitzian. The converse does not necessarily hold. For example, $\varphi : \mathbb{R}^2 \to \mathbb{R}$, $\varphi(u, x) = \sqrt{|u|} + x$ is uniformly Lipschitzian with respect to the second variable at $x = 0$, but it is not locally Lipschitzian around $(0, 0)$. Moreover, Theorem 2.3 and 2.5 cannot be deduced from each other. Neither the conditions used in these theorems imply each other. This can be seen by the following examples.

1) Let $U = X = c_0$ and let $\varphi : c_0 \times c_0 \to 1\mathrm{R}$ defined by $\varphi(u, x) = \sqrt{\|u\|} + \|x\|$. Then $\varphi(., .)$ is uniformly Lipschitzian in the variable x around $(0, 0)$. Direct calculation shows that $\partial^F \varphi(0, 0) = l_1 \times B_{l_1}$. Hence φ is not sequentially normally epi-compact at $(0, 0)$.

2) Consider the function $\varphi : 1\mathrm{R} \times c_0 \to 1\mathrm{R}$ defined by $\varphi(u, x) = \sqrt{|u|} + \|x\|$ and $F : 1\mathrm{R} \to c_0$; $F(u) = \Big((\sin nu)/n\Big)_n$. Then φ is uniformly Lipschitzian and F is Lipschitzian. Observe that $(1, 0) \in \partial^\infty \varphi(0, 0)$. On the other hand, denote by $\{e_1, e_2, ..., e_n, ...\}$ the usual *basis* of the topological dual l_1 of c_0. Then $e_n \overset{w^*}{\to} 0$ and for every $u \in 1\mathrm{R}$, $(\cos nu, -e_n) \in N^F(\text{graph})(u, F(u))$. Hence, by taking $u_n = \pi/n$, one has $(-1, -e_n) \in N^F(\text{graph})(u_n, F(u_n))$. Consequently, $(-1, 0) \in (-\partial^\infty \varphi(0, 0)) \cap \hat{N}(\text{graph})(0, 0)$.

3) Let $U = X = 1\mathrm{R}$. Let $\varphi : 1\mathrm{R} \to 1\mathrm{R} \cup \{+\infty\}$ defined by

$$\varphi(u, x) = \begin{cases} u - \sqrt{x} & \text{if} \quad x \geq 0 \\ +\infty & \quad \text{otherwise,} \end{cases}$$

$$F(u) = \begin{cases} \{0\} & \text{if} \quad u = 0 \\ [-\sqrt{|u|}, 0] & \quad \text{otherwise.} \end{cases}$$

We see that φ is not uniformly Lipschitzian and F is not Lipschitzian around zero. Despite of this, all the conditions of Theorem 2.3 are satisfied.
Observe also that when $F(u) = C$ for all $u \in U$ with C a nonempty closed subset of X, Theorem 2.5 is an improvement of Theorem 2.18 in [9], in which φ is required to be Lipschitzian in both variables.

Corollary 2.6 *Let U, X and F be as in Theorem 2.5. If φ is lower semicontinuous in (u, x), continuous and linear in x for u in a small neighborhood U_0 of u_0, bounded on U_0 for each $x \in X$ and if condition ii) of Theorem 2.3 is verified, then the conclusion of Theorem 2.5 remains true.*

Proof. By virtue of the Banach-Steinhaus theorem , $\varphi(u, .)$ is uniformly bounded on U_0, that is, there is $M_0 > 0$ such that $\|\varphi(u, .)\|_{X^*} \leq M_0$ for all $u \in U_0$. Hence $|\varphi(u, x_1) - \varphi(u, x_2)| \leq M_0\|x_1 - x_2\|$ for all $u \in U_0$ and $x_1, x_2 \in X$. This shows that $\varphi(u, .)$ is uniformly Lipschitzian on U_0. Apply Theorem 2.5 to achieve the proof.$\triangle$

3. $\epsilon-$SUBDIFFERENTIAL OF COMPOSITE FUNCTIONS

A calculus rule for the limiting Fréchet $\epsilon-$subdifferential of the composition of a locally Lipschitzian function with a Fréchet differentiable mapping has been established in [9]. A similar result for the Kruger-Mordukhovich subdifferential has been obtained in [19] for the composition of a normally compact function with a strictly Lipschitzian mapping. The concept of strict Lipschitzianity is an infinite dimension version of locally Lipschitzian mappings and is actually equivalent to the concept of compact Lipschitianity of Jourani and Thibault [13]. In this section, we wish to extend the chain rules of [9] for the Fréchet $\epsilon-$subdifferential and the limiting Fréchet $\epsilon-$subdifferential to a broader class of functions. Let us first introduce the notion of strictly compactly Lipschitzian mappings.

Definition 3.1 *Let X and Y be Banach spaces. A mapping $F : X \to Y$ is said to be strictly compactly Lipschitzian at $\bar{x} \in X$ if for each sequences $x_n \to \bar{x}$, $h_n \to 0$, $h_n \neq 0$, the sequence*

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} \quad n = 1, 2, ...$$

has a norm convergent subsequence.

Recall [19] that a mapping $F : X \to Y$ is said to be *strictly Lipschitzian* at $\bar{x} \in X$ if it is Lipschitzian at $\bar{x}$ and the sequence

$$\frac{F(x_n + t_n v) - F(x_n)}{t_n} \quad n = 1, 2, ...$$

has a convergent subsequence in the norm topology of Y for each $v \in X$, $x_n \to \bar{x}$ and $t_n \downarrow 0$ as $n \to \infty$.

It is obvious that a strictly compactly Lipschitzian mapping is strictly Lipschitzian, hence locally Lipschitzian. The converse is also true if Y is finite dimensional. In general, a strictly Lipschitian mapping fails to be strictly compactly Lipschitian, as the examplke of the mapping $F : c_0 \to c_0$ given by $F(\{x_n\}_{n\in \mathbb{N}}) := \{sin x_n\}_{n\in \mathbb{N}}$ shows. Moreover, if F is strictly Fréchet differentiable and its derivative F' is a compact operator, or if F is a composition $G \circ F_0$ where G is strictly differentiable with G' being a compact operator and F_0 is Lipschitzian, then F is strictly compactly Lipschitzian. The class of mappings with the above properties is quite large. It includes for instance Fredholm integral operators with Lipschitzian kernels.

