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SKETCHES AND SPECIFICATIONS

REFERENCE MANUAL

Third part:
Models

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SKETCHES AND SPECIFICATIONS — REFERENCE MANUAL

SKETCHES AND SPECIFICATIONS is a common denomination for several papers which deal with applications of Ehresmann's sketch theory to computer science. These papers can be considered as the first steps towards a unified theory for software engineering. However, their aim is not to advocate a unification of computer languages; they are designed to build a frame for the study of notions which arise from several areas in computer science.

These papers are arranged in two complementary families:

REFERENCE MANUAL and USER'S GUIDE.

The *reference manual* provides general definitions and results, with comprehensive proofs. On the other hand, the *user's guide* places emphasis on motivations and gives a detailed description of several examples. These two families, though complementary, can be read independently. No prerequisite is assumed; however, it can prove helpful to be familiar either with specification techniques in computer science or with category theory in mathematics.

These papers are under development, they are, or will be, available at:
`http://www.unilim.fr/laco/rapports`.

REFERENCE MANUAL:

First Part: Compositive Graphs

Second Part: Projective Sketches

Third Part: Models

USER'S GUIDE:

First Part: Wefts for Explicit Specification

Second Part: Mosaics for Implicit Specification

Third Part: Functional and Imperative Programs

In addition, further papers about APPLICATIONS are in progress, with several co-authors. They deal with various topics, including the notion of state in computer science [etat], overloading, coercions and subsorts.

These articles owe a great deal to the working group *sketches and computer algebra*; we would like to thank its participants, specially Catherine Oriat and Jean-Claude Reynaud, as well as the CNRS.

These papers have been processed with LaTeX and XY-pic.

Third part: Models

This paper is the third part of the reference manual [ref]. The aim of this third part is to introduce the *models* and *counter-models* of the projective sketches. The only prerequisites to read this paper are [ref1] and [ref2].

1 Introduction

In a family of papers [guide1,guide2,guide3], we put forward the use of *wefts* and *mosaics* for describing, studying, handling and building *specifications*. Indeed, we first show that the *terms* of a weft yield a relevant representation of the *programs*, in the frame of functional programming languages. Then (and this is the most interesting part), thanks to mosaics, we extend this to imperative programming and various *implicit* features (side-effects, errors treatment, state management,...), which do not fit easily into earlier algebraic specification paradigms.

Although it may prove helpful, for reading the family of papers [guide1,guide2,guide3], to have some acquaintance with algebraic specifications (see for instance [Goguen *et al.* 78] or [Astesiano *et al.* 99]) and category theory (see for instance [Mac Lane 71]), no prerequisite is assumed. Indeed, in our papers, we proceed carefully from examples to general definitions and results.

On the other hand, we believe that this family of papers, which forms the "*user's guide*" for our point of view on specifications, has to come with another family of papers, which forms the corresponding "*reference manual*". The aim of the reference manual is to give a purely formal presentation, with the optimal level of generality, of "all" the definitions, methods and results which are necessary for building a rigorous basis for the user's guide.

The aim of this third part is to introduce the *theory of models (and counter-models) of projective sketches.*

In the category of sets, the finality and initiality properties of (projective as well as inductive) limits can easily be given a concrete meaning: they correspond to the *satisfaction of some first-order formulae*, by the elements of the base sets, or of the vertex set, of these limit cones.

More globally, let us consider a functor from a projective sketch towards the category of sets, and let us assume that it assigns to each distinguished projective cone a projective limit cone. Then, this functor yields the following interpretations:

- each point of the projective sketch can be interpreted as a *sort* of elements,
- each arrow can be interpreted as a composition law between the elements of the sorts,
- each equality between composed arrows (*i.e.* each commutative diagram) can be interpreted as a (universally quantified) equation which is satisfied by the composition laws,
- and each distinguished projective cone can be interpreted as an (essentially "Horn-universal") formula which is satisfied by the composition laws.

So, such a functor can be identified to a set-valued model (in the sense of logicians) of some first-order logic theory, which is associated to the given projective sketch.

This can be extended (§ 2) from the category of sets to *any* category. The *models* of a projective sketch towards any category are the functors from the projective sketch towards the category which map distinguished projective cones to projective limit cones. The *homomorphisms* between these models are the natural transformations between the functors. Dually, the *counter-models* of a projective sketch towards any category are the contravariant functors which map distinguished projective cones to inductive limit cones. The *counter-homomorphisms* between these counter-models are the natural transformations between the functors.

Of course, the representations between projective sketches, the models of a projective sketch towards a category, and the functors between categories which preserve some types of projective limits, are *formally* different kinds of data. However, fundamentally, they are closely related (§ 3).

Let us consider a "kind" of (*i.e.* a set of) compositive graphs. Then, we may *extrapolate* from each category a projective prototype of this kind. This projective prototype has the given category for support, and it has, for distinguished projective cones, the projective *limit* cones of the category with base of the prescribed kind (*i.e.* in the prescribed set). In this way, the functors between categories, which preserve the projective limit cones of this kind, can be identified to the representations between the extrapolating projective prototypes.

Similarly, the models of a projective sketch of a given kind towards a category are, formally, data between quite heterogeneous types of structures. However, they can be identified to representations which *uniformize* them, *i.e.* to the representations from this projective sketch towards the projective prototype of this kind, which extrapolates this category.

It has been proven in [ref2] that each projective sketch generates a projective prototype. Then, it is easy to prove that the category of models of a projective sketch towards a category is canonically isomorphic to the category of models of the generated projective prototype towards the same category.

Projective sketches can be limited to a given kind and a given *size* (which means that the size of their sublying compositive graphs is relative to a *universe*, as defined in [ref1]). Categories, also, can be limited to the same size. Then (§ 4), there is an *enriched system* (by the tensor system of compositive graphs and categories studied in [ref1]) *of projective sketches and categories.* This is made up of:

- the category with points (*i.e.* objects) the projective sketches of this kind and size, and with arrows the representations between these projective sketches,
- the category with points (*i.e.* objects) the categories of this size, and with arrows the functors between these categories, which preserve the projective limit cones of this kind,
- a system of enrichments of both these categories, separated as well as intermediate, which means:
  - an additionnal structure on the "Hom's" of the first category, *i.e.* the representations,
  - an additionnal structure on the "Hom's" of the second category, *i.e.* the functors which preserve the projective limit cones of the prescribed kind,
  - an additionnal structure on the "intermediate Hom's", *i.e.* the models of the points of the first category towards the points of the second category.

In this way, we get the complete structure of the various actual composition modes. These composition modes include, of course, the composition of representations, models and limit-preserving functors, and also the composition of metamorphoses, homomorphisms and natural transformations.

With such a detailed description of this system, the effects, on the categories of models, of the forthcoming constructions (of "inductive lax-limits"), are fully known.

This family of papers is a Reference Manual. For this reason, motivations and examples will not be found here: they are in our User's Guide. For this reason too, it is essentially self-contained. Hence, this paper can be read with only some familiarity with the common use of category theory, and with the first and second parts of this Reference Manual, [ref1] and [ref2].

2 Models and homomorphisms

2.1 Models

2.1.a. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. A *model of* $\mathbf{E}$ *towards* $\mathcal{A}$:

$$\mu : \mathbf{E} \to \mathcal{A}$$

is made up of:

- a *sublying* functor $\mathtt{subl}(\mu) : \underline{\mathbf{E}} \to \mathcal{A}$
  (we may write ${}_{\wedge}\underline{\mu} = \mathtt{subl}(\mu) : \underline{\mathbf{E}} \to \mathcal{A}$)

such that:

- for each distinguished projective cone c of $\mathbf{E}$, the projective cone ${}_{\wedge}\underline{\mu}(c)$, image of c, is a projective limit cone of $\mathcal{A}$.

