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SKETCHES AND SPECIFICATIONS

REFERENCE MANUAL

First part:
Compositive Graphs

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SKETCHES AND SPECIFICATIONS — REFERENCE MANUAL

SKETCHES AND SPECIFICATIONS is a common denomination for several papers which deal with applications of Ehresmann's sketch theory to computer science. These papers can be considered as the first steps towards a unified theory for software engineering. However, their aim is not to advocate a unification of computer languages; they are designed to build a frame for the study of notions which arise from several areas in computer science.

These papers are arranged in two complementary families:

REFERENCE MANUAL and USER'S GUIDE.

The *reference manual* provides general definitions and results, with comprehensive proofs. On the other hand, the *user's guide* places emphasis on motivations and gives a detailed description of several examples. These two families, though complementary, can be read independently. No prerequisite is assumed; however, it can prove helpful to be familiar either with specification techniques in computer science or with category theory in mathematics.

These papers are under development, they are, or will be, available at: `http://www.unilim.fr/laco/rapports`.

REFERENCE MANUAL:

First Part: Compositive Graphs

Second Part: Projective Sketches

Third Part: Models

USER'S GUIDE:

First Part: Wefts for Explicit Specification

Second Part: Mosaics for Implicit Specification

Third Part: Functional and Imperative Programs

In addition, further papers about APPLICATIONS are in progress, with several co-authors. They deal with various topics, including the notion of state in computer science [etat], overloading, coercions and subsorts.

These articles owe a great deal to the working group *sketches and computer algebra*; we would like to thank its participants, specially Catherine Oriat and Jean-Claude Reynaud, as well as the CNRS.

These papers have been processed with LaTeX and XY-pic.

First Part: Compositive Graphs

This paper is the first part of this reference manual [ref]. The aim of this first part is to introduce the compositive graphs, which will be used in [ref2] in order to define the projective sketches. This paper includes a detailed study of various tensor products and enrichments of compositive graphs and the description of the category freely generated by a given compositive graph. No prerequisite is assumed to read it.

1 Introduction

In a family of papers [guide1,guide2,guide3], we put forward the use of *wefts* and *mosaics* for describing, studying, handling and building *specifications*. Indeed, we first show that the *terms* of a weft yield a relevant representation of the *programs*, in the frame of functional programming languages. Then (and this is the most interesting part), thanks to mosaics, we extend this to imperative programming and various *implicit* features (side-effects, errors treatment, state management,...), which do not fit easily into earlier algebraic specification paradigms.

Although it may prove helpful, for reading the family of papers [guide1,guide2,guide3], to have some acquaintance with algebraic specifications (see for instance [Goguen *et al.* 78] or [Astesiano *et al.* 99]) and category theory (see for instance [Mac Lane 71]), no prerequisite is assumed. Indeed, in our papers, we proceed carefully from examples to general definitions and results.

On the other hand, we believe that this family of papers, which forms the "*user's guide*" for our point of view on specifications, has to come with another family of papers, which forms the corresponding "*reference manual*". The aim of the reference manual is to give a purely formal presentation, with the optimal level of generality, of "all" the definitions, methods and results which are necessary for building a rigorous basis for the user's guide.

Compositive graphs were first introduced in [Ehresmann 65], where they were called "multiplicative graphs", and were somewhat more particuliar than here. For our purpose, they are mainly the *supports* of *projective sketches* (we will see in [ref2] that a projective sketch is a compositive graph with "distinguished projective cones", and, in [ref3], that projective sketches have *models*; such models, in turn, are the supports of the wefts).

Formally (§ 2), compositive graphs are structures which are weaker than categories. Like a category, a compositive graph is made up of objects (which we call *points*) and *arrows*, but two consecutive arrows *need not be composable*. For instance, a directed graph is a compositive graph where *nothing* is composable, whereas a category is a compositive graph where *everything* (if consecutive) is composable.

As in the special case of categories, the *functors* between compositive graphs are exactly the homomorphisms between these structures, while the *natural transformations* between functors (from a compositive graph towards another) are comparisons between these functors: these comparisons are stated thanks to commutative diagrams, however it is necessary to stipulate, first, that the consecutive vertices of these diagrams are composable.

Compositive graphs with a given *size* are defined with respect to a *universe*, *i.e.* a set of sets which is stable for the "usual" operations on sets (in this way we avoid the paradox of "the set of all sets"). Then we have (§ 3 and § 4) the *tensor system* (which could have been presented in terms of bicategories) and the *enriched system* of compositive graphs with this size. This means that there are:

- a category such that its points are the compositive graphs with this size and its arrows are the functors between these compositive graphs;
- a system of three *tensor products* of compositive graphs: the first one is called the *hollow* tensor product, the second one is called the *full* tensor product, and the third one is the *cartesian product*;
- a system of *enrichments* of this category, *i.e.* a structure over its arrows (or "Hom's"); indeed the set of functors from a compositive graph towards another one (both with the

right size) is a third compositive graph (with the same size): it is called the *exponentiation* of the second by the first, and its arrows are the natural transformations between these functors;

- and *enriched* compositions, which work on various tensor products of these "enriched Hom's".

In this way, we get the complete structure of the various actual composition modes. These composition modes include the composition of two functors of course, but also the composition of a functor and a natural transformation, and the composition of two natural transformations. In addition, their values can be either in compositive graphs, in categories, or in directed graphs.

With such a detailed description of these "systems", the forthcoming constructions (of "inductive lax-limits") are fully legitimated.

The image of a functor from a category to another is a compositive graph which *is not*, in general, a category. However, when it is "saturated by composition", it generates a subcategory of the codomain category of this functor. In a more structural way, we consider (§ 5) the *paths* (*i.e.* the families), of consecutive arrows in a compositive graph, and their equivalence classes modulo the potential equalities of composed arrows; this does always "freely" generate a category.

So, a compositive graph can be considered as the *presentation of a category by generators and relations*: the generators are the arrows of the compositive graph, and the relations are the equations which result from the partial composition of these arrows.

This family of papers is a reference manual. For this reason, motivations and examples will not be found here: they are in our user's guide. For this reason too, it is essentially self-contained; it can be read with only some familiarity with the common use of category theory.

2 Compositive graphs, functors and natural transformations

2.1 Compositive graphs

2.1.a. A *compositive graph* $\mathcal{G}$ is made up of:

- a set of *points* $\mathtt{Pt}(\mathcal{G})$,
- a set of *arrows* $\mathtt{Ar}(\mathcal{G})$,
- a set of *identity arrows* $\mathtt{IdAr}(\mathcal{G}) \subseteq \mathtt{Ar}(\mathcal{G})$,
- a set of *composable pairs (of arrows)* $\mathtt{CompP}(\mathcal{G}) \subseteq \mathtt{Ar}(\mathcal{G}) \times \mathtt{Ar}(\mathcal{G})$,
- a *selection of domains* map $\mathtt{seldom}(\mathcal{G}) : \mathtt{Ar}(\mathcal{G}) \to \mathtt{Pt}(\mathcal{G})$
  (for all arrow g of $\mathcal{G}$, we may write $\mathtt{seldom}(\mathcal{G})(g) = \mathtt{dom}(g)$),
- a *selection of codomains* map $\mathtt{selcodom}(\mathcal{G}) : \mathtt{Ar}(\mathcal{G}) \to \mathtt{Pt}(\mathcal{G})$
  (for all arrow g of $\mathcal{G}$, we may write $\mathtt{selcodom}(\mathcal{G})(g) = \mathtt{codom}(g)$),
- a *composition* map $\mathtt{comp}(\mathcal{G}) : \mathtt{CompP}(\mathcal{G}) \to \mathtt{Ar}(\mathcal{G})$
  (for all composable pair (g_1, g_2) of $\mathcal{G}$, we may write $\mathtt{comp}(\mathcal{G})(g_1, g_2) = g_2 \centerdot g_1$; in some cases, another composition symbol, more usual than "$\centerdot$", can be used),

with the following properties:

- for all identity arrow g of $\mathcal{G}$:
  $$\mathtt{dom}(g) = \mathtt{codom}(g) \,,$$
  so that there is a *selection of points* map (for the identity arrows):
  $$\mathtt{selpt}(\mathcal{G}) : \mathtt{IdAr}(\mathcal{G}) \to \mathtt{Pt}(\mathcal{G})$$
  $$=$$
  $$\mathtt{IdAr}(\mathcal{G}) \xrightarrow{\subseteq} \mathtt{Ar}(\mathcal{G}) \xrightarrow{\mathtt{seldom}(\mathcal{G})} \mathtt{Pt}(\mathcal{G})$$
  $$=$$
  $$\mathtt{IdAr}(\mathcal{G}) \xrightarrow{\subseteq} \mathtt{Ar}(\mathcal{G}) \xrightarrow{\mathtt{selcodom}(\mathcal{G})} \mathtt{Pt}(\mathcal{G})$$
  (for all identity arrow g of $\mathcal{G}$, we may write $\mathtt{selpt}(\mathcal{G})(g) = \mathtt{pt}(g)$),
- for all composable pair (g_1, g_2) of $\mathcal{G}$:
  $$\begin{aligned} \mathtt{codom}(g_1) &= \mathtt{dom}(g_2) \,, \\ \mathtt{dom}(g_2 \centerdot g_1) &= \mathtt{dom}(g_1) \,, \\ \mathtt{codom}(g_2 \centerdot g_1) &= \mathtt{codom}(g_2) \,. \end{aligned}$$

Then, if G_1 and G_2 are two points of $\mathcal{G}$ and if g is an arrow of $\mathcal{G}$ such that $\mathtt{dom}(g) = G_1$ and $\mathtt{codom}(g) = G_2$, we write $g : G_1 \to_{\mathcal{G}} G_2$, or simply $g : G_1 \to G_2$.

More precisely, if G is a point of $\mathcal{G}$ and if g is an identity arrow of $\mathcal{G}$ such that $\mathtt{pt}(g) = G$, we write $g : G_1 \Longrightarrow_{\mathcal{G}} G_2$, or simply $g : G_1 \Longrightarrow G_2$.