The following proposition provides another characterization of strictly compactly Lipschitian mappings. (see Thibault [23] for a similar characterization of strictly Lipchitz mappings

Proposition 3.2 *Let X and Y be Banach spaces. Then a mapping $F : X \to Y$ is strictly compactly Lipschitian at $\bar{x} \in X$ if and only if there is a set-valued mapping $K : X \rightrightarrows Y$ and a function $r : X \times X \to [0, +\infty)$ such that*

(i) $\lim_{x\to\bar{x},h\to 0} \frac{r(x,h)}{\|h\|} = 0$;

(ii) There is $\alpha > 0$ such that for all $h \in \alpha B_X$, $x \in \bar{x} + \alpha B_X$ one has

$$F(x + h) - F(x) \in K(h)\|h\| + r(x, h)B_Y \quad \forall x \in \bar{x} + \alpha B_X,\, h \in \alpha B_X;$$

(iii) $\bigcup_{\|h\|<\alpha} K(h)$ is compact in Y.

Proof. Let K and r be as in the proposition. Let $x_n \to x$, $h \to 0$. Then there are $y_n \in K(h_n)$, $a_n \in B_Y$ such that

$$\frac{F(x_n + h_n) - F(x_n)}{\|h_n\|} = y_n + \frac{r(x_n, h_n)}{\|h_n\|} a_n.$$

Since $\bigcup_{h\in\alpha B} K(h)$ is compact, the sequence $\{y_n\}_{n\in \mathbb{N}}$ has a convergent subsequence. This implies that $\left\{\frac{F(x_n+h_n)-F(x_n)}{\|h_n\|}\right\}_{n\in \mathbb{N}}$ has a convergent subsequence.

Conversely, suppose that F is strictly compactly Lipschitian at $x \in X$. Define

$$K := \left\{ y \in Y : \exists x_n \to \bar{x},\, h_n \to 0, \quad y = \lim_{n\to\infty} \frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \right\},$$

$$r(x,h) = \begin{cases} \|h\| \Big(d_K \big(\frac{F(x+h)-F(x)}{\|h\|} \big) + \|h\| \Big) & \text{if} \quad h \neq 0 \\ 0 & \text{if} \quad h = 0. \end{cases}$$

Obviously,

$$F(x+h) - F(x) \in K\|h\| + r(x,h)B_Y$$

and (i) holds. We claim that K is compact. Indeed, let $\{k_n\}_{n\in\mathbb{N}}$ be a sequence in K. For every n, there are sequences $x_i^n \to x$, $h_i^n \to 0$ as $i \to \infty$ such that

$$k_n = \lim_{i\to\infty} \frac{F(x_i^n + h_i^n) - F(x_i^n)}{\|h_i^n\|}.$$

Hence, there exist $x_n \to \bar{x}$, $h_n \to 0$ such that

$$\left\| k_n - \frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \right\| < \frac{1}{n}.$$

Since F is strictly compactly Lipschitian, the sequence $\left\{ \frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, $\{k_n\}$ has a convergent subsequence too. The proof is complete. △

Below we give a characterization of strictly Lipschitian mappings.

Proposition 3.3 *Let X and Y be Banach spaces. Then $F : X \to Y$ is strictly Lipschitian at $x \in X$ if and only if F is Lipschitian at x and for each sequences $x_n \to x$ and $h_n \to 0$ such that $\left\{ \frac{h_n}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a norm convergent subsequence, the sequence*

$$\frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \quad n = 1, 2, ...$$

has a norm convergent subsequence.

Proof. The "if" part is obvious. For the "only if" part, suppose that F is strictly Lipschitian. Let $x_n \to x$ and $h_n \to 0$, ($h_n \neq 0$) such that the sequence $\left\{ \frac{h_n}{\|h_n\|} \right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Without loss of generality, we may assume that $\frac{h_n}{\|h_n\|} \to v$. Since F is strictly Lipschitian, the sequence

$$\frac{F(x_n + \|h_n\|v) - F(x_n)}{\|h_n\|} \quad n = 1, 2, ...$$

has a convergent subsequence. On the other hand, one has

$$\left\| \frac{F(x_n + \|h_n\|v) - F(x_n)}{\|h_n\|} - \frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \right\| \leq M \left\| v - \frac{h_n}{\|h_n\|} \right\|,$$

where M is a Lipschitz constant of F. It follows that the sequence

$$\frac{F(x_n+h_n)-F(x_n)}{\|h_n\|} \quad n = 1, 2, ...$$

also has a convergent subsequence and the proof is complete. △

Observe that for mappings from a finite dimensional space to a Banach space there is no distinction between strict Lipschitzianity and strictly compact Lipschitzianity. Neverthless, the class of strictly Lipschitzian mappings does not coincide with the class of Lipschitzian mappings. For example, the mapping $F : \mathbb{R} \to c_0$ defined by $F(x) = ((\sin nx)/n)_n$ is Lipschitzian, but not strictly Lipschitzian. Moreover, one can show without any difficulties that the class of strictly compactly Lipschitzian mappings is a vector space, that is, if F and H are strictly compactly Lipschitzian, then tF and $F + H$ are strictly compactly Lipschitzian for all $t \in \mathbb{R}$; the product of two strictly compactly Lipschitzian mappings is strictly compactly Lipschitzian, and so is the composition of a strictly compactly Lipschitzian mapping with a Lipschitzian mapping.

Another charaterization of strictly Lipschitzian and strictly compactly Lipschitzian mappings is given in terms of Fréchet normal cones.

Proposition 3.4 *Let X and Y be Banach spaces and $F : X \to Y$ be a Lipschitzian mapping at $x \in X$. The following assertions hold:*
(i) If F is strictly Lipschitzian at x, then for each sequences $x_n \to x$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$ with $y_n^* \xrightarrow{w^*} 0$, one has $x_n^* \xrightarrow{w^*} 0$.
(ii) If F is strictly compactly Lipschitzian at x, then for each sequences $x_n \to x$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$, with $y_n^* \xrightarrow{w^*} 0$, one has $x_n^* \to^s 0$.
Moreover, if in addition X is an Asplund space and Y is reflexive, then the converse of (i) and (ii) is true.