Then, we write:

$$\mathtt{Pt}(\mu) = \mathtt{Pt}({}_{\wedge}\underline{\mu})$$

and, for all point E of $\mathbf{E}$:

$$\mu(E) = \mathtt{Pt}(\mu)(E) = \mathtt{Pt}({}_{\wedge}\underline{\mu})(E) = \mathtt{subl}(\mu)(E) = {}_{\wedge}\underline{\mu}(E)\ .$$

Similarly, we write:

$$\mathtt{Ar}(\mu) = \mathtt{Ar}({}_{\wedge}\underline{\mu})$$

and, for all arrow e of $\mathbf{E}$:

$$\mu(e) = \mathtt{Ar}(\mu)(e) = \mathtt{Ar}({}_{\wedge}\underline{\mu})(e) = \mathtt{subl}(\mu)(e) = {}_{\wedge}\underline{\mu}(e)\ .$$

Moreover, we write:

$$\mathtt{IdAr}(\mu) = \mathtt{IdAr}({}_{\wedge}\underline{\mu})\ ,$$

$$\mathtt{ConsP}(\mu) = \mathtt{ConsP}({}_{\wedge}\underline{\mu})\ ,$$

$$\mathtt{CompP}(\mu) = \mathtt{CompP}({}_{\wedge}\underline{\mu})\ .$$

Finally, for all projective cone c (either ditinguished or not), with values in $\mathbf{E}$, we write:

$$\mu(c) = {}_{\wedge}\underline{\mu}(c)\ .$$

2.1.b. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. A *counter-model of* $\mathbf{E}$ *towards* $\mathcal{A}$:

$$\kappa : \mathbf{E} \mathrel{+\!\!\!\longrightarrow} \mathcal{A}$$

is a model $\kappa : \mathbf{E} \to \mathcal{A}^{op}$ of $\mathbf{E}$ towards $\mathcal{A}^{op}$. The name (counter-model) and the notation ($\mathrel{+\!\!\!\longrightarrow}$) allow us to refer "concretely" to $\mathcal{A}$.

2.1.c. Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ a category and $\mu' : \mathbf{E}' \to \mathcal{A}$ a model. The *composed model of* μ' *with* ρ:

$$\mu' \circ \rho : \mathbf{E} \to \mathcal{A}$$

is the obviously obtained model such that:

- $\underline{\mu' \circ \rho} = \underline{\mu'} \circ \underline{\rho}$.

Similarly, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones and $\mu : \mathbf{E} \to \mathcal{A}$ a model. The *composed model of* ϕ *with* μ:

$$\phi \circ \mu : \mathbf{E} \to \mathcal{A}'$$

is the obviously obtained model such that:

- $\underline{\phi \circ \mu} = \phi \circ \underline{\mu}$.

2.2 Homomorphisms

2.2.a. Let $\mathbf{E}$ be a projective sketch, $\mathcal{A}$ a category and $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ two models. A *homomorphism from* μ_1 *towards* μ_2:

$$h : \mu_1 \Rightarrow \mu_2 : \mathbf{E} \to \mathcal{A}$$

(or simply $h : \mu_1 \Rightarrow \mu_2$), is made up of:

- a *sublying* natural transformation:

$$\texttt{subl}(h) : \texttt{subl}(\mu_1) \Rightarrow \texttt{subl}(\mu_2) : \underline{\mathbf{E}} \to \mathcal{A}\,,$$

  which we may denote:

$$\underline{h} = \texttt{subl}(h) : \underline{\mu}_1 = \texttt{subl}(\mu_1) \Rightarrow \underline{\mu}_2 = \texttt{subl}(\mu_2) : \underline{\mathbf{E}} \to \mathcal{A}\,.$$

2.2.b. Let $\mathbf{E}$ be a projective sketch, $\mathcal{A}$ a category and $\kappa_1, \kappa_2 : \mathbf{E} \longleftrightarrow \mathcal{A}$ two counter-models of $\mathbf{E}$ towards $\mathcal{A}$ (*i.e.* $\kappa_1, \kappa_2 : \mathbf{E} \to \mathcal{A}^{op}$ are two models of $\mathbf{E}$ towards $\mathcal{A}^{op}$). A *counter-homomorphism from the counter-model* κ_2 *towards the counter-model* κ_1:

$$q : \kappa_2 \Longrightarrow \kappa_1 : \mathbf{E} \longleftrightarrow \mathcal{A}$$

is a homomorphism $q : \kappa_2 \Rightarrow \kappa_1 : \mathbf{E} \to \mathcal{A}^{op}$ from the model κ_2 towards the model κ_1. The name (counter-homomorphism) and the notation ($\Longrightarrow$) allow us to refer "concretely" to $\mathcal{A}$.

2.2.c. Let $\mathbf{E}$ be a projective sketch, $\mathcal{A}$ a category, $\mu_1, \mu_2, \mu_3 : \mathbf{E} \to \mathcal{A}$ three models, $h_1 : \mu_1 \Rightarrow \mu_2$ and $h_2 : \mu_2 \Rightarrow \mu_3$ two homomorphisms. The *natural composed homomorphism of h_2 with h_1*:

$$h_2 \centerdot h_1 : \mu_1 \Rightarrow \mu_3 : \mathbf{E} \to \mathcal{A}$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$(h_2 \centerdot h_1)(E) = h_2(E) \centerdot h_1(E)$$
(which is meaningful, since h_2 and h_1 take their values in a category),

or, equivalently:

- ${}_{\wedge}\underline{h_2 \centerdot h_1} = {}_{\wedge}\underline{h_2} \centerdot {}_{\wedge}\underline{h_1}$.

Similarly, let $\mathbf{E}$ be a projective sketch, $\mathcal{A}$ a category, and $\mu : \mathbf{E} \to \mathcal{A}$ a model. The *identity homomorphism at μ*:

$$\mathrm{id}(\mu) : \mu \Rightarrow \mu : \mathbf{E} \to \mathcal{A}$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$\mathrm{id}(\mu)(E) = \mathrm{id}(\mu(E))$$
(which is meaningful, since μ take its values in a category),

or, equivalently:

- ${}_{\wedge}\underline{\mathrm{id}(\mu)} = \mathrm{id}({}_{\wedge}\underline{\mu})$.

2.2.d. Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ a category, $\mu'_1, \mu'_2 : \mathbf{E}' \to \mathcal{A}$ two models and $h' : \mu'_1 \Rightarrow \mu'_2$ a homomorphism. The *composed homomorphism of h' with ρ*:

$$h' \circ \rho : \mu'_1 \circ \rho \Rightarrow \mu'_2 \circ \rho : \mathbf{E} \to \mathcal{A}$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$(h' \circ \rho)(E) = h'(\rho(E))\,,$$

or, equivalently:

- ${}_{\wedge}\underline{h' \circ \rho} = {}_{\wedge}\underline{h'} \circ \underline{\rho}$.

Similarly, let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho_1, \rho_2 : \mathbf{E} \to \mathbf{E}'$ two representations, $m : \rho_1 \Rightarrow \rho_2 : \mathbf{E} \to \mathbf{E}'$ a natural metamorphosis, $\mathcal{A}$ a category, and $\mu' : \mathbf{E}' \to \mathcal{A}$ a model. The *composed homomorphism of μ' with m*:

$$\mu' \circ m : \mu' \circ \rho_1 \Rightarrow \mu' \circ \rho_2 : \mathbf{E} \to \mathcal{A}$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$(\mu' \circ m)(E) = \mu'(m(E))\,,$$

or, equivalently:

- ${}_\wedge\underline{\mu' \circ m} = {}_\wedge\underline{\mu'} \circ \underline{m}$.