If G_1 and G_2 are two points of $\mathcal{G}$, we write:

$$\mathrm{Hom}_{\mathcal{G}}(G_1, G_2) = \mathcal{G}(G_1, G_2) = \{g \in \mathtt{Ar}(\mathcal{G}) \mid g : G_1 \to G_2\} \ .$$

Finally, the set of pairs of *consecutive* arrows of $\mathcal{G}$ is:

$$\mathtt{ConsP}(\mathcal{G}) = \{(g_1, g_2) \in \mathtt{Ar}(\mathcal{G}) \times \mathtt{Ar}(\mathcal{G}) \mid \mathtt{codom}(g_1) = \mathtt{dom}(g_2)\} \ ,$$

so that $\mathtt{CompP}(\mathcal{G}) \subseteq \mathtt{ConsP}(\mathcal{G})$.

2.1.b. Let $\mathcal{G}$ be a compositive graph. As for categories, the *dual compositive graph of* $\mathcal{G}$:

$$\mathcal{G}^{op}$$

is the obviously obtained compositive graph such that:

- $\mathtt{Pt}(\mathcal{G}^{op}) = \mathtt{Pt}(\mathcal{G})$,
- $\mathtt{Ar}(\mathcal{G}^{op}) = \mathtt{Ar}(\mathcal{G})$,
- $\mathtt{IdAr}(\mathcal{G}^{op}) = \mathtt{IdAr}(\mathcal{G})$,
- $\mathtt{CompP}(\mathcal{G}^{op}) = \{(g_2, g_1) \mid (g_1, g_2) \in \mathtt{CompP}(\mathcal{G})\}$,
- $\mathtt{seldom}(\mathcal{G}^{op}) = \mathtt{selcodom}(\mathcal{G})$,
- $\mathtt{selcodom}(\mathcal{G}^{op}) = \mathtt{seldom}(\mathcal{G})$,
- for all composable pair (g_2, g_1) of $\mathcal{G}^{op}$:

$$\mathtt{comp}(\mathcal{G}^{op})(g_2, g_1) = \mathtt{comp}(\mathcal{G})(g_1, g_2) \ .$$

2.1.c. A *category* $\mathcal{A}$ is (*i.e.* can be identified to) a compositive graph such that:

- the map $\mathtt{selpt}(\mathcal{A}) : \mathtt{IdAr}(\mathcal{A}) \to \mathtt{Pt}(\mathcal{A})$ is a bijection, so that there is a *selection of identities (of points)* map:

$$\mathtt{selid}(\mathcal{A}) : \mathtt{Pt}(\mathcal{A}) \to \mathtt{Ar}(\mathcal{A})$$
$$=$$
$$\mathtt{Pt}(\mathcal{A}) \xrightarrow{\mathtt{selpt}(\mathcal{A})^{-1}} \mathtt{IdAr}(\mathcal{A}) \xrightarrow{\subseteq} \mathtt{Ar}(\mathcal{A})$$

  (for all point A of $\mathcal{A}$, we may write $\mathtt{selid}(\mathcal{A})(A) = \mathrm{id}(A)$),
- each pair of consecutive arrows is composable, *i.e.* $\mathtt{CompP}(\mathcal{A}) = \mathtt{ConsP}(\mathcal{A})$,
- identity arrows are (local) units for the composition of arrows,
- the composition of arrows is associative.

2.1.d. A *directed graph* $\mathcal{R}$ is (*i.e.* can be identified to) a compositive graph such that:

- there is no identity arrow, *i.e.* $\mathtt{IdAr}(\mathcal{R})=\emptyset$,
- there is no composable pair, *i.e.* $\mathtt{CompP}(\mathcal{R})=\emptyset$.

Then, to each compositive graph $\mathcal{G}$ is assigned its *sublying directed graph*, which has the same points and arrows as $\mathcal{G}$, *i.e.* the directed graph $\mathtt{subl}(\mathcal{G})$ such that:

- $\mathtt{Pt}(\mathtt{subl}(\mathcal{G})) = \mathtt{Pt}(\mathcal{G})$,
- $\mathtt{Ar}(\mathtt{subl}(\mathcal{G})) = \mathtt{Ar}(\mathcal{G})$,
- $\mathtt{seldom}(\mathtt{subl}(\mathcal{G})) = \mathtt{seldom}(\mathcal{G})$,
- $\mathtt{selcodom}(\mathtt{subl}(\mathcal{G})) = \mathtt{selcodom}(\mathcal{G})$.

2.2 Functors

2.2.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. A *functor* $\phi : \mathcal{G} \to \mathcal{G}'$ *from* $\mathcal{G}$ *to* $\mathcal{G}'$ is made up of:

- a map $\mathtt{Pt}(\phi) : \mathtt{Pt}(\mathcal{G}) \to \mathtt{Pt}(\mathcal{G}')$
  (for all point G of $\mathcal{G}$, we may write $\mathtt{Pt}(\phi)(G)=\phi(G)$),
- a map $\mathtt{Ar}(\phi) : \mathtt{Ar}(\mathcal{G}) \to \mathtt{Ar}(\mathcal{G}')$
  (for all arrow g of $\mathcal{G}$, we may write $\mathtt{Ar}(\phi)(g)=\phi(g)$),

with the following properties:

- for all arrow g of $\mathcal{G}$:

  $$\phi(\mathtt{dom}(g)) = \mathtt{dom}(\phi(g)) \qquad \text{and} \qquad \phi(\mathtt{codom}(g)) = \mathtt{codom}(\phi(g))\,,$$

  which means that the following diagrams (of maps) are commutative:

  $$\begin{array}{ccc} \mathtt{Ar}(\mathcal{G}) & \xrightarrow{\mathtt{seldom}(\mathcal{G})} & \mathtt{Pt}(\mathcal{G}) \\ {\scriptstyle \mathtt{Ar}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{Pt}(\phi)} \\ \mathtt{Ar}(\mathcal{G}') & \xrightarrow[\mathtt{seldom}(\mathcal{G}')]{} & \mathtt{Pt}(\mathcal{G}') \end{array} \qquad \begin{array}{ccc} \mathtt{Ar}(\mathcal{G}) & \xrightarrow{\mathtt{selcodom}(\mathcal{G})} & \mathtt{Pt}(\mathcal{G}) \\ {\scriptstyle \mathtt{Ar}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{Pt}(\phi)} \\ \mathtt{Ar}(\mathcal{G}') & \xrightarrow[\mathtt{selcodom}(\mathcal{G}')]{} & \mathtt{Pt}(\mathcal{G}') \end{array}$$

  and then, it follows that the map:

  $$\mathtt{Ar}(\phi) \times \mathtt{Ar}(\phi) : \mathtt{Ar}(\mathcal{G}) \times \mathtt{Ar}(\mathcal{G}) \to \mathtt{Ar}(\mathcal{G}') \times \mathtt{Ar}(\mathcal{G}')$$

  can be restricted to:

  $$\mathtt{ConsP}(\phi) : \mathtt{ConsP}(\mathcal{G}) \to \mathtt{ConsP}(\mathcal{G}')\,,$$

  so that the following diagram (of maps) is commutative:

  $$\begin{array}{ccc} \mathtt{ConsP}(\mathcal{G}) & \xrightarrow{\subseteq} & \mathtt{Ar}(\mathcal{G}) \times \mathtt{Ar}(\mathcal{G}) \\ {\scriptstyle \mathtt{ConsP}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{Ar}(\phi)\times\mathtt{Ar}(\phi)} \\ \mathtt{ConsP}(\mathcal{G}') & \xrightarrow[\subseteq]{} & \mathtt{Ar}(\mathcal{G}') \times \mathtt{Ar}(\mathcal{G}') \end{array}$$

- for all identity arrow g of $\mathcal{G}$, the arrow $\phi(g)$ is an identity arrow of $\mathcal{G}'$, which means that the map:

  $$\mathtt{Ar}(\phi) : \mathtt{Ar}(\mathcal{G}) \to \mathtt{Ar}(\mathcal{G}')$$

  can be restricted to:

  $$\mathtt{IdAr}(\phi) : \mathtt{IdAr}(\mathcal{G}) \to \mathtt{IdAr}(\mathcal{G}')$$

  so that the following diagram (of maps) is commutative:

  $$\begin{array}{ccc} \mathtt{IdAr}(\mathcal{G}) & \xrightarrow{\subseteq} & \mathtt{Ar}(\mathcal{G}) \\ {\scriptstyle \mathtt{IdAr}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{Ar}(\phi)} \\ \mathtt{IdAr}(\mathcal{G}') & \xrightarrow[\subseteq]{} & \mathtt{Ar}(\mathcal{G}') \end{array}$$

- for all composable pair (g_1, g_2) of $\mathcal{G}$, the pair $(\phi(g_1), \phi(g_2))$ is a composable pair of $\mathcal{G}'$ and:

  $$\phi(g_2 \centerdot g_1) = \phi(g_2) \centerdot \phi(g_1)\,,$$

  which means that the map:

  $$\mathtt{ConsP}(\phi) : \mathtt{ConsP}(\mathcal{G}) \to \mathtt{ConsP}(\mathcal{G}')$$

  can be restricted to:

  $$\mathtt{CompP}(\phi) : \mathtt{CompP}(\mathcal{G}) \to \mathtt{CompP}(\mathcal{G}')$$

  and the following diagrams (of maps) are commutative:

  $$\begin{array}{ccc} \mathtt{CompP}(\mathcal{G}) & \xrightarrow{\subseteq} & \mathtt{ConsP}(\mathcal{G}) \\ {\scriptstyle \mathtt{CompP}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{ConsP}(\phi)} \\ \mathtt{CompP}(\mathcal{G}') & \xrightarrow[\subseteq]{} & \mathtt{ConsP}(\mathcal{G}') \end{array} \qquad \begin{array}{ccc} \mathtt{CompP}(\mathcal{G}) & \xrightarrow{\mathtt{comp}(\mathcal{G})} & \mathtt{Ar}(\mathcal{G}) \\ {\scriptstyle \mathtt{CompP}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathtt{Ar}(\phi)} \\ \mathtt{CompP}(\mathcal{G}') & \xrightarrow[\mathtt{comp}(\mathcal{G}')]{} & \mathtt{Ar}(\mathcal{G}') \end{array}$$

2.2.b. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs and $\phi : \mathcal{G} \to \mathcal{G}'$ a functor. The *dual functor of* ϕ:

$$\phi^{op} : \mathcal{G}'^{op} \to \mathcal{G}^{op}$$

is the obviously obtained functor such that:

- $\mathtt{Pt}(\phi^{op}) = \mathtt{Pt}(\phi)$,
- $\mathtt{Ar}(\phi^{op}) = \mathtt{Ar}(\phi)$.