Proof. The first assertion was proven in Abdouni-Thibault [1; Lemma 2.5] and [19]. For the converse assertion, suppose X is an Asplund space and Y is reflexive. Let $h \in X$ and $x_n \to x$, $t_n \downarrow 0$. We show that the sequence

$$y_n := \frac{F(x_n + t_n h) - F(x_n)}{t_n} \quad n = 1, 2, ...$$

has a norm convergent subsequence. Since Y is reflexive, and $\{y_n\}_{n\in\mathbb{N}}$ is bounded (because F is Lipschitzian at x), the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a weak-convergent subsequence. We may assume that $y_n \xrightarrow{w} y$. By the Hahn-Banach theorem, for each n, there exists $y_n^* \in Y^*$ such that

$$\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2, \quad \|y_n^*\| = \|y_n - y\|.$$

Then, the sequence $\{y_n^*\}_{n\in\mathbb{N}}$ is bounded and we may assume that $y_n^* \xrightarrow{w^*} y^*$. Using the mean value theorem ([16],[19],[26]), there are $v_n \in B([x_n, x_n + t_n h], \frac{1}{n})$ and $v_n^* \in \partial^F (y_n^* - y^*)F(v_n)$ such that

$$\langle y_n^* - y^*, F(x_n + t_n h) - F(x_n)\rangle - \frac{t_n}{n} \leq \langle v_n^*, t_n h\rangle$$

where $B([x_n, x_n + t_n h], \frac{1}{n}) := \{x \in X : d_{[x_n, x_n+t_n h]}(x) \leq \frac{1}{n}\}$. Hence,

$$\langle y_n^* - y^*, y_n\rangle - \frac{1}{n} \leq \langle v_n^*, h\rangle.$$

0bserve that

$$(x^*, -y^*) \in \partial_\epsilon N^F(x, F(x)) \Longleftrightarrow x^* \in \partial^F(y^* F)(x).$$

Thus $(v_n^*, y^* - y_n^*) \in N^F(\text{graph}, .)(v_n, F(v_n))$. Since $v_n \to x$, by the assumption, $v_n^* \xrightarrow{w^*} 0$. Therefore

$$\limsup_{n\to\infty} \langle y_n^* - y^*, y_n\rangle \le 0.$$

On the other hand,

$$\langle y_n^* - y^*, y_n\rangle = \langle y_n^*, y_n - y\rangle + \langle y_n^* - y^*, y\rangle - \langle y^*, y_n - y\rangle.$$

Since $y_n^* \xrightarrow{w^*} y^*$, $y_n \xrightarrow{w} y$ and $\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2$, the above shows that $y_n \xrightarrow{s} y$.

For the second assertion, let $x_n \to x$, $(x_n^*, -y_n^*) \in N^F(\text{graph}F, .)(x_n, F(x_n))$, and $y_n^* \xrightarrow{w^*} 0$. We want to show that $\|x_n^*\| \to 0$. For every n, take $h_n \in X$ with $\|h_n\| = 1$ such that $\langle x_n^*, h_n\rangle > \|x_n^*\| - 1/n$. Pick a sequence $\delta_n \downarrow 0$ such that for all $x \in x_n + \delta_n B_X$, one has

$$\langle x_n^*, x - x_n\rangle - \langle y_n^*, F(x) - F(x_n)\rangle \le 1/n(\|x - x_n\| + \|F(x) - F(x_n)\|).$$

Consequently, by taking $x = x_n + \delta_n h_n$, we obtain

$$\langle x_n^*, h_n\rangle - \left\langle y_n^*, \frac{F(x_n + \delta_n h_n) - F(x_n)}{\delta_n}\right\rangle \le 1/n\left(1 + \left\|\frac{F(x_n + \delta_n h_n) - F(x_n)}{\delta_n}\right\|\right).$$

Since F is strictly compactly Lipschitzian, we may assume that the sequence $\{(F(x_n + \delta_n h_n) - F(x_n))/\delta_n\}_{n\in\mathbb{N}}$ is norm convergent. Therefore,

$$\lim_{n\to\infty}\left\langle y_n^*, \frac{F(x_n + \delta_n h_n) - F(x_n)}{\delta_n}\right\rangle = 0$$

and by this $\langle x_n^*, h_n\rangle \to 0$ as $n \to \infty$. Thus, $x_n^* \xrightarrow{s} 0$.

Conversely, let $x_n \to x$, $h_n \to 0$. In order to show that the sequence $\{y_n\}_{n\in\mathbb{N}}$ defined by $y_n := (F(x_n + h_n) - F(x_n))/\|h_n\|$ has a norm convergent subsequence, we use the same argument as the one developed in the converse part of the first assertion. Indeed, we may assume that $y_n \xrightarrow{w} y$. Take y_n^* such that $\langle y_n^*, y_n - y\rangle = \|y_n - y\|^2$, $\|y_n^*\| = \|y_n - y\|$ and $y_n^* \xrightarrow{w^*} y^*$. By the mean value theorem, there are $v_n \in X$, $v_n^* \in \partial^F(y_n^* - y^*)F(v_n)$ such that

$$\langle y_n^* - y^*, y_n\rangle - \frac{1}{n} \le \left\langle v_n^*, \frac{h_n}{\|h_n\|}\right\rangle$$

and $v_n \to x$, $v_n^* \xrightarrow{s} 0$. Hence, $\limsup_{n\to\infty}\langle y_n^* - y^*, y_n\rangle \le 0$. This yields $y_n \xrightarrow{s} y$. The proof is complete. △

We are now ready to obtain the main result of this section. Recall that $\delta_{\text{graph}F}(.,.)$ is the indicator function of the graph of F.