Now, let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho_1, \rho_2 : \mathbf{E} \to \mathbf{E}'$ two representations, $m : \rho_1 \Rightarrow \rho_2 : \mathbf{E} \to \mathbf{E}'$ a natural metamorphosis, $\mathcal{A}$ a category, $\mu'_1, \mu'_2 : \mathbf{E}' \to \mathcal{A}$ two models and $h' : \mu'_1 \Rightarrow \mu'_2$ a homomorphism. Then, it is easy to check, since $\mathcal{A}$ is a category, that the following diagram (of homomorphisms) is commutative:

$$\begin{array}{ccc}
\mu'_1 \circ \rho_1 & \xRightarrow{\mu'_1 \circ m} & \mu'_1 \circ \rho_2 \\
{\scriptstyle h' \circ \rho_1}\Big\Downarrow & & \Big\Downarrow{\scriptstyle h' \circ \rho_2} \\
\mu'_2 \circ \rho_1 & \xRightarrow[\mu'_2 \circ m]{} & \mu'_2 \circ \rho_2
\end{array}$$

The diagonal of this diagram defines the *composed homomorphism of h' with m*:

$$h' \circ m : \mu'_1 \circ \rho_1 \Rightarrow \mu'_2 \circ \rho_2 : \mathbf{E} \to \mathcal{A}\,.$$

2.2.e. Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones, $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ two models and $h : \mu_1 \Rightarrow \mu_2$ a homomorphism. The *composed homomorphism of ϕ with h*:

$$\phi \circ h : \phi \circ \mu_1 \Rightarrow \phi \circ \mu_2 : \mathbf{E} \to \mathcal{A}'$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$(\phi \circ h)(E) = \phi(h(E))\,,$$

or, equivalently:

- ${}_\wedge\underline{\phi \circ h} = \phi \circ {}_\wedge\underline{h}$.

Similarly, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ two functors which preserve the $\mathbb{B}$-projective limit cones, $t : \phi_1 \Rightarrow \phi_2 : \mathcal{A} \to \mathcal{A}'$ a natural transformation and $\mu : \mathbf{E} \to \mathcal{A}$ a model. The *composed homomorphism of t with μ*:

$$t \circ \mu : \phi_1 \circ \mu \Rightarrow \phi_2 \circ \mu : \mathbf{E} \to \mathcal{A}'$$

is the obviously obtained homomorphism such that:

- for all point E of $\mathbf{E}$:
$$(t \circ \mu)(E) = t(\mu(E))\,,$$

or, equivalently:

- ${}_\wedge\underline{t \circ \mu} = t \circ {}_\wedge\underline{\mu}$.

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ two functors which preserve the $\mathbb{B}$-projective limit cones, $t : \phi_1 \Rightarrow \phi_2 : \mathcal{A} \to \mathcal{A}'$ a natural transformation, $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ two models and $h : \mu_1 \Rightarrow \mu_2$ a homomorphism. Then, it is easy to check, since $\mathcal{A}$ and $\mathcal{A}'$ are categories, that the following diagram (of homomorphisms) is commutative:

$$\begin{array}{ccc} \phi_1 \circ \mu_1 & \xRightarrow{\phi_1 \circ h} & \phi_1 \circ \mu_2 \\ {\scriptstyle t\circ\mu_1}\Downarrow & & \Downarrow{\scriptstyle t\circ\mu_2} \\ \phi_2 \circ \mu_1 & \xRightarrow[\phi_2 \circ h]{} & \phi_2 \circ \mu_2 \end{array}$$

The diagonal of this diagram defines the *composed homomorphism of* t *with* h:

$$t \circ h : \phi_1 \circ \mu_1 \Rightarrow \phi_2 \circ \mu_2 : \mathbf{E} \to \mathcal{A}' \ .$$

3 Categories of models and of limit-preserving functors

3.1 Categories of models

3.1.a. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. The set of models of $\mathbf{E}$ towards $\mathcal{A}$ is denoted:

$$\mathrm{Mod}(\mathbf{E}, \mathcal{A}) \ .$$

Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation and $\mathcal{A}$ a category. The *right-composition with* ρ map:

$$\mathrm{Mod}(\rho, \mathcal{A}) : \mathrm{Mod}(\mathbf{E}', \mathcal{A}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A})$$

is the map such that:

- for all model $\mu' : \mathbf{E}' \to \mathcal{A}$:
$$\mathrm{Mod}(\rho, \mathcal{A})(\mu') = \mu' \circ \rho \ .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. The *left-composition with* ϕ map:

$$\mathrm{Mod}(\mathbf{E}, \phi) : \mathrm{Mod}(\mathbf{E}, \mathcal{A}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}')$$

is the map such that:

- for all model $\mu : \mathbf{E} \to \mathcal{A}$:
$$\mathrm{Mod}(\mathbf{E}, \phi)(\mu) = \phi \circ \mu \ .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ and $\mathbf{E}'$ two $\mathbb{B}$-projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, it is easy to check that the following diagram (of maps) is commutative:

$$\begin{array}{ccc}
\mathrm{Mod}(\mathbf{E}', \mathcal{A}) & \xrightarrow{\mathrm{Mod}(\mathbf{E}', \phi)} & \mathrm{Mod}(\mathbf{E}', \mathcal{A}') \\
{\scriptstyle \mathrm{Mod}(\rho, \mathcal{A})} \downarrow & & \downarrow {\scriptstyle \mathrm{Mod}(\rho, \mathcal{A}')} \\
\mathrm{Mod}(\mathbf{E}, \mathcal{A}) & \xrightarrow[\mathrm{Mod}(\mathbf{E}, \phi)]{} & \mathrm{Mod}(\mathbf{E}, \mathcal{A}')
\end{array}$$

The diagonal of this diagram defines the map:

$$\mathrm{Mod}(\rho, \phi) : \mathrm{Mod}(\mathbf{E}', \mathcal{A}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}') \ .$$

3.1.b. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. The set of counter-models of $\mathbf{E}$ towards $\mathcal{A}$ is also denoted (in order to refer "concretely" to $\mathcal{A}$):

$$\mathrm{CtrMod}(\mathbf{E}, \mathcal{A}) = \mathrm{Mod}(\mathbf{E}, \mathcal{A}^{op}) \ .$$

Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation and $\mathcal{A}$ a category. The *right-composition (of counter-models) with* ρ map is:

$$\mathrm{CtrMod}(\rho, \mathcal{A}) : \mathrm{CtrMod}(\mathbf{E}', \mathcal{A}) \to \mathrm{CtrMod}(\mathbf{E}, \mathcal{A})$$
$$=$$
$$\mathrm{Mod}(\rho, \mathcal{A}^{op}) : \mathrm{Mod}(\mathbf{E}', \mathcal{A}^{op}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}^{op}) \; .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}^{op}$-inductive limit cones (which means that the functor $\phi^{op} : \mathcal{A}^{op} \to \mathcal{A}'^{op}$ preserves the $\mathbb{B}$-projective limit cones). The *left-composition (of counter-models) with* ϕ map is:

$$\mathrm{CtrMod}(\mathbf{E}, \phi) : \mathrm{CtrMod}(\mathbf{E}, \mathcal{A}) \to \mathrm{CtrMod}(\mathbf{E}, \mathcal{A}')$$
$$=$$
$$\mathrm{Mod}(\mathbf{E}, \phi^{op}) : \mathrm{Mod}(\mathbf{E}, \mathcal{A}^{op}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}'^{op}) \; .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ and $\mathbf{E}'$ two $\mathbb{B}$-projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}^{op}$-inductive limit cones (which means that the functor $\phi^{op} : \mathcal{A}^{op} \to \mathcal{A}'^{op}$ preserves the $\mathbb{B}$-projective limit cones). Then we also denote:

$$\mathrm{CtrMod}(\rho, \phi) : \mathrm{CtrMod}(\mathbf{E}', \mathcal{A}) \to \mathrm{CtrMod}(\mathbf{E}, \mathcal{A}')$$
$$=$$
$$\mathrm{Mod}(\rho, \phi^{op}) : \mathrm{Mod}(\mathbf{E}', \mathcal{A}^{op}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}'^{op}) \; .$$

3.1.c. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. The *category of models of* $\mathbf{E}$ *towards* $\mathcal{A}$:

$$\mathcal{M}od(\mathbf{E}, \mathcal{A})$$

is the canonically obtained category such that:

- its points are the models of $\mathbf{E}$ towards $\mathcal{A}$,
- its arrows are the homomorphisms between these models,
- its composition law is the natural composition (of consecutive pairs of homomorphisms).

Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation and $\mathcal{A}$ a category. The *right-composition with* ρ *(of models and homomorphisms)* functor:

$$\mathcal{M}od(\rho, \mathcal{A}) : \mathcal{M}od(\mathbf{E}', \mathcal{A}) \to \mathcal{M}od(\mathbf{E}, \mathcal{A})$$

is the obviously obtained functor such that:

- for all model $\mu' : \mathbf{E}' \to \mathcal{A}$:
$$\mathcal{M}od(\rho, \mathcal{A})(\mu') = \mu' \circ \rho \, ,$$
- for all models $\mu'_1, \mu'_2 : \mathbf{E}' \to \mathcal{A}$ and all homomorphism $h' : \mu'_1 \Rightarrow \mu'_2$:
$$\mathcal{M}od(\rho, \mathcal{A})(h') = h' \circ \rho \; .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. The *left-composition with ϕ (of models and homomorphisms)* functor:

$$\mathrm{Mod}(\mathbf{E}, \phi) : \mathrm{Mod}(\mathbf{E}, \mathcal{A}) \to \mathrm{Mod}(\mathbf{E}, \mathcal{A}')$$

is the obviously obtained functor such that:

- for all model $\mu : \mathbf{E} \to \mathcal{A}$:
$$\mathcal{M}od(\mathbf{E}, \phi)(\mu) = \phi \circ \mu \,,$$
- for all models $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ and all homomorphism $h : \mu_1 \Rightarrow \mu_2$:
$$\mathcal{M}od(\mathbf{E}, \phi)(h) = \phi \circ h \ .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ and $\mathbf{E}'$ two $\mathbb{B}$-projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, it is easy to check that the following diagram (of functors) is commutative:

$$\begin{array}{ccc}
\mathcal{M}od(\mathbf{E}', \mathcal{A}) & \xrightarrow{\mathcal{M}od(\mathbf{E}', \phi)} & \mathcal{M}od(\mathbf{E}', \mathcal{A}') \\
{\scriptstyle \mathcal{M}od(\rho, \mathcal{A})} \downarrow & & \downarrow {\scriptstyle \mathcal{M}od(\rho, \mathcal{A}')} \\
\mathcal{M}od(\mathbf{E}, \mathcal{A}) & \xrightarrow[\mathcal{M}od(\mathbf{E}, \phi)]{} & \mathcal{M}od(\mathbf{E}, \mathcal{A}')
\end{array}$$

The diagonal of this diagram defines the functor:

$$\mathcal{M}od(\rho, \phi) : \mathcal{M}od(\mathbf{E}', \mathcal{A}) \to \mathcal{M}od(\mathbf{E}, \mathcal{A}') \ .$$

3.1.d. Let $\mathbf{E}$ be a projective sketch and $\mathcal{A}$ a category. The category of counter-models of $\mathbf{E}$ towards $\mathcal{A}$ is:

$$\mathcal{C}tr\mathcal{M}od(\mathbf{E}, \mathcal{A}) = (\mathcal{M}od(\mathbf{E}, \mathcal{A}^{op}))^{op} \ .$$

Let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation and $\mathcal{A}$ a category. The *right-composition with ρ (of counter-models and counter-homomorphisms)* functor is:

$$\mathcal{C}tr\mathcal{M}od(\rho, \mathcal{A}) : \mathcal{C}tr\mathcal{M}od(\mathbf{E}', \mathcal{A}) \to \mathcal{C}tr\mathcal{M}od(\mathbf{E}, \mathcal{A})$$

$$=$$

$$\mathcal{M}od(\rho, \mathcal{A}^{op})^{op} : \mathcal{M}od(\mathbf{E}', \mathcal{A}^{op})^{op} \to \mathcal{M}od(\mathbf{E}, \mathcal{A}^{op})^{op} \ .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}^{op}$-inductive limit cones (which means that the functor $\phi^{op} : \mathcal{A}^{op} \to \mathcal{A}'^{op}$ preserves the $\mathbb{B}$-projective limit cones). The *left-composition with ϕ (of counter-models and counter-homomorphisms)* functor is:

$$\mathcal{C}tr\mathcal{M}od(\mathbf{E}, \phi) : \mathcal{C}tr\mathcal{M}od(\mathbf{E}, \mathcal{A}) \to \mathcal{C}tr\mathcal{M}od(\mathbf{E}, \mathcal{A}')$$

$$=$$

$$\mathcal{M}od(\mathbf{E}, \phi^{op})^{op} : \mathcal{M}od(\mathbf{E}, \mathcal{A}^{op})^{op} \to \mathcal{M}od(\mathbf{E}, \mathcal{A}'^{op})^{op} \ .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ and $\mathbf{E}'$ two $\mathbb{B}$-projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}^{op}$-inductive limit cones (which means that the functor $\phi^{op} : \mathcal{A}^{op} \to \mathcal{A}'^{op}$ preserves the $\mathbb{B}$-projective limit cones). Then we also denote:

$$\mathcal{C}tr\mathcal{M}od(\rho, \phi) : \mathcal{C}tr\mathcal{M}od(\mathbf{E}', \mathcal{A}) \to \mathcal{C}tr\mathcal{M}od(\mathbf{E}, \mathcal{A}')$$

$$=$$

$$\mathcal{M}od(\rho, \phi^{op})^{op} : \mathcal{M}od(\mathbf{E}', \mathcal{A}^{op})^{op} \to \mathcal{M}od(\mathbf{E}, \mathcal{A}'^{op})^{op} .$$

3.2 Categories of limit-preserving functors

3.2.a. Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$ and $\mathcal{A}'$ two categories. The set of functors from $\mathcal{A}$ to $\mathcal{A}'$ which preserve the $\mathbb{B}$-projective limit cones is denoted:

$$\mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') .$$

So:

$$\mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \subseteq \mathrm{Func}(\mathcal{A}, \mathcal{A}') .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ three categories and $\phi' : \mathcal{A}' \to \mathcal{A}''$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, clearly, the map:

$$\mathrm{Func}(\mathcal{A}, \phi') : \mathrm{Func}(\mathcal{A}, \mathcal{A}') \to \mathrm{Func}(\mathcal{A}, \mathcal{A}'')$$

can be restricted as:

$$\mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \phi') : \mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \to \mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}'') .$$

Similarly, let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ three categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, clearly, the map:

$$\mathrm{Func}(\phi, \mathcal{A}'') : \mathrm{Func}(\mathcal{A}', \mathcal{A}'') \to \mathrm{Func}(\mathcal{A}, \mathcal{A}'')$$

can be restricted as:

$$\mathrm{Func}_{\mathbb{B}}(\phi, \mathcal{A}'') : \mathrm{Func}_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') \to \mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}'') .$$

Finally, let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ four categories, $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$ two functors which preserve the $\mathbb{B}$-projective limit cones. Then, clearly, the map:

$$\mathrm{Func}(\phi, \phi'') : \mathrm{Func}(\mathcal{A}', \mathcal{A}'') \to \mathrm{Func}(\mathcal{A}, \mathcal{A}''')$$

can be restricted as:

$$\mathrm{Func}_{\mathbb{B}}(\phi, \phi'') : \mathrm{Func}_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') \to \mathrm{Func}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}''') .$$