2.2.c. Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs, $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{G}''$ two functors. The *composed functor of* ϕ' *with* ϕ:

$$\phi' \circ \phi : \mathcal{G} \to \mathcal{G}''$$

is the obviously obtained functor such that:

- $\mathtt{Pt}(\phi' \circ \phi) = \mathtt{Pt}(\phi') \circ \mathtt{Pt}(\phi)$,

- $\mathtt{Ar}(\phi' \circ \phi) = \mathtt{Ar}(\phi') \circ \mathtt{Ar}(\phi)$.

Let $\mathcal{G}$ be a compositive graph. In a similar way, the *identity functor of* $\mathcal{G}$:

$$\mathrm{id}(\mathcal{G}) : \mathcal{G} \to \mathcal{G}$$

is the obviously obtained functor such that:

- $\mathtt{Pt}(\mathrm{id}(\mathcal{G})) = \mathrm{id}(\mathtt{Pt}(\mathcal{G}))$,
- $\mathtt{Ar}(\mathrm{id}(\mathcal{G})) = \mathrm{id}(\mathtt{Ar}(\mathcal{G}))$.

2.3 Natural transformations

2.3.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs and $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ two functors. A *natural transformation* $t : \phi_1 \Rightarrow \phi_2 : \mathcal{G} \to \mathcal{G}'$ (or simply $t : \phi_1 \Rightarrow \phi_2$) *from* ϕ_1 *to* ϕ_2 is made up of:

- a family of arrows of $\mathcal{G}'$:

$$\mathtt{fam}(t) = (\, \mathtt{fam}(t)(G) : \phi_1(G) \to \phi_2(G)\,)_{G \in \mathtt{Pt}(\mathcal{G})}$$

  (for all point G of $\mathcal{G}$, we may write $\mathtt{fam}(t)(G) = t(G)$),

with the following property:

- for all arrow $g : G_1 \to G_2$ of $\mathcal{G}$, the pairs $(t(G_1), \phi_2(g))$ and $(\phi_1(g), t(G_2))$ are composable pairs of $\mathcal{G}'$ and:

$$t(G_2) \centerdot \phi_1(g) = \phi_2(g) \centerdot t(G_1)\,,$$

  which means that the following diagram (in $\mathcal{G}'$) is commutative:

$$\begin{array}{ccc} \phi_1(G_1) & \xrightarrow{\phi_1(g)} & \phi_1(G_2) \\ {\scriptstyle t(G_1)}\big\downarrow & & \big\downarrow{\scriptstyle t(G_2)} \\ \phi_2(G_1) & \xrightarrow[\phi_2(g)]{} & \phi_2(G_2) \end{array}$$

2.3.b. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs, $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ two functors and $t : \phi_1 \Rightarrow \phi_2 : \mathcal{G} \to \mathcal{G}'$ a natural transformation. The *dual natural transformation of* t:

$$t^{op} : \phi_2^{op} \Rightarrow \phi_1^{op} : \mathcal{G}^{op} \to \mathcal{G}'^{op}$$

is the obviously obtained natural transformation such that:

- for all point G of $\mathcal{G}$:

$$t^{op}(G) = t(G)\;.$$

2.3.c. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs, $\phi_1, \phi_2, \phi_3 : \mathcal{G} \to \mathcal{G}'$ three functors, $t_1 : \phi_1 \Rightarrow \phi_2$ and $t_2 : \phi_2 \Rightarrow \phi_3$ two natural transformations. The pair (t_1, t_2) is a *naturally composable pair (of natural transformations)* if:

- for all point G of $\mathcal{G}$, the pair $(t_1(G), t_2(G))$ is a composable pair of $\mathcal{G}'$,
- for all arrow $g : G_1 \to G_2$ of $\mathcal{G}$, the pair $(\phi_1(G), t_2(G_2) \centerdot t_1(G_2))$ and the pair $(t_2(G_1) \centerdot t_1(G_1), \phi_3(G))$ are composable pairs of $\mathcal{G}'$ and their composed arrows are equal, *i.e.* :

$$(t_2(G_2) \centerdot t_1(G_2)) \centerdot \phi_1(G) = \phi_3(G) \centerdot (t_2(G_1) \centerdot t_1(G_1)) \ .$$

Then the *composed natural transformation of* t_2 *with* t_1:

$$t_2 \centerdot t_1 : \phi_1 \Rightarrow \phi_3 : \mathcal{G} \to \mathcal{G}'$$

is the obviously obtained natural transformation such that:

- for all point G of $\mathcal{G}$:
$$(t_2 \centerdot t_1)(G) = t_2(G) \centerdot t_1(G) \ .$$

When $\mathcal{G}' = \mathcal{A}'$ is a category, each pair of consecutive natural transformations (t_1, t_2) is naturally composable (indeed, for all point G of $\mathcal{G}$, the pair $(t_1(G), t_2(G))$ is a composable pair of $\mathcal{A}'$, and the composition of arrows in $\mathcal{A}'$ is associative).

Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs and $\phi : \mathcal{G} \to \mathcal{G}'$ a functor. In a similar way, a natural transformation $t : \phi \Rightarrow \phi$ is *identitary* if:

- for all point G of $\mathcal{G}$, the arrow $t(G) : \phi(G) \to \phi(G)$ is an identity arrow of $\mathcal{G}'$.

When $\mathcal{G}' = \mathcal{A}'$ is a category, the *identity natural transformation at* ϕ:

$$\mathrm{id}(\phi) : \phi \Rightarrow \phi : \mathcal{G} \to \mathcal{A}'$$

is the unique obviously obtained identitary natural transformation from ϕ to ϕ such that:

- for all point G of $\mathcal{G}$:
$$\mathrm{id}(\phi)(G) = \mathrm{id}(\phi(G)) \ .$$

2.3.d. Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs, $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{G}''$ three functors and $t : \phi_1 \Rightarrow \phi_2$ a natural transformation. The *composed of* ϕ' *with* t:

$$\phi' \circ t : \phi' \circ \phi_1 \Rightarrow \phi' \circ \phi_2 : \mathcal{G} \to \mathcal{G}''$$

is the obviously obtained natural transformation such that:

- for all point G of $\mathcal{G}$:
$$(\phi' \circ t)(G) = \phi'(t(G)) \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs, $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'_1, \phi'_2 : \mathcal{G}' \to \mathcal{G}''$ three functors and $t' : \phi'_1 \Rightarrow \phi'_2$ a natural transformation. In a similar way, the *composed of* t' *with* ϕ:

$$t' \circ \phi : \phi'_1 \circ \phi \Rightarrow \phi'_2 \circ \phi : \mathcal{G} \to \mathcal{G}''$$

is the obviously obtained natural transformation such that:

- for all point G of $\mathcal{G}$:
$$(t' \circ \phi)(G) = t'(\phi(G)) \ .$$

Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs, $\mathcal{A}''$ a category, $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ and $\phi'_1, \phi'_2 : \mathcal{G}' \to \mathcal{A}''$ four functors, $t : \phi_1 \Rightarrow \phi_2$ and $t' : \phi'_1 \Rightarrow \phi'_2$ two natural transformations. Then it is easy to check, since $\mathcal{A}''$ is a category, that the following diagram (of natural transformations) is commutative:

$$\begin{array}{ccc} \phi'_1 \circ \phi_1 & \xRightarrow{\phi'_1 \circ t} & \phi'_1 \circ \phi_2 \\ {\scriptstyle t' \circ \phi_1} \Downarrow & & \Downarrow {\scriptstyle t' \circ \phi_2} \\ \phi'_2 \circ \phi_1 & \xRightarrow[\phi'_2 \circ t]{} & \phi'_2 \circ \phi_2 \end{array}$$

The diagonal of this diagram defines the natural transformation *composed of t' with t*:

$$t' \circ t : \phi'_1 \circ \phi_1 \Rightarrow \phi'_2 \circ \phi_2 : \mathcal{G} \to \mathcal{A}'' \ .$$

3 Exponentiation and tensorisations of compositive graphs

3.1 Exponentiation

3.1.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. The set of functors from $\mathcal{G}$ to $\mathcal{G}'$ is denoted:

$$\mathrm{Func}(\mathcal{G}, \mathcal{G}') \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs and $\phi' : \mathcal{G}' \to \mathcal{G}''$ a functor. The *left-composition with* ϕ' map:

$$\mathrm{Func}(\mathcal{G}, \phi') : \mathrm{Func}(\mathcal{G}, \mathcal{G}') \to \mathrm{Func}(\mathcal{G}, \mathcal{G}'')$$

is the map such that:

- for all functor $\phi : \mathcal{G} \to \mathcal{G}'$:
$$\mathrm{Func}(\mathcal{G}, \phi')(\phi) = \phi' \circ \phi \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs and $\phi : \mathcal{G} \to \mathcal{G}'$ a functor. In a similar way, the *right-composition with* ϕ map:

$$\mathrm{Func}(\phi, \mathcal{G}'') : \mathrm{Func}(\mathcal{G}', \mathcal{G}'') \to \mathrm{Func}(\mathcal{G}, \mathcal{G}'')$$

is the map such that:

- for all functor $\phi' : \mathcal{G}' \to \mathcal{G}''$:
$$\mathrm{Func}(\phi, \mathcal{G}'')(\phi') = \phi' \circ \phi \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$ and $\mathcal{G}'''$ be four compositive graphs, $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'' : \mathcal{G}'' \to \mathcal{G}'''$ two functors. It is easy to check that the following diagram (of maps) is commutative:

$$\begin{array}{ccc}
\mathrm{Func}(\mathcal{G}', \mathcal{G}'') & \xrightarrow{\mathrm{Func}(\mathcal{G}', \phi'')} & \mathrm{Func}(\mathcal{G}', \mathcal{G}''') \\
\downarrow{\scriptstyle \mathrm{Func}(\phi, \mathcal{G}'')} & & \downarrow{\scriptstyle \mathrm{Func}(\phi, \mathcal{G}''')} \\
\mathrm{Func}(\mathcal{G}, \mathcal{G}'') & \xrightarrow[\mathrm{Func}(\mathcal{G}, \phi'')]{} & \mathrm{Func}(\mathcal{G}, \mathcal{G}''')
\end{array}$$

So, the diagonal of this diagram defines the map:

$$\mathrm{Func}(\phi, \phi'') : \mathrm{Func}(\mathcal{G}', \mathcal{G}'') \to \mathrm{Func}(\mathcal{G}, \mathcal{G}''') \ .$$

3.1.b. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. The *compositive graph of functors from* $\mathcal{G}$ *to* $\mathcal{G}'$ (or *exponentiation of* $\mathcal{G}'$ *by* $\mathcal{G}$):

$$\mathcal{F}unc(\mathcal{G}, \mathcal{G}')$$

is the canonically obtained compositive graph such that:

- its points are the functors from $\mathcal{G}$ to $\mathcal{G}'$,
- its arrows are the natural transformations between these functors,

- its identity arrows are the identitary natural transformations between these functors,
- its composition law is the composition of the pairs of natural transformations (between these functors) which are naturally composable.