Theorem 3.5 *Let X be an Asplund space, let $F : X \to Y$ be a mapping from X to Y, and let $g : Y \to \mathbb{R}\cup\{+\infty\}$ be a lower semicontinuous function. Let $\bar{x} \in X$, $\bar{y} := F(\bar{x}) \in \text{Dom}g$. Assume that the fuzzy sum rule is satisfied for $g(v) + \delta_{\text{graph}F}(u, v)$. The following assertions hold:*

(a) If F is strictly Lipschitzian at $\bar{x}$, then

$$\hat{\partial}_\epsilon(g\circ F)(\bar{x}) \subset \bigcup_{y^*\in\hat{\partial}g\left(F(\bar{x})\right)} \hat{\partial}_\epsilon(y^*F)(\bar{x})\,; \tag{3.1}$$

(b) If F is strictly compactly Lipschitzian at $\bar{x}$, then for each $\gamma > 0$, $\delta > 0$, one has

$$\partial^F_\epsilon (g \circ F)(\bar{x}) \subset \bigcup_{y^* \in \hat{\partial} g\big(F(\bar{x})\big)} \Big\{ \partial^F_\epsilon (y^* F)(x) + \gamma B^*_X : x \in \bar{x} + \delta B_X \Big\}. \quad (3.2)$$

Proof. In order to show (3.1), let us define

$$h(u, v) := g(v) + \delta_{\text{graph}F}(u, v), \quad \text{and} \quad f(u) := \inf_{v \in Y} h(u, v) = g\big(F(u)\big).$$

Observe that

$$(x^*, -y^*) \in \partial^F_\epsilon \delta_{\text{graph}F}(x, F(x)) \Longleftrightarrow x^* \in \partial^F_\epsilon (y^* F)(x).$$

Assertion (a) is a consequence of a similar formula for limiting Fréchet subdifferential ([19]); we include its proof for sake of completness. Let $x^* \in \hat{\partial}_\epsilon f(\bar{x}) := \hat{\partial}_\epsilon (g \circ F(\bar{x})$. By definition, there are sequences $x_n \overset{g \circ F}{\to} x$ and $x^*_n \in \partial^F_\epsilon f(x_n)$ such that $x^*_n \overset{w^*}{\to} x^*$. Since $x^*_n \overset{w^*}{\to} x^*$, the sequence $\{x^*_n\}_{n \in \mathbb{N}}$ is bounded. Thus, for some $M > 0$, we have $\|x^*_n\| \le M$ for all $n \in \mathbb{N}$. As $x^*_n \in \hat{\partial}_\epsilon f(\bar{x}_n)$, one has

$$(x^*_n, 0) \in \partial^F_\epsilon \big(g(.) + \delta_{\text{graph}F}(.,.)\big)(x_n, F(x_n)).$$

By using the fuzzy sum rule with $\delta = \frac{1}{n}$, $b_1, b_2 > aM + 3$, there are sequences $\{u_n\}_{n \in \mathbb{N}} \subset X$, $\{v_n\}_{n \in \mathbb{N}} \subset Y$ such that $\|u_n - x_n\| < \frac{1}{n}$, $\|F(u_n) - F(x_n)\| < \frac{1}{n}$, $\|v_n - F(x_n)\| < \frac{1}{n}$, $v^*_n \in \partial^F g(v_n)$, $(u^*_n, -y^*_n) \in \partial^F_\epsilon \delta_{\text{graph}F}(u_n, F(u_n))$, $\|(u^*_n, -y^*_n)\| \le b_1$, $\|v^*_n\| \le b_2$ and

$$(x^*_n, 0) \in (0, v^*_n) + (u^*_n, -y^*_n) + \frac{1}{n}(b_1 + b_2) B^*_X \times B^*_Y.$$

The latter inclusion is equivalent to

$$x^*_n \in u^*_n + \frac{1}{n}(b_1 + b_2) B^*_X, \quad (3.3)$$

and

$$y^*_n \in v^*_n + \frac{1}{n}(b_1 + b_2) B^*_Y. \quad (3.4)$$

Observe that the sequence $\{y^*_n\}_{n \in N}$ is bounded. Since X is Asplund, the closed unit ball in X^* is weak*-sequentially compact. Hence we may assume that $y^*_n \overset{w^*}{\to} y^*$. Therefore, $v^*_n \overset{w^*}{\to} y^*$, yielding $y^* \in \hat{\partial} g(F(\bar{x}))$. As noticed above, since $(u^*_n, -y^*_n) \in \partial^F_\epsilon \delta_{\text{graph}F}(u_n, F(u_n))$, one has $u^*_n \in \partial^F_\epsilon (y^*_n F)(u_n)$. The fuzzy sum rule applied to $y^*_n F = (y^*_n - y^*)F + y^* F$, with $\delta = \gamma = \frac{1}{n}$, yields the existence of sequences $\{u^1_n\}_{n \in \mathbb{N}}$, $\{u^2_n\}_{n \in \mathbb{N}}$ such that $\|u^1_n - u_n\| < \frac{1}{n}$, $\|u^2_n - u_n\| < \frac{1}{n}$, $u^{1*}_n \in \partial^F (y^*_n - y^*) F(u^1_n)$, $u^{2*}_n \in \partial^F_\epsilon y^* F(u^1_n)$, and $u^*_n = u^{1*}_n + u^{2*}_n$. By virtue of Proposition 3.4 and as $y^*_n \overset{w^*}{\to} y^*$, we have $u^{1*}_n \overset{w^*}{\to} 0$. Since $x^*_n \overset{w^*}{\to} x^*$, the inclusion (3.3) yields $u^*_n \overset{w^*}{\to} x^*$. Hence, $u^{2*}_n \overset{w^*}{\to} x^*$. This shows that $x^* \in \partial^F_\epsilon (y^* F)(\bar{x})$, the inclusion (3.1) is obtained.