3.2.b. Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$ and $\mathcal{A}'$ two categories. The full subcategory of the category $\mathcal{F}unc(\mathcal{A}, \mathcal{A}')$, with objects the functors from $\mathcal{A}$ to $\mathcal{A}'$ which preserve the $\mathbb{B}$-projective limit cones, is denoted:

$$\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \ .$$

So, there is a canonical injection functor:

$$\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \subseteq \mathcal{F}unc(\mathcal{A}, \mathcal{A}') : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \to \mathcal{F}unc(\mathcal{A}, \mathcal{A}') \ .$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ three categories and $\phi' : \mathcal{A}' \to \mathcal{A}''$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, clearly, the functor:

$$\mathcal{F}unc(\mathcal{A}, \phi') : \mathcal{F}unc(\mathcal{A}, \mathcal{A}') \to \mathcal{F}unc(\mathcal{A}, \mathcal{A}'')$$

can be restricted as:

$$\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \phi') : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \to \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}'') \ .$$

Similarly, let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ three categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, clearly, the functor:

$$\mathcal{F}unc(\phi, \mathcal{A}'') : \mathcal{F}unc(\mathcal{A}', \mathcal{A}'') \to \mathcal{F}unc(\mathcal{A}, \mathcal{A}'')$$

can be restricted as:

$$\mathcal{F}unc_{\mathbb{B}}(\phi, \mathcal{A}'') : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') \to \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}'') \ .$$

Finally, let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ four categories, $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$ two functors which preserve the $\mathbb{B}$-projective limit cones. Then, clearly, the functor:

$$\mathcal{F}unc(\phi, \phi'') : \mathcal{F}unc(\mathcal{A}', \mathcal{A}'') \to \mathcal{F}unc(\mathcal{A}, \mathcal{A}''')$$

can be restricted as:

$$\mathcal{F}unc_{\mathbb{B}}(\phi, \phi'') : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') \to \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}''') \ .$$

3.3 Extrapolations and uniformisations

3.3.a. Let $\mathbb{B}$ be a set of compositive graphs and $\mathcal{A}$ a category. The $\mathbb{B}$-*projective prototype canonically associated to* $\mathcal{A}$:

$$\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})$$

is the obviously obtained $\mathbb{B}$-projective prototype such that:

- $\underline{\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})} = \mathcal{A}$,
- $\mathrm{DistPC}(\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}))$ is the set of $\mathbb{B}$-projective limit cones, with values in $\mathcal{A}$.

3.3.b. Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$ and $\mathcal{A}'$ two categories and $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. The representation *extrapolating* ϕ *(to the* $\mathbb{B}$*-projective prototypes canonically associated to its domain and codomain)*:

$$\mathrm{Extra}_{\mathbb{B}}(\phi) = \mathrm{CanPpr}_{\mathbb{B}}(\phi) : \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}) \to \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}')$$

is the obviously obtained representation such that:

- $\underline{\mathrm{Extra}_{\mathbb{B}}(\phi)} = \phi$.

Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ two functors which preserve the $\mathbb{B}$-projective limit cones and $t : \phi_1 \Rightarrow \phi_2 : \mathcal{A} \to \mathcal{A}'$ a natural transformation. The natural metamorphosis *extrapolating* t:

$$\mathrm{Extra}_{\mathbb{B}}(t) : \mathrm{Extra}_{\mathbb{B}}(\phi_1) \Rightarrow \mathrm{Extra}_{\mathbb{B}}(\phi_2) : \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}) \to \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}')$$

is the obviously obtained natural metamorphosis such that:

- $\underline{\mathrm{Extra}_{\mathbb{B}}(t)} = t$.

Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$ and $\mathcal{A}'$ two categories. The functor:

$$\mathrm{extra}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \to \mathcal{R}ep(\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}), \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}'))$$

is the obviously obtained functor, clearly invertible, such that:

- for all functor $\phi : \mathcal{A} \to \mathcal{A}'$ which preserves the $\mathbb{B}$-projective limit cones:

$$\mathrm{extra}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}')(\phi) = \mathrm{Extra}_{\mathbb{B}}(\phi)\ ,$$

- for all functors $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ which preserve the $\mathbb{B}$-projective limit cones and all natural transformation $t : \phi_1 \Rightarrow \phi_2$:

$$\mathrm{extra}_{\mathbb{B}}(\mathcal{A}, \mathcal{A}')(t) = \mathrm{Extra}_{\mathbb{B}}(t)\ .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ four categories, $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$ two functors which preserve the $\mathbb{B}$-projective limit cones. Then, clearly, the following diagram (of functors) is commutative:

$$\begin{array}{ccc}
\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') & \xrightarrow{\mathrm{extra}_{\mathbb{B}}(\mathcal{A},\mathcal{A}')} & \mathcal{R}ep(\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}'), \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}'')) \\
{\scriptstyle \mathcal{F}unc_{\mathbb{B}}(\phi',\phi'')}\downarrow & & \downarrow{\scriptstyle \mathcal{R}ep(\mathrm{Extra}_{\mathbb{B}}(\phi),\mathrm{Extra}_{\mathbb{B}}(\phi''))} \\
\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}''') & \xrightarrow[\mathrm{extra}_{\mathbb{B}}(\mathcal{A},\mathcal{A}')]{} & \mathcal{R}ep(\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}), \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}'''))
\end{array}$$

3.3.c. Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ a category and $\mu : \mathbf{E} \to \mathcal{A}$ a model. The representation *uniformizing* μ:

$$\mathrm{Unif}_{\mathbb{B}}(\mu) : \mathbf{E} \to \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})$$

is the obviously obtained representation such that:

- $\underline{\mathrm{Unif}_{\mathbb{B}}(\mu)} = {}_{\wedge}\underline{\mu}$.

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch, $\mathcal{A}$ a category, $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ two models and $h : \mu_1 \Rightarrow \mu_2 : \mathbf{E} \to \mathcal{A}$ a homomorphism. The natural metamorphosis *uniformizing* h:

$$\mathrm{Unif}_{\mathbb{B}}(h) : \mathrm{Unif}_{\mathbb{B}}(\mu_1) \Rightarrow \mathrm{Unif}_{\mathbb{B}}(\mu_2) : \mathbf{E} \to \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})$$

is the obviously obtained natural metamorphosis such that:

- $\underline{\mathrm{Unif}_{\mathbb{B}}(h)} = {}_{\wedge}\underline{h}$.

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch and $\mathcal{A}$ a category. The functor:

$$\mathrm{unif}_{\mathbb{B}}(\mathbf{E}, \mathcal{A}) : \mathcal{M}od(\mathbf{E}, \mathcal{A}) \to \mathcal{R}ep(\mathbf{E}, \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}))$$

is the obviously obtained functor, clearly invertible, such that:

- for all model $\mu : \mathbf{E} \to \mathcal{A}$:
$$\mathrm{unif}_{\mathbb{B}}(\mathbf{E}, \mathcal{A})(\mu) = \mathrm{Unif}_{\mathbb{B}}(\mu)\,,$$
- for all models $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$ and all homomorphism $h : \mu_1 \Rightarrow \mu_2$:
$$\mathrm{unif}_{\mathbb{B}}(\mathbf{E}, \mathcal{A})(h) = \mathrm{Unif}_{\mathbb{B}}(h)\ .$$

Now, let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ and $\mathbf{E}'$ two $\mathbb{B}$-projective sketches, $\rho : \mathbf{E} \to \mathbf{E}'$ a representation, $\mathcal{A}$ and $\mathcal{A}'$ two categories, $\phi : \mathcal{A} \to \mathcal{A}'$ a functor which preserves the $\mathbb{B}$-projective limit cones. Then, clearly, the following diagram (of functors) is commutative:

$$\begin{array}{ccc}
\mathcal{M}od(\mathbf{E}', \mathcal{A}) & \xrightarrow{\mathrm{unif}_{\mathbb{B}}(\mathbf{E}', \mathcal{A})} & \mathcal{R}ep(\mathbf{E}', \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})) \\
{\scriptstyle \mathcal{M}od(\rho, \phi)} \downarrow & & \downarrow {\scriptstyle \mathcal{R}ep(\rho, \mathrm{Extra}_{\mathbb{B}}(\phi)) \,=\, \mathcal{R}ep(\rho, \mathrm{CanPpr}_{\mathbb{B}}(\phi))} \\
\mathcal{M}od(\mathbf{E}, \mathcal{A}') & \xrightarrow[\mathrm{unif}_{\mathbb{B}}(\mathbf{E}, \mathcal{A}')]{} & \mathcal{R}ep(\mathbf{E}, \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}'))
\end{array}$$

3.3.d. The following result follows easily from the previous contructions and from proposition 5 in [ref2].