When $\mathcal{G}' = \mathcal{A}'$ is a category, the compositive graph $\mathcal{F}unc(\mathcal{G}, \mathcal{A}')$ is clearly a category.
When $\mathcal{G}' = \mathcal{R}'$ is a directed graph, the compositive graph $\mathcal{F}unc(\mathcal{G}, \mathcal{R}')$ is clearly a directed graph.

Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs and $\phi' : \mathcal{G}' \to \mathcal{G}''$ a functor. The *left-composition with* ϕ' functor:

$$\mathcal{F}unc(\mathcal{G}, \phi') : \mathcal{F}unc(\mathcal{G}, \mathcal{G}') \to \mathcal{F}unc(\mathcal{G}, \mathcal{G}'')$$

is the obviously obtained functor such that:

- for all functor $\phi : \mathcal{G} \to \mathcal{G}'$:
$$\mathcal{F}unc(\mathcal{G}, \phi')(\phi) = \phi' \circ \phi \,,$$
- for all natural transformation $t : \phi_1 \Rightarrow \phi_2 : \mathcal{G} \to \mathcal{G}'$:
$$\mathcal{F}unc(\mathcal{G}, \phi')(t) = \phi' \circ t \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs and $\phi : \mathcal{G} \to \mathcal{G}'$ a functor. In a similar way, the *right-composition with* ϕ functor:

$$\mathcal{F}unc(\phi, \mathcal{G}'') : \mathcal{F}unc(\mathcal{G}', \mathcal{G}'') \to \mathcal{F}unc(\mathcal{G}, \mathcal{G}'')$$

is the obviously obtained functor such that:

- for all functor $\phi' : \mathcal{G}' \to \mathcal{G}''$:
$$\mathcal{F}unc(\phi, \mathcal{G}'')(\phi') = \phi' \circ \phi \,,$$
- for all natural transformation $t' : \phi'_1 \Rightarrow \phi'_2 : \mathcal{G}' \to \mathcal{G}''$:
$$\mathcal{F}unc(\phi, \mathcal{G}'')(t') = t' \circ \phi \ .$$

Let $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$ and $\mathcal{G}'''$ be four compositive graphs, $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'' : \mathcal{G}'' \to \mathcal{G}'''$ two functors. It is easy to check that the following diagram (of functors) is commutative:

$$\begin{array}{ccc}
\mathcal{F}unc(\mathcal{G}', \mathcal{G}'') & \xrightarrow{\mathcal{F}unc(\mathcal{G}', \phi'')} & \mathcal{F}unc(\mathcal{G}', \mathcal{G}''') \\
{\scriptstyle \mathcal{F}unc(\phi, \mathcal{G}'')}\downarrow & & \downarrow{\scriptstyle \mathcal{F}unc(\phi, \mathcal{G}''')} \\
\mathcal{F}unc(\mathcal{G}, \mathcal{G}'') & \xrightarrow[\mathcal{F}unc(\mathcal{G}, \phi'')]{} & \mathcal{F}unc(\mathcal{G}, \mathcal{G}''')
\end{array}$$

So, the diagonal of this diagram defines the functor:

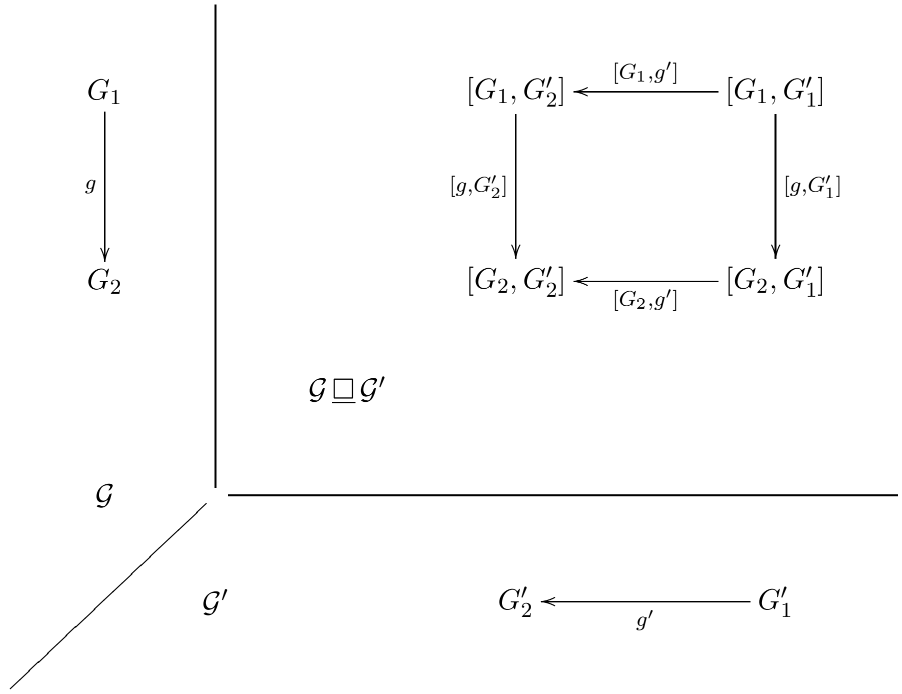
$$\mathcal{F}unc(\phi, \phi'') : \mathcal{F}unc(\mathcal{G}', \mathcal{G}'') \to \mathcal{F}unc(\mathcal{G}, \mathcal{G}''') \ .$$

3.2 Hollow tensorisation

3.2.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. The *hollow tensor product of* $\mathcal{G}$ *with* $\mathcal{G}'$:

$$\mathcal{G} \,\underline{\square}\, \mathcal{G}'$$

is the obviously obtained compositive graph such that, as schematically represented below:



- its points are the $[G, G']$, where G is a point of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
- its arrows are:
  - the $[g, G'] : [G_1, G'] \to [G_2, G']$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
  - the $[G, g'] : [G, G'_1] \to [G, G'_2]$, where G is a point of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$,
- its identity arrows are:
  - the $[g, G'] : [G, G'] \Longrightarrow [G, G']$, where $g : G \Longrightarrow G$ is an identity arrow of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
  - the $[G, g'] : [G, G'] \Longrightarrow [G, G']$, where G is a point of $\mathcal{G}$ and $g' : G' \Longrightarrow G'$ is an identity arrow of $\mathcal{G}'$,
- its composable pairs are:
  - the $([g_1, G'], [g_2, G'])$, where (g_1, g_2) is a composable pair of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$, in which case $[g_2, G'] \centerdot [g_1, G'] = [g_2 \centerdot g_1, G']$,

- the $([G, g'_1], [G, g'_2])$, where G is a point of $\mathcal{G}$ and (g'_1, g'_2) is a composable pair of $\mathcal{G}'$, in which case $[G, g'_2] \centerdot [G, g'_1] = [G, g'_2 \centerdot g'_1]$.

When $\mathcal{G} = \mathcal{R}$ and $\mathcal{G}' = \mathcal{R}'$ are two directed graphs, $\mathcal{R} \,\underline{\square}\, \mathcal{R}'$ is, of course, a directed graph.

3.2.b. Let $\mathcal{G}_1$, $\mathcal{G}'_1$, $\mathcal{G}_2$ and $\mathcal{G}'_2$ be four compositive graphs, $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\phi' : \mathcal{G}'_1 \to \mathcal{G}'_2$ two functors. The *hollow tensor product of ϕ with ϕ'*:

$$\phi \,\underline{\square}\, \phi' : \mathcal{G}_1 \,\underline{\square}\, \mathcal{G}'_1 \to \mathcal{G}_2 \,\underline{\square}\, \mathcal{G}'_2$$

is the obviously obtained functor such that:

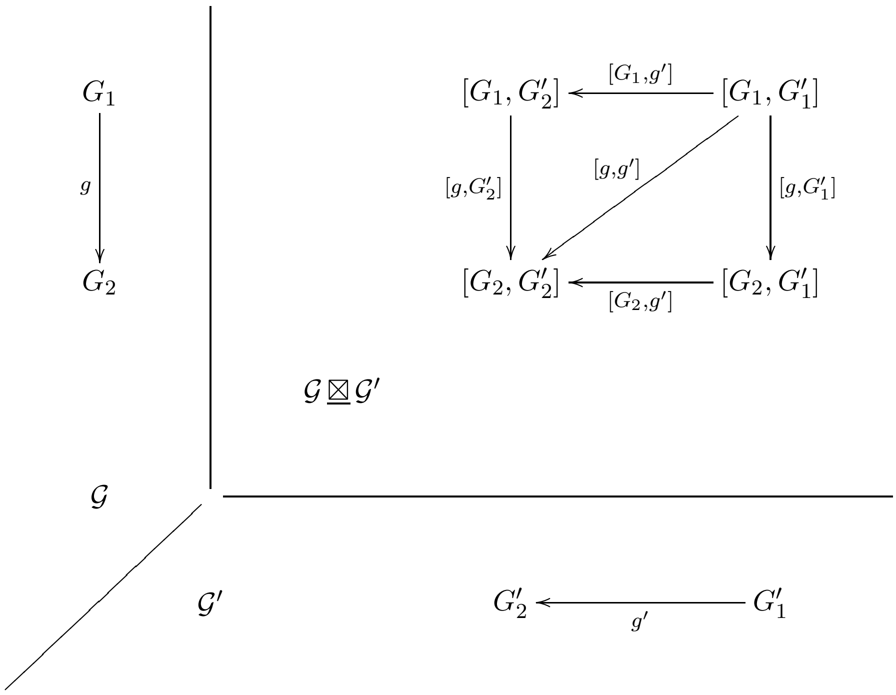
- for all point G_1 of $\mathcal{G}_1$ and all point G'_1 of $\mathcal{G}'_1$:
$$(\phi \,\underline{\square}\, \phi')([G_1, G'_1]) = [\phi(G_1), \phi'(G'_1)] \,,$$
- for all arrow g_1 of $\mathcal{G}_1$ and all point G'_1 of $\mathcal{G}'_1$:
$$(\phi \,\underline{\square}\, \phi')([g_1, G'_1]) = [\phi(g_1), \phi'(G'_1)] \,,$$
- for all point G_1 of $\mathcal{G}_1$ and all arrow g'_1 of $\mathcal{G}'_1$:
$$(\phi \,\underline{\square}\, \phi')([G_1, g'_1]) = [\phi(G_1), \phi'(g'_1)] \,.$$