To prove (3.2), let $x^* \in \partial^F_\epsilon f(\bar{x}) := \partial^F_\epsilon (g \circ F)(\bar{x})$. Similarly to the proof of (3.1), there are sequences $\{u_n\}_{n \in N}$, $\{(u^*_n, -y^*_n)\}_{n \in N}$ such that $\|u_n - \bar{x}\| < \frac{1}{n}$, $(u^*_n, -y^*_n) \in \partial^F_\epsilon I(u_n, F(u_n))$; $x^* \in u^*_n + \frac{1}{n}(b_1 + b_2) B^*_X$; and $y^*_n \overset{w^*}{\to} y^*$ with $y^* \in \hat{\partial} g(F(\bar{x}))$. Let us now apply the fuzzy sum rule to $y^*_n F = (y^*_n - y^*)F + y^* F$. There are sequences $u^1_n \to \bar{x}$, $u^2_n \to \bar{x}$, $u^{1*}_n \in \partial^F (y^*_n - y^*) F(u^1_n)$, $u^{2*}_n \in \partial^F_\epsilon y^* F(u^2_n)$ such that $u^*_n = u^{1*}_n + u^{2*}_n$. By Proposition 3.5, $u^{1*}_n \overset{s}{\to} 0$, hence $u^{2*}_n \overset{s}{\to} x^*$ and (3.2) follows. The proof is complete. △

We mention that the fuzzy sum rule is satisfied for $g(v) + \delta_{\text{graph}F}(u,v)$ if g is locally Lipschitzian or more generally, if g is sequentially normally epi-compact and the following qualification condition is verified:

$$\Big[y^* \in \partial^\infty g\big(F(\bar{x})\big) \quad \& \quad (0,-y^*) \in \hat{N}\big(\text{graph}F,(\bar{x},F(\bar{x}))\big)\Big] \Longrightarrow y^* = 0.$$

Corollary 3.6 *Let X be an Asplund space, let $f_i : X \to 1\text{R}$, $i = 1,2,...,n$, be locally Lipschitzian at $\bar{x}$. Let $f(x) := \max\{f_i(x) : i = 1,2,...n\}$. Then for every $\gamma > 0$ and $\delta > 0$ one has the inclusions:*

$$\partial^F_\epsilon f(\bar{x}) \subset \text{co}\bigcup\Big\{\partial^F_\epsilon f_i(x) + \delta B^* : x \in \bar{x} + \gamma B, \quad i \in I(\bar{x})\Big\}; \tag{3.5}$$

and

$$\hat{\partial}_\epsilon f(\bar{x}) \subset \text{co}\Big\{\hat{\partial}_\epsilon f_i(\bar{x}) : i \in I(\bar{x})\Big\}, \tag{3.6}$$

where "co" denotes the convex hull of a set and $I(\bar{x}) := \{i : f_i(\bar{x}) = f(\bar{x})\}$.

Proof. Observe that $f = g \circ F$, where $g : 1\text{R}^n \to 1\text{R}$ with $g(x_1, x_2, ..., x_n) := \max_i x_i$, and $F : X \to 1\text{R}^n$ with $F(x) := \big(f_1(x), f_2(x), ..., f_n(x)\big)$. Clearly, F is locally Lipschitzian. Hence it is strictlly compactly Lipschitzian. Note also that g is convex Lipschitzian with

$$\hat{\partial} g(y) = \partial g(y) = \left\{(\lambda_i) : \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1, \; \lambda_i = 0 \quad \text{if} \quad i \notin I(y)\right\}.$$

Applying Theorem 3.5 to the functions g and F above, we obtain (3.5) and (3.6) due to the fuzzy sum rule and the sum rule of the limiting Fréchet $\epsilon-$subdifferential. △

For the purpose of applications we derive the following corollary.

Corollary 3.7 *Let X and Y be Asplund spaces. Let $F : X \to Y$ be strictly compactly Lipschitzian at $\bar{x} \in X$. Let $K \subset Y^*$ be a nonempty weak*-compact convex subset and*

$$f(x) := \max\{\langle y^*, F(x)\rangle : y^* \in K\}.$$

Then for each $\gamma > 0$, $\delta > 0$ one has

$$\partial^F_\epsilon f(\bar{x}) \subset \bigcup\Big\{\partial^F_\epsilon(y^*F)(x) + \gamma B^* : \text{ for } y^* \in K \text{ with } y^*F(\bar{x}) = f(\bar{x}), \; x \in \bar{x} + \delta B\Big\}.$$

Proof. Let $g(y) := \max\{\langle y^*, y\rangle : y \in K\}$ be the support functional of K. Obviously, g is convex Lipschitzian and

$$\hat{\partial} g(y) = \partial g(y) = \{y^* \in K : \langle y^*, y\rangle = g(y)\}.$$

So $f = g \circ F$ and the corollary follows immediately from Theorem 3.5. △

4. Application to optimality conditions

Let f be a function from X to $1\text{R} \cup \{+\infty\}$ and $x \in X$. Recall [9] that x is an $\epsilon-$minimizer of f if

$$f(y) \geq f(x) - \epsilon \quad \text{for all} \quad y \in X, \tag{4.1}$$

and x is $\epsilon\|.\|-$minimizer if

$$f(y) \geq f(x) - \epsilon\|y - x\| \quad \text{for all} \quad y \in X. \tag{4.2}$$

A necessary condition for x to be an $\epsilon\|.\|-$minimizer of f is that x satisfies the inclusion $0 \in \partial_\epsilon^F f(x)$. Certainly, this holds when (4.2) is satisfied for all y in some neighbourhood of x.

We shall say that x is *a local* $\epsilon-$*minimizer* (respectively $\epsilon\|.\|-$*minimizer*) of f if (4.1) (respectively (4.2)) is satisfied in some neighbourhood of x. Similarly, x is said to be *an* $\epsilon-$*minimizer* (respectively $\epsilon\|.\|-$*minimizer*) of f on C if (4.1) (respectively (4.2)) is satisfied for all $y \in C$.

By using the Ekeland variational principle, the following relation between $\epsilon-$minimum and $\epsilon\|.\|-$minimum points has been given in [9]: Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be lower semicontinuous. If x_0 is an $\epsilon-$minimizer of f on a nonempty set $C \subseteq X$, then for every $\delta > 0$, there exists $\bar{x} \in B(x_0, \delta)$ such that $\bar{x}$ is an $\epsilon/\delta\|.\|-$ minimizer of f on C.

For convex functions, it is well known that every local minimum is a global minimum. For $\epsilon-$convex functions, a similar property can be expected. Recall [9] that a function $f : X \to \mathbb{R} \cup \{+\infty\}$ is $\epsilon-$convex if it satisfies the following inequality for every $x, y \in X$ and $\lambda \in (0, 1)$:

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) + \epsilon\lambda(1-\lambda)\|x-y\|.$$

Proposition 4.1. *Let* ϵ_1 *and* $\epsilon_2 > 0$ *and let* f *be an* ϵ_1*-convex function. Then every local* $\epsilon_2\|.\|-$*minimizer of* f *is a global* $(\epsilon_1+\epsilon_2)\|.\|-$*minimizer of* f.