Proposition 1 *Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a $\mathbb{B}$-projective sketch and $\mathcal{A}$ a category. Then the canonical representation (from the $\mathbb{B}$-projective sketch $\mathbf{E}$ towards the $\mathbb{B}$-projective prototype* $\mathrm{GenPpr}(\mathbf{E})$ *generated by* $\mathbf{E}$*):*

$$\mathbf{E} \mid \mathrm{GenPpr}(\mathbf{E}) : \mathbf{E} \to \mathrm{GenPpr}(\mathbf{E})$$

yields an isomorphism:

$$\mathcal{M}od(\mathbf{E} \mid \mathrm{GenPpr}(\mathbf{E}), \mathcal{A}) : \mathcal{M}od(\mathrm{GenPpr}(\mathbf{E}), \mathcal{A}) \to \mathcal{M}od(\mathbf{E}, \mathcal{A})\ .$$

Proof. Indeed, the following diagram (of functors) is commutative:

$$\begin{array}{ccc}
\mathcal{M}od(\mathrm{GenPpr}(\mathbf{E}), \mathcal{A}) & \xrightarrow[\cong]{\mathrm{unif}_{\mathbb{B}}\ldots} & \mathcal{R}ep(\mathrm{GenPpr}(\mathbf{E}), \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})) \\
{\scriptstyle \mathcal{M}od(\mathbf{E}|\mathrm{GenPpr}(\mathbf{E}),\mathcal{A})}\downarrow & & \cong\downarrow{\scriptstyle \mathcal{R}ep(\mathbf{E}|\mathrm{GenPpr}(\mathbf{E}),\mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}))} \\
\mathcal{M}od(\mathbf{E}, \mathcal{A}) & \xrightarrow[\mathrm{unif}_{\mathbb{B}}\ldots]{\cong} & \mathcal{R}ep(\mathbf{E}, \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A}))
\end{array}$$

End of proof.

4 The enriched system of projective sketches and categories

4.1 The category of categories with preserved limits

4.1.a. Let $\mathbb{U}$ be a universe and $\mathbb{B}$ a set of compositive graphs. The *category of* $\mathbb{U}$*-small categories with preserved* $\mathbb{B}$*-projective limit cones*:

$$\mathcal{C}at_{\mathbb{U},\mathbb{B}}$$

is the obviously obtained (clearly not full) subcategory of $\mathcal{C}at_{\mathbb{U}}$ such that:

- its points are the $\mathbb{U}$-small categories (as for $\mathcal{C}at_{\mathbb{U}}$),
- its arrows are the functors between these categories which preserve the $\mathbb{B}$-projective limit cones.

Then, obviously, there is a canonical injection functor:

$$\mathcal{C}at_{\mathbb{U},\mathbb{B}} \subseteq \mathcal{C}at_{\mathbb{U}} : \mathcal{C}at_{\mathbb{U},\mathbb{B}} \to \mathcal{C}at_{\mathbb{U}} \ .$$

4.1.b. Let $\mathbb{U}$ be a universe and $\mathbb{B}$ a set of compositive graphs. The functor:

$$\mathrm{CanPpr}_{\mathbb{U},\mathbb{B}}(-) : \mathcal{C}at_{\mathbb{U},\mathbb{B}} \to \mathcal{P}pr_{\mathbb{U},\mathbb{B}}$$

is the obviously obtained functor such that:

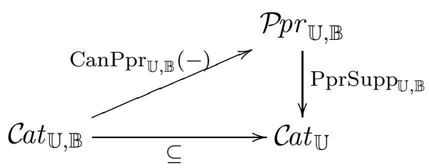
- for all $\mathbb{U}$-small category $\mathcal{A}$:

$$\mathrm{CanPpr}_{\mathbb{U},\mathbb{B}}(-)(\mathcal{A}) = \mathrm{CanPpr}_{\mathbb{B}}(\mathcal{A})\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$ and all functor $\phi : \mathcal{A} \to \mathcal{A}'$:

$$\mathrm{CanPpr}_{\mathbb{U},\mathbb{B}}(-)(\phi) = \mathrm{CanPpr}_{\mathbb{B}}(\phi)\ (\ = \mathrm{Extra}_{\mathbb{B}}(\phi)\ )\ .$$

Then, it is easy to check that the following diagram (of functors) is commutative:



4.2 The enriched system of projective sketches and categories with preserved limits

4.2.a. Let $\mathbf{E}$, $\mathbf{E}'$ and $\mathbf{E}''$ be three $\mathbb{B}$-projective sketches. Then, as in [ref2]:

$$-\circ\,\square_{\mathbf{E},\mathbf{E}',\mathbf{E}''}- : \mathcal{R}ep(\mathbf{E},\mathbf{E}')\,\square\,\mathcal{R}ep(\mathbf{E}',\mathbf{E}'') \to \mathcal{R}ep(\mathbf{E},\mathbf{E}'')$$

is the obviously obtained functor such that:

- for all representations $\rho : \mathbf{E} \to \mathbf{E}'$ and $\rho' : \mathbf{E}' \to \mathbf{E}''$:

$$(-\circ_{\underline{\square}_{\mathbf{E},\mathbf{E}',\mathbf{E}''}}-)([\rho, \rho']) = \rho' \circ \rho\,,$$

- for all representations $\rho_1, \rho_2 : \mathbf{E} \to \mathbf{E}'$ and $\rho' : \mathbf{E}' \to \mathbf{E}''$ and all natural metamorphosis $m : \rho_1 \Rightarrow \rho_2$:

$$(-\circ_{\underline{\square}_{\mathbf{E},\mathbf{E}',\mathbf{E}''}}-)([m, \rho']) = \rho' \circ m\,,$$

- for all representations $\rho : \mathbf{E} \to \mathbf{E}'$ and $\rho'_1, \rho'_2 : \mathbf{E}' \to \mathbf{E}''$ and all natural metamorphosis $m' : \rho'_1 \Rightarrow \rho'_2$:

$$(-\circ_{\underline{\square}_{\mathbf{E},\mathbf{E}',\mathbf{E}''}}-)([\rho, m']) = m' \circ \rho\,.$$

Similarly, let $\mathbf{E}$ and $\mathbf{E}'$ be two projective sketches and $\mathcal{A}$ a category. Then:

$$-\circ_{\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}}}- : \mathcal{R}ep(\mathbf{E}, \mathbf{E}') \underline{\boxtimes} \mathcal{M}od(\mathbf{E}', \mathcal{A}) \to \mathcal{M}od(\mathbf{E}, \mathcal{A})$$

is the obviously obtained functor such that:

- for all representation $\rho : \mathbf{E} \to \mathbf{E}'$ and all model $\mu' : \mathbf{E}' \to \mathcal{A}$:

$$(-\circ_{\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}}}-)([\rho, \mu']) = \mu' \circ \rho\,,$$

- for all representations $\rho_1, \rho_2 : \mathbf{E} \to \mathbf{E}'$, all model $\mu' : \mathbf{E}' \to \mathcal{A}$ and all natural metamorphosis $m : \rho_1 \Rightarrow \rho_2$:

$$(-\circ_{\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}}}-)([m, \mu']) = \mu' \circ m\,,$$