3.3 Full tensorisation

3.3.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. The *full tensor product of $\mathcal{G}$ with $\mathcal{G}'$*:

$$\mathcal{G} \,\underline{\boxtimes}\, \mathcal{G}'$$

is the obviously obtained compositive graph such that, as schematically represented below:



- its points are the $[G, G']$, where G is a point of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
- its arrows are:
  - the $[g, G'] : [G_1, G'] \to [G_2, G']$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
  - the $[G, g'] : [G, G'_1] \to [G, G'_2]$, where G is a point of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$,
  - the $[g, g'] : [G_1, G'_1] \to [G_2, G'_2]$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$,
- its identity arrows are:
  - the $[g, G'] : [G, G'] \Longrightarrow [G, G']$, where $g : G \Longrightarrow G$ is an identity arrow of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
  - the $[G, g'] : [G, G'] \Longrightarrow [G, G']$, where G is a point of $\mathcal{G}$ and $g' : G' \Longrightarrow G'$ is an identity arrow of $\mathcal{G}'$,
  - the $[g, g'] : [G, G'] \Longrightarrow [G, G']$, where $g : G \Longrightarrow G$ is an identity arrow of $\mathcal{G}$ and $g' : G' \Longrightarrow G'$ is an identity arrow of $\mathcal{G}'$,
- its composable pairs are:
  - the $([g_1, G'], [g_2, G'])$, where (g_1, g_2) is a composable pair of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$, in which case $[g_2, G'] \cdot [g_1, G'] = [g_2 \cdot g_1, G']$,
  - the $([G, g'_1], [G, g'_2])$, where G is a point of $\mathcal{G}$ and (g'_1, g'_2) is a composable pair of $\mathcal{G}'$, in which case $[G, g'_2] \cdot [G, g'_1] = [G, g'_2 \cdot g'_1]$,
  - the $([g, G'_1], [G_2, g'])$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$, in which case $[G_2, g'] \cdot [g, G'_1] = [g, g']$,
  - the $([G_1, g'], [g, G'_2])$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$, in which case $[g, G'_2] \cdot [G_1, g'] = [g, g']$,
  - the $([g_1, g'_1], [g_2, g'_2])$, where (g_1, g_2) is a composable pair of $\mathcal{G}$ and (g'_1, g'_2) is a composable pair of $\mathcal{G}'$, in which case $[g_2, g'_2] \cdot [g_1, g'_1] = [g_2 \cdot g_1, g'_2 \cdot g'_1]$.

3.3.b. Let $\mathcal{G}_1$, $\mathcal{G}'_1$, $\mathcal{G}_2$ and $\mathcal{G}'_2$ be four compositive graphs, $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\phi' : \mathcal{G}'_1 \to \mathcal{G}'_2$ two functors. The *full tensor product of ϕ with ϕ'*:

$$\phi \boxtimes \phi' : \mathcal{G}_1 \boxtimes \mathcal{G}'_1 \to \mathcal{G}_2 \boxtimes \mathcal{G}'_2$$

is the obviously obtained functor such that:

- for all point G_1 of $\mathcal{G}_1$ and all point G'_1 of $\mathcal{G}'_1$:

$$(\phi \boxtimes \phi')([G_1, G'_1]) = [\phi(G_1), \phi'(G'_1)]\,,$$

- for all arrow g_1 of $\mathcal{G}_1$ and all point G'_1 of $\mathcal{G}'_1$:

$$(\phi \boxtimes \phi')([g_1, G'_1]) = [\phi(g_1), \phi'(G'_1)]\,,$$

- for all point G_1 of $\mathcal{G}_1$ and all arrow g'_1 of $\mathcal{G}'_1$:

$$(\phi \boxtimes \phi')([G_1, g'_1]) = [\phi(G_1), \phi'(g'_1)] \,,$$

- for all arrow g_1 of $\mathcal{G}_1$ and all arrow g'_1 of $\mathcal{G}'_1$:

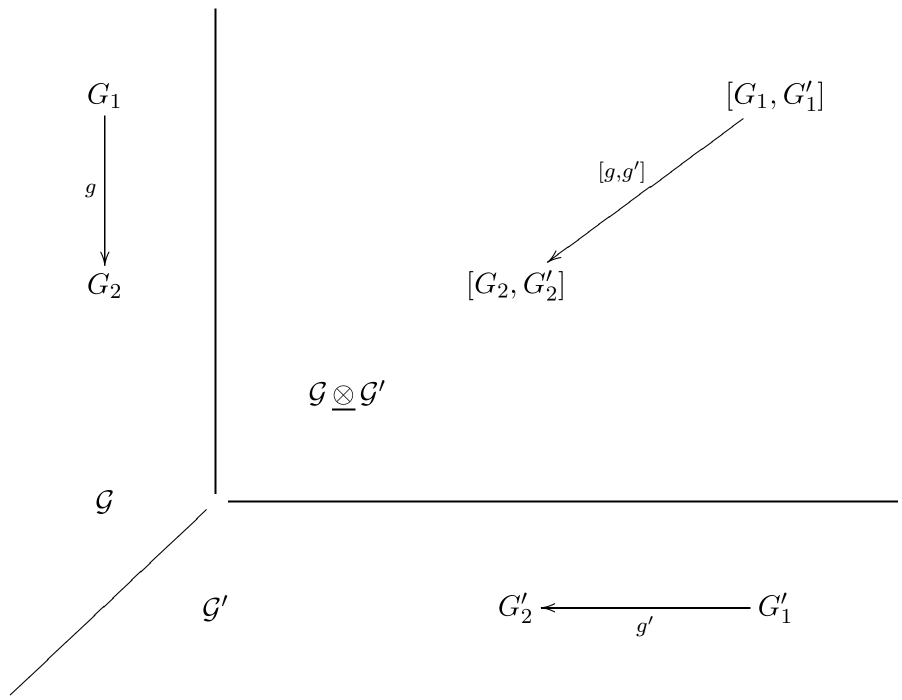
$$(\phi \boxtimes \phi')([g_1, g'_1]) = [\phi(g_1), \phi'(g'_1)] \,.$$

3.4 Cartesian tensorisation

3.4.a. Let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs. The *cartesian tensor product of* $\mathcal{G}$ *with* $\mathcal{G}'$:

$$\mathcal{G} \underline{\otimes} \mathcal{G}' = \mathcal{G} \times \mathcal{G}'$$

is the obviously obtained compositive graph such that, as schematically represented below:



- its points are the $[G, G']$, where G is a point of $\mathcal{G}$ and G' is a point of $\mathcal{G}'$,
- its arrows are the $[g, g'] : [G_1, G'_1] \to [G_2, G'_2]$, where $g : G_1 \to G_2$ is an arrow of $\mathcal{G}$ and $g' : G'_1 \to G'_2$ is an arrow of $\mathcal{G}'$,
- the $[g, g'] : [G, G'] \Longrightarrow [G, G']$, where $g : G \Longrightarrow G$ is an identity arrow of $\mathcal{G}$ and $g' : G' \Longrightarrow G'$ is an identity arrow of $\mathcal{G}'$,
- its composable pairs are the $([g_1, g'_1], [g_2, g'_2])$, where (g_1, g_2) is a composable pair of $\mathcal{G}$ and (g'_1, g'_2) is a composable pair of $\mathcal{G}'$, in which case $[g_2, g'_2] \centerdot [g_1, g'_1] = [g_2 \centerdot g_1, g'_2 \centerdot g'_1]$.

When $\mathcal{G} = \mathcal{A}$ and $\mathcal{G}' = \mathcal{A}'$ are two categories, it is easy to check that $\mathcal{A} \underline{\otimes} \mathcal{A}' = \mathcal{A} \times \mathcal{A}'$ is a category. From here on, we often use cartesian products of compositive graphs and/or categories. However we use their tensor product "status" (which will be defined later) only for categories.

3.4.b. Let $\mathcal{G}_1$, $\mathcal{G}'_1$, $\mathcal{G}_2$ and $\mathcal{G}'_2$ be four compositive graphs, $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\phi' : \mathcal{G}'_1 \to \mathcal{G}'_2$ two functors. The *cartesian tensor product of ϕ with ϕ'*:

$$\phi \underline{\otimes} \phi' : \mathcal{G}_1 \underline{\otimes} \mathcal{G}'_1 \to \mathcal{G}_2 \underline{\otimes} \mathcal{G}'_2$$

$$=$$

$$\phi \times \phi' : \mathcal{G}_1 \times \mathcal{G}'_1 \to \mathcal{G}_2 \times \mathcal{G}'_2$$

is the obviously obtained functor such that:

- for all point G_1 of $\mathcal{G}_1$ and all point G'_1 of $\mathcal{G}'_1$:

$$(\phi \underline{\otimes} \phi')([G_1, G'_1]) = (\phi \times \phi')([G_1, G'_1]) = [\phi(G_1), \phi'(G'_1)]\,,$$

- for all arrow g_1 of $\mathcal{G}_1$ and all arrow g'_1 of $\mathcal{G}'_1$:

$$(\phi \underline{\otimes} \phi')([g_1, g'_1]) = (\phi \times \phi')([g_1, g'_1]) = [\phi(g_1), \phi'(g'_1)]\,.$$

4 The systems of compositive graphs

4.1 Universes

4.1.a. A *universe* $\mathbb{U}$ is a set (of sets) such that:

- the set $\mathbb{N}$ of natural numbers belongs to $\mathbb{U}$,
- if $\mathbb{X}_1$ and $\mathbb{X}_2$ are two sets belonging to $\mathbb{U}$, then the cartesian product $\mathbb{X}_1 \times \mathbb{X}_2$ belongs to $\mathbb{U}$,
- if $\mathbb{Y}$ is a set belonging to $\mathbb{U}$ and if $(\mathbb{X}_y)_{y\in\mathbb{Y}}$ is a family of sets belonging to $\mathbb{U}$, then the union $\cup_{y\in\mathbb{Y}}\mathbb{X}_y$ belongs to $\mathbb{U}$,
- if $\mathbb{X}$ is a set belonging to $\mathbb{U}$ and if $\mathbb{X}'$ is a subset of $\mathbb{X}$, then $\mathbb{X}'$ belongs to $\mathbb{U}$,
- if $\mathbb{X}$ is a set belonging to $\mathbb{U}$, then the set $\wp(\mathbb{X})$ of the subsets of $\mathbb{X}$ belongs to $\mathbb{U}$.