Proof. The proof is similar to the convex case. Let x be a local $\epsilon_2\|.\|-$minimizer of f. There is $\delta > 0$ such that

$$f(y) \geq f(x) - \epsilon_2\|y-x\| \quad \text{for all} \quad y \in x + \delta B.$$

Let $y \in X$, $y \notin x + \delta B$. Then $x + \delta\frac{y-x}{\|y-x\|} \in x + \delta B$ and

$$f\left(x + \delta\frac{y-x}{\|y-x\|}\right) \geq f(x) - \epsilon_2\|y-x\|.$$

Since f is $\epsilon_1\|.\|-$convex, one has

$$\begin{aligned} f\left(x + \delta\frac{y-x}{\|y-x\|}\right) \leq \quad & \left(1 - \tfrac{\delta}{\|y-x\|}\right) f(x) + \tfrac{\delta}{\|y-x\|} f(y) \\ & + \epsilon_1 \left(1 - \tfrac{\delta}{\|y-x\|}\right) \tfrac{\delta}{\|y-x\|}\|x-y\|, \end{aligned}$$

and therefore $f(y) \geq f(x) - (\epsilon_1+\epsilon_2)\|y-x\|$ for all $y \in X$. The proof is complete. △

Proposition 4.2 *Let* C *be a nonempty closed subset of* X. *Let* $f : X \to \mathbb{R} \cup \{+\infty\}$ *be a lower semicontinuous function, bounded from below on* $C \subset X$. *Then for all* $\epsilon > 0$, *the function* f *has at least an* $\epsilon\|.\|-$*minimizer on* C.

Proof. Invoke the Ekeland variational principle [2]. △

Proposition 4.3 *Let* X *be an Asplund space and* C *be a nonempty closed subset of* X. *Assume that* $f : X \to \mathbb{R}$ *is Lipschitzian at* $\bar{x} \in C$. *Then a necessary condition for* $\bar{x}$ *to be an* $\epsilon\|.\|-$*minimizer of* f *on* C *is that*

$$0 \in \hat{\partial}_\epsilon f(\bar{x}) + \hat{N}_C(\bar{x}).$$

Conversely, if f is $\epsilon'-$convex for some $\epsilon' \geq 0$ and C is convex, then the inclusion above is a sufficient condition for $\bar{x}$ to be an $(\epsilon+\epsilon')\|.\|-$minimizer of f on C.

Proof. Let $\bar{x} \in C$ be an $\epsilon\|.\|-$minimizer of f on C which is Lipschitzian at $\bar{x}$ with a Lipschitz constant κ. By Lemma 2.4, $\bar{x}$ is a local $\epsilon\|.\|-$minimizer of the function $h(x) := f(x) + \kappa d_C(x)$. Therefore $0 \in \hat{\partial}_\epsilon(f(.) + \kappa d_C(.))(\bar{x})$. By the sum rule and Corollary 3.2, we obtain

$$0 \in \hat{\partial}_\epsilon f(\bar{x}) + \hat{N}_C(\bar{x}).$$

Now, let f be an $\epsilon'-$convex function, and let C be convex. It is obvious that $\hat{N}_C(\bar{x})$ is the normal cone in the sense of convex analysis, that is,

$$\hat{N}_C(\bar{x}) = N_C(\bar{x}) = \{x^* \in X^* : \langle x^*, x-\bar{x}\rangle \leq 0 \quad \text{for all}\, x \in C\}.$$

If $0 \in \hat{\partial}_\epsilon f(\bar{x}) + N_C(\bar{x})$, then there is $x^* \in \hat{\partial}_\epsilon f(\bar{x})$ such that $-x^* \in N_C(\bar{x})$. By virtue of Lemma 3.5 in [9], we obtain

$$f(x) \geq f(\bar{x}) + \langle x^*, x-\bar{x}\rangle - (\epsilon+\epsilon')\|x-\bar{x}\| \quad \forall x \in X.$$

Since $-x^* \in N_C(\bar{x})$, one has $\langle x^*, x-\bar{x}\rangle \geq 0$ for all $x \in C$. Combining these two inequalities, we obtain $f(x) \geq f(\bar{x}) - (\epsilon+\epsilon')\|x-\bar{x}\|$ for all $x \in C$. The proof is complete. △

Let us now consider a general constrained minimization problem:

$$\min f(x) \quad \text{s.t.} \quad F(x) \in -S, \tag{CP}$$

where $f : X \to \mathbb{R}$, $F : X \to Y$.

Assume that X and Y are Asplund spaces, and $S \subseteq Y$ is a nonempty convex closed cone. Denote by $C := \{x \in X : F(x) \in -S\}$ the feasible set of (CP) and by $S^* := \{y^* \in Y^* : \langle y^*, y\rangle \geq 0 \quad \forall y \in S\}$ the dual cone to S.

We say that $x \in C$ is *a local $\epsilon-$solution (resp., $\epsilon\|.\|-$solution)* of (CP) if x is a local $\epsilon-$minimizer (resp., $\epsilon\|.\|-$minimizer) of f on the feasible set C of (CP).

Now we can state the main result of this section about a necessary condition for (CP) to have a local $\epsilon\|.\|-$solution via Fréchet $\epsilon-$subdifferential and limiting Fréchet $\epsilon-$subdifferential.

Theorem 4.4 *Let X and Y be Asplund spaces. Assume that f is Lipschitzian at $\bar{x} \in C$ and F is strictly compactly Lipschitzian in some neighborhood of $\bar{x}$. If $\bar{x}$ is a local $\epsilon\|.\|-$solution of (CP), then for each sequence of positive numbers $\delta_n \downarrow 0$, there exist sequences $t_n := (\lambda_n, y_n^*) \in [0,+\infty) \times S^*$, $x_n^1 \to \bar{x}$, $x_n^2 \to \bar{x}$, such that*

$$\lambda_n + \|y_n^*\| = 1;$$

$$0 \in \lambda_n \partial_\epsilon^F f(x_n^1) + \partial^F \langle y_n^*, F\rangle(x_n^2) + \delta_n B_X^*; \tag{4.3}$$

$$\lim_{n\to\infty} \langle y_n^*, F(\bar{x})\rangle = 0. \tag{4.4}$$

Moreover, if $t := (\lambda, y^*)$ is a weak-limit point of the sequence $\{t_n\}_{n\in\mathbb{N}}$, then*

$$0 \in \lambda\hat{\partial}_\epsilon f(\bar{x}) + \hat{\partial}\langle y^*, F\rangle(\bar{x}); \tag{4.5}$$

$$\langle y^*, F(\bar{x})\rangle = 0. \tag{4.6}$$

Proof. The proof we present here is based on Clarke [3]. Let us consider the following set:

$$T := \Big\{(\lambda, y^*) \in R_+ \times S^* : |\lambda| + \|y^*\| \leq 1\Big\}.$$