- for all representation $\rho : \mathbf{E} \to \mathbf{E}'$, all models $\mu'_1, \mu'_2 : \mathbf{E}' \to \mathcal{A}$ and all homomorphism $h' : \mu'_1 \Rightarrow \mu'_2$:

$$(-\circ_{\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}}}-)([\rho, h']) = h' \circ \rho\,,$$

- for all representations $\rho_1, \rho_2 : \mathbf{E} \to \mathbf{E}'$, all models $\mu'_1, \mu'_2 : \mathbf{E}' \to \mathcal{A}$, all natural metamorphosis $m : \rho_1 \Rightarrow \rho_2$ and all homomorphism $h' : \mu'_1 \Rightarrow \mu'_2$:

$$(-\circ_{\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}}}-)([m, h']) = h' \circ m\,.$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathbf{E}$ a projective sketch, $\mathcal{A}$ and $\mathcal{A}'$ two categories. Then:

$$-\circ_{\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A},\mathcal{A}'}}- : \mathcal{M}od(\mathbf{E}, \mathcal{A}) \underline{\boxtimes} \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \to \mathcal{M}od(\mathbf{E}, \mathcal{A}')$$

is the obviously obtained functor such that:

- for all model $\mu : \mathbf{E} \to \mathcal{A}$ and all functor $\phi : \mathcal{A} \to \mathcal{A}'$ which preserves the $\mathbb{B}$-projective limit cones:

$$(-\circ_{\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A},\mathcal{A}'}}-)([\mu, \phi]) = \phi \circ \mu\,,$$

- for all models $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$, all functor $\phi : \mathcal{A} \to \mathcal{A}'$ which preserves the $\mathbb{B}$-projective limit cones and all homomorphism $h : \mu_1 \Rightarrow \mu_2$:

$$(-\circ_{\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A},\mathcal{A}'}}-)([h, \phi]) = \phi \circ h\,,$$

- for all model $\mu : \mathbf{E} \to \mathcal{A}$, all functors $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ which preserve the $\mathbb{B}$-projective limit cones and all natural transformation $t : \phi_1 \Rightarrow \phi_2$:

$$(- \circ_{\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A},\mathcal{A}'}} -)([\mu, t]) = t \circ \mu \,,$$

- for all models $\mu_1, \mu_2 : \mathbf{E} \to \mathcal{A}$, all functors $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ which preserve the $\mathbb{B}$-projective limit cones, all homomorphism $h : \mu_1 \Rightarrow \mu_2$ and all natural transformation $t : \phi_1 \Rightarrow \phi_2$:

$$(- \circ_{\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A},\mathcal{A}'}} -)([h, t]) = t \circ h \,.$$

Let $\mathbb{B}$ be a set of compositive graphs, $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ three categories. Then:

$$- \circ_{\underline{\otimes}_{\mathbb{B},\mathcal{A},\mathcal{A}',\mathcal{A}''}} - : \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}') \,\underline{\otimes}\, \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}', \mathcal{A}'') \to \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A}'')$$

is the obviously obtained functor such that:

- for all functors $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi' : \mathcal{A}' \to \mathcal{A}''$ which preserve the $\mathbb{B}$-projective limit cones:

$$(- \circ_{\underline{\otimes}_{\mathbb{B},\mathcal{A},\mathcal{A}',\mathcal{A}''}} -)([\phi, \phi']) = \phi' \circ \phi \,,$$

- for all functors $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ and $\phi'_1, \phi'_2 : \mathcal{A}' \to \mathcal{A}''$ which preserve the $\mathbb{B}$-projective limit cones and all natural transformations $t : \phi_1 \Rightarrow \phi_2$ and $t' : \phi'_1 \Rightarrow \phi'_2$:

$$(- \circ_{\underline{\otimes}_{\mathbb{B},\mathcal{A},\mathcal{A}',\mathcal{A}''}} -)([t, t']) = t' \circ t \,.$$

4.2.b. Let $\mathbf{E}$ be a projective sketch. Then, as in [ref2]:

$$\underline{\mathtt{uid}}(\mathbf{E})_{\emptyset} : \mathit{1}_{\emptyset} \to \mathcal{R}ep(\mathbf{E}, \mathbf{E})$$

is the functor which maps the point $\mathit{0}$ to the representation $\mathrm{id}(\mathbf{E}) : \mathbf{E} \to \mathbf{E}$ (which is indeed a point of $\mathcal{R}ep(\mathbf{E}, \mathbf{E})$).

Let $\mathbb{B}$ be a set of compositive graphs and $\mathcal{A}$ a category. Then:

$$\underline{\mathtt{uid}}_{\mathbb{B}}(\mathcal{A}) : \mathit{1} \to \mathcal{F}unc_{\mathbb{B}}(\mathcal{A}, \mathcal{A})$$

is the functor which maps the point $\mathit{0}$ to the functor $\mathrm{id}(\mathcal{A}) : \mathcal{A} \to \mathcal{A}$, which (clearly) preserves the $\mathbb{B}$-projective limit cones.

4.2.c. Let $\mathbb{U}$ be a universe and $\mathbb{B}$ a set of compositive graphs. The *enriched system of* $\mathbb{U}$*-small* $\mathbb{B}$*-projective sketches and* $\mathbb{U}$*-small categories with preserved* $\mathbb{B}$*-projective limit cones*:

$$\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}}$$

is the obviously obtained categorical $\mathsf{CompCat}_{\mathbb{U}}$-enriched $\mathit{2}$-system such that:

- $(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathit{0}} = \mathtt{Pt}(\mathcal{P}sk_{\mathbb{U},\mathbb{B}})$ is the set of $\mathbb{U}$-small $\mathbb{B}$-projective sketches,
- $(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathit{1}} = \mathtt{Pt}(\mathcal{C}at_{\mathbb{U},\mathbb{B}})$ is the set of $\mathbb{U}$-small categories,

- for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$ and $\mathbf{E}'$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{0})}(\mathbf{E},\mathbf{E}') = \mathcal{R}ep(\mathbf{E},\mathbf{E}')\,,$$

- for all $\mathbb{U}$-small $\mathbb{B}$-projective sketch $\mathbf{E}$ and all $\mathbb{U}$-small category $\mathcal{A}$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{(\mathit{0},\mathit{1})}(\mathbf{E},\mathcal{A}) = \mathcal{M}od(\mathbf{E},\mathcal{A})\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}') = \mathcal{F}unc_{\mathbb{B}}(\mathcal{A},\mathcal{A}')\,,$$

- for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$, $\mathbf{E}'$ and $\mathbf{E}''$:

$$\begin{array}{ccc}
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{0})}(\mathbf{E},\mathbf{E}') \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} (\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{0})}(\mathbf{E}',\mathbf{E}'') & & \mathcal{R}ep(\mathbf{E},\mathbf{E}')\,\underline{\Box}\,\mathcal{R}ep(\mathbf{E}',\mathbf{E}'') \\
\Big\downarrow {\scriptstyle \mathtt{comp}_{\mathbf{E},\mathrm{id}(\mathit{0}),\mathbf{E}',\mathrm{id}(\mathit{0}),\mathbf{E}''}} & = & \Big\downarrow {\scriptstyle -\circ\,\underline{\Box}_{\mathbf{E},\mathbf{E}',\mathbf{E}''}-} \\
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{0})}(\mathbf{E},\mathbf{E}'') & & \mathcal{R}ep(\mathbf{E},\mathbf{E}'')
\end{array}$$