From here on, we reason in a model of the Zermelo-Fraenkel theory of sets with the axiom of choice and the axiom of universes ("there is at least one universe").

4.1.b. Let $\mathbb{U}$ be a universe. A set $\mathbb{X}$ is $\mathbb{U}$-*small* if:

- $\mathbb{X}$ belongs to $\mathbb{U}$.

The *category of* $\mathbb{U}$-*small sets*:

$$\mathcal{S}et_{\mathbb{U}}$$

is the canonically obtained category such that:

- its points are the $\mathbb{U}$-small sets,
- its arrows are the maps between $\mathbb{U}$-small sets.

4.2 The category of $\mathbb{U}$-small compositive graphs

4.2.a. Let $\mathbb{U}$ be a universe. A compositive graph $\mathcal{G}$ is *locally* $\mathbb{U}$-*small* if:

- for all points G_1 and G_2 of $\mathcal{G}$, the set $\mathrm{Hom}_{\mathcal{G}}(G_1, G_2)$ is $\mathbb{U}$-small.

From here on, we consider only *locally* $\mathbb{U}$-*small categories*, *i.e.* categories which are locally $\mathbb{U}$-small as compositive graphs. However, generally, we must take care of the fact that the category which is generated (as explained in § 5) by a locally $\mathbb{U}$-small compositive graph is *not necessarily* locally $\mathbb{U}$-small.

4.2.b. Let $\mathbb{U}$ be a universe. A compositive graph $\mathcal{G}$ is $\mathbb{U}$-*small* if:

- $\mathtt{Pt}(\mathcal{G})$ is a $\mathbb{U}$-small set,
- $\mathtt{Ar}(\mathcal{G})$ is a $\mathbb{U}$-small set,

so that, of course, the sets $\mathtt{IdAr}(\mathcal{G})$ and $\mathtt{CompP}(\mathcal{G})$ are also $\mathbb{U}$-small.

The *category of* $\mathbb{U}$-*small compositive graphs*:

$$\mathcal{C}omp_{\mathbb{U}}$$

is the canonically obtained category (which is clearly locally $\mathbb{U}$-small) such that:

- its points are the $\mathbb{U}$-small compositive graphs,
- its arrows are the functors between $\mathbb{U}$-small compositive graphs.

Similarly, the *category of* $\mathbb{U}$-*small categories* is the full subcategory:

$$\mathcal{C}at_{\mathbb{U}}$$

of $\mathcal{C}omp_{\mathbb{U}}$ such that its points are the $\mathbb{U}$-*small categories*, *i.e.* the categories which are $\mathbb{U}$-small as compositive graphs. The canonical injection functor is denoted:

$$\mathcal{C}at_{\mathbb{U}} \subseteq \mathcal{C}omp_{\mathbb{U}} : \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}} \ .$$

Finally, the *category of* $\mathbb{U}$-*small directed graphs* is the full subcategory:

$$\mathcal{D}ir_{\mathbb{U}}$$

of $\mathcal{C}omp_{\mathbb{U}}$ such that its points are the $\mathbb{U}$-*small directed graphs*, *i.e.* the directed graphs which are $\mathbb{U}$-small as compositive graphs. The canonical injection functor is denoted:

$$\mathcal{D}ir_{\mathbb{U}} \subseteq \mathcal{C}omp_{\mathbb{U}} : \mathcal{D}ir_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}} \ .$$

4.3 Tensor systems

4.3.a. Let $\mathcal{J}$ be a category. A $\mathcal{J}$-*tensor system* V is made up of:

- for all arrow j of $\mathcal{J}$, a category $\mathrm{Cpt}(\mathsf{V}, j)$ *component of* V *at* j (we may write $\mathsf{V}_j = \mathrm{Cpt}(\mathsf{V}, j)$ and, if $j = \mathrm{id}(J) : J \Longrightarrow J$ is an identity arrow, $\mathsf{V}_J = \mathsf{V}_{\mathrm{id}(J)}$),
- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive (hence composable) arrows of $\mathcal{J}$, a *tensor product at* (j, j') *functor*:

  $$\mathtt{tensfunc}(\mathsf{V}, (j, j')) : \mathsf{V}_j \times \mathsf{V}_{j'} \to \mathsf{V}_{j' \centerdot j}$$

  (we may write $- \otimes_{j,j'} - = \mathtt{tensfunc}(\mathsf{V}, (j, j'))$ and even $- \otimes_{\dots} - = - \otimes_{j,j'} -$).

Then, if $(j : J \to J', j' : J' \to J'', j'' : J'' \to J''')$ is a triple of consecutive (hence composable) arrows of $\mathcal{J}$, we say that:

- V is *left-to-right quasi-associative at* (j, j', j'') if there is a *left-to-right-quasi-associativity rewriting*, *i.e.* a natural transformation:

$$\mathtt{qass}_{j,j',j''} : (- \otimes_{j,j'} -) \otimes_{j'\centerdot j,j''} - \Rightarrow - \otimes_{j,j''\centerdot j'} (- \otimes_{j',j''} -) : \mathsf{V}_j \times \mathsf{V}_{j'} \times \mathsf{V}_{j''} \to \mathsf{V}_{j''\centerdot j'\centerdot j} ,$$

- V is *associative at* (j, j', j'') if there is an *associativity rewriting*, *i.e.* a natural equivalence:

$$\mathtt{ass}_{j,j',j''} : (- \otimes_{j,j'} -) \otimes_{j'\centerdot j,j''} - \Rightarrow - \otimes_{j,j''\centerdot j'} (- \otimes_{j',j''} -) : \mathsf{V}_j \times \mathsf{V}_{j'} \times \mathsf{V}_{j''} \to \mathsf{V}_{j''\centerdot j'\centerdot j} ,$$

the definition of *right-to-left quasi-associativity*, which will not be used, is left to the reader.

Similarly, if $j : J \to J'$ is an arrow of $\mathcal{J}$, we say that:

- V *has the point* V *of* V_J *as a quasi-unit at the domain of* j if there is a *quasi-unitarity-at-the-domain rewriting*, *i.e.* a natural transformation:

$$\mathtt{qunit}_{V,j} : \mathrm{id}(\mathsf{V}_j) \Rightarrow (V \otimes_{\mathrm{id}(J),j} -) : \mathsf{V}_j \to \mathsf{V}_j ,$$

- V *has the point* V *of* V_J *as a unit at the domain of* j if there is a *unitarity-at-the-domain rewriting*, *i.e.* a natural equivalence:

$$\mathtt{unit}_{V,j} : \mathrm{id}(\mathsf{V}_j) \Rightarrow (V \otimes_{\mathrm{id}(J),j} -) : \mathsf{V}_j \to \mathsf{V}_j ,$$

- V *has the point* V' *of* V_J *as a quasi-unit at the codomain of* j if there is a *quasi-unitarity-at-the-codomain rewriting*, *i.e.* a natural transformation:

$$\mathtt{qunit}_{j,V'} : \mathrm{id}(\mathsf{V}_j) \Rightarrow (- \otimes_{j,\mathrm{id}(J')} V') : \mathsf{V}_j \to \mathsf{V}_j ,$$

- V *has the point* V' *of* V_J *as a unit at the codomain of* j if there is a *unitarity-at-the-codomain rewriting*, *i.e.* a natural equivalence:

$$\mathtt{unit}_{j,V'} : \mathrm{id}(\mathsf{V}_j) \Rightarrow (- \otimes_{j,\mathrm{id}(J')} V') : \mathsf{V}_j \to \mathsf{V}_j .$$

In a more global way, we say that:

- V is *left-to-right quasi-associative* if:
  - V is left-to-right quasi-associative at each triple (j, j', j'') of consecutive arrows of $\mathcal{J}$,
  - all the left-to-right-quasi-associativity rewriting diagrams are commutative (then we say that these various rewritings are *coherent*),
- V is *associative* if:
  - V is associative at each triple (j, j', j'') of consecutive arrows of $\mathcal{J}$,
  - all the associativity rewriting diagrams are commutative.

Similarly, we say that:

- V is *quasi-unitary* and its *family of units* is $(U_J(\mathsf{V}))_{J\in\mathtt{Pt}(\mathcal{J})}$ if:
  - for all point J of $\mathcal{J}$, the element $U_J(\mathsf{V})$ is a point of V_J,
  - V has $U_J(\mathsf{V})$ as a quasi-unit at the domain of each arrow of $\mathcal{J}$ with domain J,

  - V has $U_J(\mathsf{V})$ as a quasi-unit at the codomain of each arrow of $\mathcal{J}$ with codomain J,
  - all the quasi-unitarity rewriting diagrams are commutative
    (then we say that these various rewritings are *coherent*),

- V is *unitary* and its *family of units* is $(U_J(\mathsf{V}))_{J\in\mathtt{Pt}(\mathcal{J})}$ if:
  - for all point J of $\mathcal{J}$, the element $U_J(\mathsf{V})$ is a point of V_J,
  - V has $U_J(\mathsf{V})$ as a unit at the domain of each arrow of $\mathcal{J}$ with domain J,
  - V has $U_J(\mathsf{V})$ as a unit at the codomain of each arrow of $\mathcal{J}$ with codomain J,
  - all the unitarity rewriting diagrams are commutative.

Finally, we say that:

- V is *(left-to-right) quasi-categorical* if:
  - V is left-to-right quasi-associative,
  - V is quasi-unitary,
  - all the diagrams made up of left-to-right-quasi-associativity rewritings and quasi--unitarity rewritings (possibly mixed together) are commutative
    (then we say that these various rewritings are *coherent*),
- V is *categorical* if:
  - V is associative,
  - V is unitary,
  - all the diagrams made up of associativity rewritings and unitarity rewritings (possibly mixed together) are commutative.