Since this set is weak*-compact, by the Asplund property, it is weak*-sequentially compact. Fix a sequence $\{\delta_n\}_{n\in N}$ such that $\delta_n \downarrow 0$ and $t := (\lambda, y^*) \in T$. Consider the mappings defined by

$$L_{\delta_n}(x,t) := \lambda\big(f(x) - f(\bar{x}) + \epsilon\|x-\bar{x}\| + \delta_n^2/4\big) + \langle y^*, F(x)\rangle$$

and

$$G_{\delta_n}(x) := \max_{t\in T} L_{\delta_n}(x,t).$$

Observe that $G_{\delta_n}(.)$ is lower semicontinuous, $\lim_{x\to\bar{x},n\to\infty} G_{\delta_n}(x) = 0$, and $G_{\delta_n}(\bar{x}) = \frac{\delta_n^2}{4}$. Moreover $G_{\delta_n}(x) > 0$ for every $x \in X$. Indeed, if for some $x \in X$, $G_{\delta_n}(x)$ was negative, then as S is a convex closed cone, x would be a feasible solution and $f(x) < f(\bar{x}) - \epsilon\|x-\bar{x}\|$, which is a contradiction. In this way, $G_{\delta_n}(\bar{x}) \le \inf G_{\delta_n} + \delta_n^2/4$. By the Ekeland variational principle, there exists $u_n \in \bar{x} + \delta_n B_X$ such that

$$G_{\delta_n}(u_n) - \frac{\delta_n}{4}\|u_n - x\| \le G_{\delta_n}(x) \quad \text{for all} \quad x \in X.$$

It follows that u_n is a minimum point of the function $G_{\delta_n}(.) + \frac{\delta_n}{4}\|. - u_n\|$ and therefore

$$0 \in \partial^F(G_{\delta_n} + \frac{\delta_n}{4}\|. - u_n\|)(u_n).$$

Let us apply the fuzzy sum rule, to obtain

$$0 \in \left\{\bigcup \partial^F G_{\delta_n}(x) + \frac{\delta_n}{2}B^* : x \in B(u_n,\delta_n)\right\}. \tag{4.7}$$

Let $t_x = (\lambda_x, y_x^*)$ be a point at which the maximum defining $G(x)$ is attained. We have $\|t_x\| = 1$. Indeed, if $\|t_x\| < 1$, then as $G_{\delta_n}(x) > 0$, we obtain

$$G_{\delta_n}(x) < \frac{1}{\|t_x\|}G_{\delta_n}(x) = G_{\delta_n}(x)$$

a contradiction. It is clear that

$$0 \le \langle y_x^*, F(x)\rangle \le G_{\delta_n}(x). \tag{4.8}$$

Now, using Corollary 3.7, we obtain

$$\partial^F G_{\delta_n}(z) \subseteq \bigcup\left\{\partial^F L_{\delta_n}(x,t) + \frac{\delta_n}{4}B^* :\ x \in B(z,\delta_n), t \in T; L_{\delta_n}(z,t) = G_{\delta_n}(z)\right\}. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9) one can find sequences $t_n = (\lambda_n, y_n^*) \in T$, $x_n \to \bar{x}$, and $z_n \to \bar{x}$ such that

$$\begin{aligned}
&\lambda_n + \|y_n^*\| = 1;\\
&0 \in \partial^F L_{\delta_n}(x_n,t_n) + \frac{3\delta_n}{4}B^*;\\
&\text{and } \lim_{n\to\infty}\langle y_n^*, F(z_n)\rangle = 0.
\end{aligned}$$

Therefore $\lim_{n\to\infty}\langle y_n^*, F(\bar{x})\rangle = 0$. Applying the fuzzy sum rule to the function $L_{\delta_n}(x,t_n) = \lambda_n(f(x) - f(\bar{x}) + \epsilon\|x-\bar{x}\| + \delta_n^2/4) + \langle y_n^*, F(x)\rangle$ we obtain (4.3) and the first part is proved.

For the second part, let $t = (\lambda, y^*)$ be a weak*-limit point of $\{t_n\}_{n\in\mathbb{N}}$. Then $t \in S^*$ and $\langle y^*, F(\bar{x})\rangle = \lim_{n\to\infty}\langle y_n^*, F(z_n)\rangle = 0$, so that (4.6) is satisfied. By (4.3), there are sequences $x_n^{1*} \in \lambda_n \partial_\epsilon^F f(x_n^1)$, $x_n^{2*} \in \partial^F\langle y_n^*, F\rangle(x_n^2)$ such as

$$0 \in x_n^{1*} + x_n^{2*} + \delta_n B^*.$$

Since f is Lipschitzian, the sequence $\{x^{1*}\}_{n\in\mathbb{N}}$ is bounded. Using the Asplund property, this sequence has a weak*-limit point, and we may assume that $x_n^{1*} \to^{w^*} x^{1*}$. Therefore, $x_n^{2*} \to^{w^*} -x^{1*}$. Consequently, $x^{1*} \in \lambda\hat{\partial}_\epsilon f(\bar{x})$ and similarly to the proof of Theorem 3.5, we have $-x^{1*} \in \hat{\partial}\langle y^*, F\rangle(\bar{x})$, which completes the proof. △

Note that the sequence $\{t_n\}_{n\in\mathbb{N}}$ used in the first part of Theorem 4.4, may have no nonzero weak*-limit points. In this case, the second part of the theorem is trivial and does not give any information . It was established by Loewen in [15] that if S^* is locally compact (in particular, if Y is finite dimensional or Int$S$ is nonempty) then the sequence $\{t_n\}_{n\in\mathbb{N}}$ has a nonzero weak*-limit point. Next, we give a condition on the function F, which ensures that the sequence $\{t_n\}_{n\in\mathbb{N}}$ has a nonzero weak*-limit point.