- for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$ and $\mathbf{E}'$ and all $\mathbb{U}$-small category $\mathcal{A}''$:

$$\begin{array}{ccc}
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{0})}(\mathbf{E},\mathbf{E}') \otimes_{\mathrm{id}(\mathit{0}),(\mathit{0},\mathit{1})} (\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{(\mathit{0},\mathit{1})}(\mathbf{E}',\mathcal{A}'') & & \mathcal{R}ep(\mathbf{E},\mathbf{E}')\,\underline{\boxtimes}\,\mathcal{M}od(\mathbf{E}',\mathcal{A}'') \\
\Big\downarrow {\scriptstyle \mathtt{comp}_{\mathbf{E},\mathrm{id}(\mathit{0}),\mathbf{E}',(\mathit{0},\mathit{1}),\mathcal{A}''}} & = & \Big\downarrow {\scriptstyle -\circ\,\underline{\boxtimes}_{\mathbf{E},\mathbf{E}',\mathcal{A}''}-} \\
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{(\mathit{0},\mathit{1})}(\mathbf{E},\mathcal{A}'') & & \mathcal{M}od(\mathbf{E},\mathcal{A}'')
\end{array}$$

- for all $\mathbb{U}$-small $\mathbb{B}$-projective sketch $\mathbf{E}$ and all $\mathbb{U}$-small categories $\mathcal{A}'$ and $\mathcal{A}''$:

$$\begin{array}{ccc}
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{(\mathit{0},\mathit{1})}(\mathbf{E},\mathcal{A}') \otimes_{(\mathit{0},\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{1})}(\mathcal{A}',\mathcal{A}'') & & \mathcal{M}od(\mathbf{E},\mathcal{A}')\,\underline{\boxtimes}\,\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}',\mathcal{A}'') \\
\Big\downarrow {\scriptstyle \mathtt{comp}_{\mathbf{E},(\mathit{0},\mathit{1}),\mathcal{A}',\mathrm{id}(\mathit{1}),\mathcal{A}''}} & = & \Big\downarrow {\scriptstyle -\circ\,\underline{\boxtimes}_{\mathbb{B},\mathbf{E},\mathcal{A}',\mathcal{A}''}-} \\
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{(\mathit{0},\mathit{1})}(\mathbf{E},\mathcal{A}'') & & \mathcal{M}od(\mathbf{E},\mathcal{A}'')
\end{array}$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$:

$$\begin{array}{ccc}
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}') \otimes_{\mathrm{id}(\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{1})}(\mathcal{A}',\mathcal{A}'') & & \mathcal{F}unc_{\mathbb{B}}(\mathcal{A},\mathcal{A}')\,\underline{\otimes}\,\mathcal{F}unc_{\mathbb{B}}(\mathcal{A}',\mathcal{A}'') \\
\Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{A},\mathrm{id}(\mathit{1}),\mathcal{A}',\mathrm{id}(\mathit{1}),\mathcal{A}''}} & = & \Big\downarrow {\scriptstyle -\circ\,\underline{\otimes}_{\mathbb{B},\mathcal{A},\mathcal{A}',\mathcal{A}''}-} \\
(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}'') & & \mathcal{F}unc_{\mathbb{B}}(\mathcal{A},\mathcal{A}'')
\end{array}$$

Then:

- the family:

$$(\,\underline{\mathtt{uid}}(\mathbf{E})_{\emptyset} : \mathit{1}_{\emptyset} \to \mathcal{R}ep(\mathbf{E},\mathbf{E})\,)_{\mathbf{E}\in \mathtt{Pt}(\mathcal{P}sk_{\mathbb{U},\mathbb{B}})}$$

  is its family of units of index the point $\mathit{0}$,

- the family:

$$(\,\underline{\mathtt{uid}}_{\mathbb{B}}(\mathcal{A}) : \mathit{1} \to \mathcal{F}unc(\mathcal{A},\mathcal{A})\,)_{\mathcal{A}\in \mathtt{Pt}(\mathcal{C}at_{\mathbb{U},\mathbb{B}})}$$

  is its family of units of index the point $\mathit{1}$,

- the category $(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathit{0}}$ (component of the enriched system $\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}}$ at $\mathit{0}$) is canonically isomorphic, then identified, to the category $\mathcal{P}sk_{\mathbb{U},\mathbb{B}}$ of $\mathbb{U}$-small $\mathbb{B}$-projective sketches,

- the category $(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathit{1}}$ (component of the enriched system $\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}}$ at $\mathit{1}$) is canonically isomorphic, then identified, to the category $\mathcal{C}at_{\mathbb{U},\mathbb{B}}$ of $\mathbb{U}$-small categories with preserved $\mathbb{B}$-projective limit cones,

- the functor:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{(\mathit{0},\mathit{1})} : (\mathcal{P}sk_{\mathbb{U},\mathbb{B}})^{op} \times \mathcal{C}at_{\mathbb{U},\mathbb{B}} \to \mathcal{C}omp_{\mathbb{U}}$$

  is such that:

  - for all $\mathbb{U}$-small $\mathbb{B}$-projective sketch $\mathbf{E}$ and all $\mathbb{U}$-small category $\mathcal{A}'$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{(\mathit{0},\mathit{1})}(\mathbf{E},\mathcal{A}') = \mathcal{M}od(\mathbf{E},\mathcal{A}')\,,$$

  - for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$ and $\mathbf{E}'$, all $\mathbb{U}$-small categories $\mathcal{A}''$ and $\mathcal{A}'''$, all representation $\rho : \mathbf{E} \to \mathbf{E}'$ and all functor $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$ which preserves the $\mathbb{B}$-projective limit cones:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{(\mathit{0},\mathit{1})}(\rho,\phi'') = \mathcal{M}od(\rho,\phi'')\,,$$

- the functor:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{P}sk_{\mathbb{U},\mathbb{B}})^{op} \times \mathcal{P}sk_{\mathbb{U},\mathbb{B}} \to \mathcal{C}omp_{\mathbb{U}}$$

  is such that:

  - for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$ and $\mathbf{E}'$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{0})}(\mathbf{E},\mathbf{E}') = \mathcal{R}ep(\mathbf{E},\mathbf{E}')\,,$$

  - for all $\mathbb{U}$-small $\mathbb{B}$-projective sketches $\mathbf{E}$, $\mathbf{E}'$, $\mathbf{E}''$ and $\mathbf{E}'''$ and all representations $\rho : \mathbf{E} \to \mathbf{E}'$ and $\rho'' : \mathbf{E}'' \to \mathbf{E}'''$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{0})}(\rho,\rho'') = \mathcal{R}ep(\rho,\rho'')\,,$$

- the functor:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{1})} : (\mathcal{C}at_{\mathbb{U},\mathbb{B}})^{op} \times \mathcal{C}at_{\mathbb{U},\mathbb{B}} \to \mathcal{C}at_{\mathbb{U}}$$

  is such that:

  - for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}') = \mathcal{F}unc_{\mathbb{B}}(\mathcal{A},\mathcal{A}')\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$, and all functors $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$ which preserve the $\mathbb{B}$-projective limit cones:

$$(\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}})^*_{\mathrm{id}(\mathit{1})}(\phi, \phi'') = \mathcal{F}\mathit{unc}_{\mathbb{B}}(\phi, \phi'')\,,$$

Obviously, there is:

- a *canonical* $\mathsf{CompCat}_{\mathbb{U}}$-enriched categorical $\mathit{2}$-morphism:

$$\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}} \mid \mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}} : \mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}} \to \mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}}$$

  (its explicitation, using the "support" operator, is left to the reader).

- a *"quasi-canonical" injection* $\mathsf{CompCat}_{\mathbb{U}}$-enriched categorical $\mathit{2}$-morphism:

$$\mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}} \subseteq \underline{\mathbb{P}\mathrm{sk}\mathbb{P}\mathrm{pr}}_{\mathbb{U},\mathbb{B}} : \mathbb{P}\mathrm{sk}\mathbb{C}\mathrm{at}_{\mathbb{U},\mathbb{B}} \to \underline{\mathbb{P}\mathrm{sk}\mathbb{P}\mathrm{pr}}_{\mathbb{U},\mathbb{B}}$$

  (its explicitation, using the "support" operator, as well as the "extrapolation" and "uniformization" operators, is left to the reader).

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