4.3.b. Let $\mathcal{J}$ be a category , V and W two $\mathcal{J}$-tensor systems. A *$\mathcal{J}$-tensor morphism* $\Omega : \mathsf{V} \to \mathsf{W}$ *from* V *to* W is made up of:

- for all arrow j of $\mathcal{J}$, a functor $\mathrm{Cpt}(\Omega, j) : \mathsf{V}_j \to \mathsf{W}_j$ *component of* Ω *at* j
  (we may write $\Omega_j(-) = \Omega_j = \mathrm{Cpt}(\Omega, j)$ and, if $j = \mathrm{id}(J) : J \Longrightarrow J$ is an identity arrow, $\Omega_J(-) = \Omega_J = \Omega_{\mathrm{id}(J)}$),
- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive (hence composable) arrows of $\mathcal{J}$, a *tensor rewriting at* (j, j'), *i.e.* a natural transformation:

$$\mathtt{tens}_{j,j'} : \Omega_j(-) \otimes_{j,j'} \Omega_{j'}(-) \Rightarrow \Omega_{j'\centerdot j}(- \otimes_{j,j'} -) : \mathsf{V}_j \times \mathsf{V}_{j'} \to \mathsf{W}_{j'\centerdot j}\ .$$

Now, assume that V and W are left-to-right quasi-associative. Then, we say that:

- $\Omega : \mathsf{V} \to \mathsf{W}$ is *left-to-right quasi-associative* if:
  - all the diagrams made up of tensor rewritings and left-to-right-quasi-associativity rewritings (possibly mixed together) are commutative
    (then we say that these various rewritings are *coherent*).

If V and W are associative, a left-to-right quasi-associative $\mathcal{J}$-tensor morphism from V to W is called *associative*.

Similarly, assume that V and W are quasi-unitary. Then, we say that:

- $\Omega : \mathsf{V} \to \mathsf{W}$ is *quasi-unitary* with the *family of quasi-unitarity-tensor rewritings*:

$$(\,\mathtt{qunittens}_J : U_J(\mathsf{W}) \to \Omega_J(U_J(\mathsf{V}))\,)_{J \in \mathtt{Pt}(\mathcal{J})}$$

  if:

  - for all point J of $\mathcal{J}$, the arrow:

$$\mathtt{qunittens}_J : U_J(\mathsf{W}) \to \Omega_J(U_J(\mathsf{V}))$$

    is an arrow of W_J,
  - all the diagrams made up of tensor rewritings, quasi-unitarity-tensor rewritings and quasi-unitarity rewritings (possibly mixed together) are commutative
    (then we say that these various rewritings are *coherent*).

If V and W are unitary, a quasi-unitary $\mathcal{J}$-tensor morphism from V to W is called *unitary*; its quasi-unitarity-tensor rewriting at each point J of $\mathcal{J}$ is called a *unitarity-tensor rewriting*, and we note:

$$\mathtt{unittens}_J = \mathtt{qunittens}_J : U_J(\mathsf{W}) \to \Omega_J(U_J(\mathsf{V}))\ .$$

Finally, assume that V and W are quasi-categorical. Then:

- $\Omega : \mathsf{V} \to \mathsf{W}$ is *quasi-categorical* if:
  - $\Omega : \mathsf{V} \to \mathsf{W}$ is left-to-right quasi-associative,
  - $\Omega : \mathsf{V} \to \mathsf{W}$ is quasi-unitary,
  - all the diagrams made up of tensor rewritings, quasi-unitarity-tensor rewritings, left-to-right-quasi-associativity rewritings and quasi-unitarity rewritings (possibly mixed together) are commutative
    (then we say that these various rewritings are *coherent*).

If V and W are categorical, a quasi-categorical $\mathcal{J}$-tensor morphism from V to W is called *categorical*.

4.3.c. Let $\mathcal{J}$ and $\mathcal{K}$ be two categories, $\theta : \mathcal{J} \to \mathcal{K}$ a functor and W a $\mathcal{K}$-tensor system. The *tensor system deduced from* W *by change of indexation*:

$$\mathsf{W}_\theta$$

is the obviously obtained $\mathcal{J}$-tensor system such that:

- for all arrow j of $\mathcal{J}$:

$$(\mathsf{W}_\theta)_j = \mathsf{W}_{\theta(j)}\ ,$$

- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive arrows of $\mathcal{J}$:

$$\mathtt{tensfunc}(\mathsf{W}_\theta, (j, j')) : (\mathsf{W}_\theta)_j \times (\mathsf{W}_\theta)_{j'} \to (\mathsf{W}_\theta)_{j'.j}$$

$$=$$

$$\mathtt{tensfunc}(\mathsf{W}, (\theta(j), \theta(j'))) : \mathsf{W}_{\theta(j)} \times \mathsf{W}_{\theta(j')} \to \mathsf{W}_{\theta(j').\theta(j)} .$$

Then, it is clear that, if W is a left-to-right quasi-associative (*resp.* associative) $\mathcal{K}$-tensor system, then W_θ is a left-to-right quasi-associative (*resp.* associative) $\mathcal{J}$-tensor system.

Similarly, if W is a quasi-unitary (*resp.* unitary) $\mathcal{K}$-tensor system, then W_θ is a quasi-unitary (*resp.* unitary) $\mathcal{J}$-tensor system.

It follows that, if W is a quasi-categorical (*resp.* categorical) $\mathcal{K}$-tensor system, then W_θ is a quasi-categorical (*resp.* categorical) $\mathcal{J}$-tensor system.

4.4 The tensor system of $\mathbb{U}$-small compositive graphs

4.4.a. From now on:

- $\mathit{1}_\emptyset$ is the directed graph with one point $\mathit{0}$ and no arrow,
- $\mathit{1}$ is the category with one point $\mathit{0}$ and one arrow $\mathrm{id}(\mathit{0})$,
- $\mathit{2}$ is the category with two points $\mathit{0}$ and $\mathit{1}$ and one non-identity arrow $(\mathit{0}, \mathit{1}) : \mathit{0} \to \mathit{1}$,
- $\mathit{1} \subseteq \mathit{2}$ is the canonical injection functor (which maps $\mathit{0}$ to $\mathit{0}$),
- $\mathit{1}$ "$\subseteq$" $\mathit{2}$ is the ("quasi-canonical") injection functor (which maps $\mathit{0}$ to $\mathit{1}$),
- $\mathit{2} \,|\, \mathit{1} : \mathit{2} \to \mathit{1}$ is the unique functor from $\mathit{2}$ to $\mathit{1}$.

4.4.b. Let $\mathbb{U}$ be a universe. The *hollow tensor product (of* $\mathbb{U}$*-small compositive graphs) functor*:

$$- \square_{\mathbb{U}} - : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$$

is the obviously obtained functor such that:

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:

$$(- \square_{\mathbb{U}} -)(\mathcal{G}, \mathcal{G}') = \mathcal{G} \square \mathcal{G}' ,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}'_1$ and $\mathcal{G}'_2$ and all functors $\phi : \mathcal{G}_1 \to \mathcal{G}_2$ and $\phi' : \mathcal{G}'_1 \to \mathcal{G}'_2$:

$$(- \square_{\mathbb{U}} -)(\phi, \phi') = \phi \square \phi' .$$

The functor $- \square_{\mathbb{U}} -$ can be restricted to:

- the *hollow tensor product (of* $\mathbb{U}$*-small compositive graphs with* $\mathbb{U}$*-small directed graphs) functor*:

$$- \square|_{\mathbb{U}} - : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}} ,$$

- the *hollow tensor product (of* $\mathbb{U}$*-small directed graphs) functor*:

$$-\,|\underline{\square}|_{\,\mathbb{U}}-\,:\mathcal{D}ir_{\mathbb{U}}\times\mathcal{D}ir_{\mathbb{U}}\to\mathcal{D}ir_{\mathbb{U}}\ .$$

Similarly, the *full tensor product (of* $\mathbb{U}$*-small compositive graphs) functor*:

$$-\,\underline{\boxtimes}_{\,\mathbb{U}}-\,:\mathcal{C}omp_{\mathbb{U}}\times\mathcal{C}omp_{\mathbb{U}}\to\mathcal{C}omp_{\mathbb{U}}$$

is the obviously obtained functor such that:

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:

$$(-\,\underline{\boxtimes}_{\,\mathbb{U}}-)(\mathcal{G},\mathcal{G}')=\mathcal{G}\,\underline{\boxtimes}\,\mathcal{G}'\,,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}_1$, $\mathcal{G}_2$, $\mathcal{G}_1'$ and $\mathcal{G}_2'$ and all functors $\phi:\mathcal{G}_1\to\mathcal{G}_2$ and $\phi':\mathcal{G}_1'\to\mathcal{G}_2'$:

$$(-\,\underline{\boxtimes}_{\,\mathbb{U}}-)(\phi,\phi')=\phi\,\underline{\boxtimes}\,\phi'\ .$$

The functor $-\,\underline{\boxtimes}_{\,\mathbb{U}}-$ can be restricted to:

- the *full tensor product (of* $\mathbb{U}$*-small compositive graphs with* $\mathbb{U}$*-small categories) functor*:

$$-\,\underline{\boxtimes}|_{\,\mathbb{U}}-\,:\mathcal{C}omp_{\mathbb{U}}\times\mathcal{C}at_{\mathbb{U}}\to\mathcal{C}omp_{\mathbb{U}}\ .$$

Finally, the *cartesian tensor product (of* $\mathbb{U}$*-small categories) functor*:

$$-\,\underline{\otimes}_{\,\mathbb{U}}-\,:\mathcal{C}at_{\mathbb{U}}\times\mathcal{C}at_{\mathbb{U}}\to\mathcal{C}at_{\mathbb{U}}$$

is the obviously obtained functor such that:

- for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

$$(-\,\underline{\otimes}_{\,\mathbb{U}}-)(\mathcal{A},\mathcal{A}')=\mathcal{A}\,\underline{\otimes}\,\mathcal{A}'\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_1'$ and $\mathcal{A}_2'$ and all functors $\phi:\mathcal{A}_1\to\mathcal{A}_2$ and $\phi':\mathcal{A}_1'\to\mathcal{A}_2'$:

$$(-\,\underline{\otimes}_{\,\mathbb{U}}-)(\phi,\phi')=\phi\,\underline{\otimes}\,\phi'\ .$$

4.4.c. Let $\mathbb{U}$ be a universe. The *tensor system of* $\mathbb{U}$*-small compositive graphs*:

$$\mathsf{Comp}_{\mathbb{U}}$$

is the obviously obtained categorical ***1***-tensor system such that:

- $(\mathsf{Comp}_{\mathbb{U}})_{\mathit{0}}=\mathcal{C}omp_{\mathbb{U}}\,,$
- $-\otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})}-\;=\;-\,\underline{\square}_{\,\mathbb{U}}-\,:\mathcal{C}omp_{\mathbb{U}}\times\mathcal{C}omp_{\mathbb{U}}\to\mathcal{C}omp_{\mathbb{U}}\ .$

Precisely, it is easy to check that:

- $\mathsf{Comp}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0}))\,,$

- $\mathsf{Comp}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a unit at the domain of $\mathrm{id}(\mathit{0})$,
- $\mathsf{Comp}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a unit at the codomain of $\mathrm{id}(\mathit{0})$.