Proposition 4.5 *Suppose that $F : X \to Y$ is strictly compactly Lipschitzian in some neighbourhood of $\bar{x} \in X$. Let $x \in X$, and $y_x^* \in S^*$ such that*

$$\langle y_x^*, F(x)\rangle = \max\{\langle y^*, F(x)\rangle : y^* \in S^*, \|y^*\| \le 1\}.$$

If the following condition is satisfied

$$\liminf_{x\to\bar{x}, F(x)\notin -S} \frac{\langle y_x^*, F(x) - F(\bar{x})\rangle}{d\big(x, F^{-1}(F(\bar{x}))\big)} > 0, \tag{4.10}$$

then the sequence $\{t_n\}_{n\in\mathbb{N}}$ used in Theorem 4.4 has a nonzero weak-limit point.*

Proof. Let $t_n := (\lambda_n, y_n^*)$. If $\{\lambda_n\}_{n\in\mathbb{N}}$ has a nonzero limit point, then we are done. Let us consider the case where $\lambda_n \to 0$. In this case, $\|y_n^*\| \to 1$. As in the proof of Theorem 4.4, there is a sequence $\{z_n\}_{n\ge 1} \subset X$ converging to $\bar{x}$ such that $L(z_n, t_n) = G(z_n)$. We claim that if n is large, then $F(z_n) \notin -S$ and

$$\langle y_n^*, F(z_n)\rangle = (1 - \lambda_n)\max\{\langle y^*, F(z_n)\rangle : y^* \in S^*, \|y^*\| \le 1\}.$$

Indeed, if $F(z_n) \in -S$, then $f(z_n) \ge f(\bar{x}) - \epsilon\|z_n - \bar{x}\|$. When n is large, one has $G(z_n) = L(z_n, t_n) < L(z_n, (1, 0))$, which is a contradiction. For every $y^* \in S^*$ with $\|y^*\| \le 1 - \lambda_n$, we have $(\lambda_n, y^*) \in T$ and $L(z_n, \lambda_n, y^*) \le L(z_n, \lambda_n, y_n^*)$, hence $\langle y^*, F(z_n)\rangle \le \langle y_n^*, F(z_n)\rangle$. Take $x_n \in X$ such that $F(x_n) = F(\bar{x})$ and $\|z_n - x_n\| < 2d\big(z_n, F^{-1}(F(\bar{x}))\big)$. Note that $z_n \to \bar{x}$, hence $x_n \to \bar{x}$. Since F is strictly compactly Lipschitzian, the sequence $\left\{\frac{F(z_n)-F(\bar{x})}{\|z_n - x_n\|}\right\}_{n\in\mathbb{N}}$ has a convergent subsequence. Hence, we may assume that $\frac{F(z_n)-F(\bar{x})}{\|z_n - x_n\|} \to y$. Let y^* be a weak*-limit point of the sequence y_n^*. According to (4.10) we obtain $\langle y^*, y\rangle > 0$. It follows that $y^* \ne 0$. The proof is complete. △

If $S = \{0\}$, then $S^* = Y^*$ and condition (4.10) takes a simpler form

$$\liminf_{x\to\bar{x}, F(x)\ne 0} \frac{\|F(x)\|}{d(x, F^{-1}(0))} > 0. \tag{4.11}$$

Recall [8], [13] that a mapping $F : X \to Y$ is said to be *metrically regular* at $x_0 \in X$ if there exist $r > 0$ and $a > 0$ such that

$$d(x, F^{-1}(y)) \leq \|y - F(x)\|$$

for all $(x, y) \in (x_0 + rB_x) \times (F(x_0) + rB_Y)$. Clearly, if F is metrically regular at $\bar{x}$, then it satisfies the condition (4.11). For more details on metric regularity, the reader is refered to [8] and [13].

Using the remark above and the argument of Theorem 4.4, we derive a necessary condition for $\epsilon\|.\|-$solutions of the following problem :

$$\min f(x) \quad \text{s.t.} \quad G(x) \in -S,\ H(x) = 0, \qquad (CP')$$

where $f : X \to \mathrm{1R}$, $G : X \to Y$, $H : X \to Z$, and $S \subset Y$ is a nonempty convex closed cone.

Theorem 4.6 *Assume that* X, Y *and* Z *are Asplund spaces,* f *is a Lipschitzian function,* G *and* H *are strictly compactly Lipschitzian mappings and*
(i) S^* *is locally compact;*
(ii) $\liminf_{z \to x, H(z) \neq 0} \frac{\|H(z)\|}{d(z, H^{-1}(0))} > 0$.
Then a necessary condition for x *to be an* $\epsilon\|.\|-$*solution of (CP') is that there exist* $\lambda \in [0, \infty)$, *and* $y^* \in S^*, z^* \in Z^*$ *not both zero such that*

$$0 \in \lambda \hat{\partial}_\epsilon f(x) + \hat{\partial} y^* G(x) + \hat{\partial} z^* H(x)\,;$$

$$\langle y^*, G(x) \rangle = 0.$$

For exact optimal solutions ($\epsilon = 0$), a similar result was obtained in [17],[20] for the case where Y is a finite dimension space (say, $Y = \mathrm{1R}^n$) and $S = \mathrm{1R}^n_+$, and in [1], [6] for Ioffe's approximate subdifferential. Note that in general, the limiting Fréchet subdifferential is smaller than the approximate subdifferential. Therefore the conclusion of Theorem 4.6 is charper than the corresponding one of [6]. The following example shows that the conclusion of Theorem 4.6 is not true if condition (ii) above is not satisfied.

Consider the problem :

$$\min f(x) \quad \text{s.t.} \quad H(x) = 0$$

with

$$f : c_0 \to \mathrm{1R}, \quad f(x) := \sum_{n=1}^{\infty} \frac{1}{n^2} x_n$$

$$H : c_0 \to c_0, \quad H(x) = \big(\frac{1}{n} x_n\big)_n.$$

The feasible set C of this problem is $\{0\}$ and obviously Theorem 4.6 fails to be true.

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