Similarly, the *tensor system of* $\mathbb{U}$*-small categories*:

$$\mathsf{Cat}_{\mathbb{U}}$$

is the obviously obtained categorical $\boldsymbol{1}$-tensor system such that:

- $(\mathsf{Cat}_{\mathbb{U}})_{\mathit{0}} = \mathcal{C}at_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} - = - \underline{\otimes}_{\mathbb{U}} - : \mathcal{C}at_{\mathbb{U}} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}$.

Precisely, it is easy to check that:

- $\mathsf{Cat}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}))$,
- $\mathsf{Cat}_{\mathbb{U}}$ has $\boldsymbol{1}$ as a unit at the domain of $\mathrm{id}(\mathit{0})$,
- $\mathsf{Cat}_{\mathbb{U}}$ has $\boldsymbol{1}$ as a unit at the codomain of $\mathrm{id}(\mathit{0})$.

Obviously, there is a categorical $\boldsymbol{1}$-tensor morphism of *canonical injection*:

$$\mathsf{Cat}_{\mathbb{U}} \subseteq \mathsf{Comp}_{\mathbb{U}} : \mathsf{Cat}_{\mathbb{U}} \to \mathsf{Comp}_{\mathbb{U}} \ .$$

Finally, the *tensor system of* $\mathbb{U}$*-small directed graphs*:

$$\mathsf{Dir}_{\mathbb{U}}$$

is the obviously obtained categorical $\boldsymbol{1}$-tensor system such that:

- $(\mathsf{Dir}_{\mathbb{U}})_{\mathit{0}} = \mathcal{D}ir_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} - = - |\underline{\square}|_{\mathbb{U}} - : \mathcal{D}ir_{\mathbb{U}} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{D}ir_{\mathbb{U}}$.

Precisely, it is easy to check that:

- $\mathsf{Dir}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}))$,
- $\mathsf{Dir}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a domain unit of $\mathrm{id}(\mathit{0})$,
- $\mathsf{Dir}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a codomain unit of $\mathrm{id}(\mathit{0})$.

Obviously, there is a categorical $\boldsymbol{1}$-tensor morphism of *canonical injection*:

$$\mathsf{Dir}_{\mathbb{U}} \subseteq \mathsf{Comp}_{\mathbb{U}} : \mathsf{Dir}_{\mathbb{U}} \to \mathsf{Comp}_{\mathbb{U}} \ .$$

4.4.d. Let $\mathbb{U}$ be a universe. The *tensor system of $\mathbb{U}$-small compositive graphs and $\mathbb{U}$-small categories*:

$$\mathsf{CompCat}_{\mathbb{U}}$$

is the obviously obtained quasi-categorical $\boldsymbol{2}$-tensor system such that:

- $(\mathsf{CompCat}_{\mathbb{U}})_{0} = \mathcal{C}omp_{\mathbb{U}}$,
- $(\mathsf{CompCat}_{\mathbb{U}})_{1} = \mathcal{C}at_{\mathbb{U}}$,
- $(\mathsf{CompCat}_{\mathbb{U}})_{(0,1)} = \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(0),\mathrm{id}(0)} - = -\underline{\Box}_{\mathbb{U}}- : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(0),(0,1)} - = -\underline{\boxtimes}_{\mathbb{U}}- : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{(0,1),\mathrm{id}(1)} - = -\underline{\boxtimes}|_{\mathbb{U}}- : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(1),\mathrm{id}(1)} - = -\underline{\otimes}_{\mathbb{U}}- : \mathcal{C}at_{\mathbb{U}} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}$.

Precisely, it is easy to check that:

- $\mathsf{CompCat}_{\mathbb{U}}$ is associative at $(\mathrm{id}(0), \mathrm{id}(0), \mathrm{id}(0))$,
- $\mathsf{CompCat}_{\mathbb{U}}$ is left-to-right quasi-associative at $(\mathrm{id}(0), \mathrm{id}(0), (0,1))$,
- $\mathsf{CompCat}_{\mathbb{U}}$ is associative at $(\mathrm{id}(0), (0,1), \mathrm{id}(1))$,
- $\mathsf{CompCat}_{\mathbb{U}}$ is left-to-right quasi-associative at $((0,1), \mathrm{id}(1), \mathrm{id}(1))$ (this needs "a sufficient number of identity arrows" of factor categories of the involved tensor products),
- $\mathsf{CompCat}_{\mathbb{U}}$ is associative at $(\mathrm{id}(1), \mathrm{id}(1), \mathrm{id}(1))$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a unit at the domain of $\mathrm{id}(0)$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a unit at the codomain of $\mathrm{id}(0)$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}_{\emptyset}$ as a unit at the domain of $(0,1)$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}$ as a unit at the codomain of $(0,1)$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}$ as a unit at the domain of $\mathrm{id}(1)$,
- $\mathsf{CompCat}_{\mathbb{U}}$ has $\boldsymbol{1}$ as a unit at the codomain of $\mathrm{id}(1)$.

Obviously, up to a change of indexation, there is a quasi-categorical $\boldsymbol{2}$-tensor morphism of *canonical injection*:

$$\mathsf{CompCat}_{\mathbb{U}} \subseteq \mathsf{Comp}_{\mathbb{U}} : \mathsf{CompCat}_{\mathbb{U}} \to (\mathsf{Comp}_{\mathbb{U}})_{\boldsymbol{2}|\boldsymbol{1}} \ .$$

In addition, up to other changes of indexation:

$$(\mathsf{CompCat}_{\mathbb{U}})_{\boldsymbol{1} \subseteq \boldsymbol{2}} = \mathsf{Comp}_{\mathbb{U}}$$

and:

$$(\mathsf{CompCat}_{\mathbb{U}})_{\boldsymbol{1} \text{``}\subseteq\text{''} \boldsymbol{2}} = \mathsf{Cat}_{\mathbb{U}} \ .$$

Similarly, the *tensor system of* $\mathbb{U}$*-small compositive graphs and* $\mathbb{U}$*-small directed graphs*:

$$\mathsf{CompDir}_{\mathbb{U}}$$

is the obviously obtained categorical $\mathit{2}$-tensor system such that:

- $(\mathsf{CompDir}_{\mathbb{U}})_{\mathit{0}} = \mathcal{C}omp_{\mathbb{U}}$,
- $(\mathsf{CompDir}_{\mathbb{U}})_{\mathit{1}} = \mathcal{D}ir_{\mathbb{U}}$,
- $(\mathsf{CompDir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})} = \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} - = - \underline{\square}_{\mathbb{U}} - : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(\mathit{0}),(\mathit{0},\mathit{1})} - = - \underline{\square}_{\mathbb{U}} - : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{(\mathit{0},\mathit{1}),\mathrm{id}(\mathit{1})} - = - \underline{\square}|_{\mathbb{U}} - : \mathcal{C}omp_{\mathbb{U}} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$,
- $- \otimes_{\mathrm{id}(\mathit{1}),\mathrm{id}(\mathit{1})} - = - |\underline{\square}|_{\mathbb{U}} - : \mathcal{D}ir_{\mathbb{U}} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{D}ir_{\mathbb{U}}$.

Precisely, it is easy to check that:

- $\mathsf{CompDir}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}))$,
- $\mathsf{CompDir}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}), \mathrm{id}(\mathit{0}), (\mathit{0}, \mathit{1}))$,
- $\mathsf{CompDir}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{0}), (\mathit{0}, \mathit{1}), \mathrm{id}(\mathit{1}))$,
- $\mathsf{CompDir}_{\mathbb{U}}$ is associative at $((\mathit{0}, \mathit{1}), \mathrm{id}(\mathit{1}), \mathrm{id}(\mathit{1}))$,
- $\mathsf{CompDir}_{\mathbb{U}}$ is associative at $(\mathrm{id}(\mathit{1}), \mathrm{id}(\mathit{1}), \mathrm{id}(\mathit{1}))$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the domain of $\mathrm{id}(\mathit{0})$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the codomain of $\mathrm{id}(\mathit{0})$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the domain of $(\mathit{0}, \mathit{1})$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the codomain of $(\mathit{0}, \mathit{1})$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the domain of $\mathrm{id}(\mathit{1})$,
- $\mathsf{CompDir}_{\mathbb{U}}$ has $\mathbf{\mathit{1}}_{\emptyset}$ as a unit at the codomain of $\mathrm{id}(\mathit{1})$.

Obviously, up to a change of indexation, there is a categorical $\mathit{2}$-tensor morphism of *canonical injection*:

$$\mathsf{CompDir}_{\mathbb{U}} \subseteq \mathsf{Comp}_{\mathbb{U}} : \mathsf{CompDir}_{\mathbb{U}} \to (\mathsf{Comp}_{\mathbb{U}})_{\mathbf{\mathit{2}}|\mathbf{\mathit{1}}} \ .$$

In addition, up to other changes of indexation:

$$(\mathsf{CompDir}_{\mathbb{U}})_{\mathbf{\mathit{1}} \subseteq \mathbf{\mathit{2}}} = \mathsf{Comp}_{\mathbb{U}}$$

and:

$$(\mathsf{CompDir}_{\mathbb{U}})_{\mathbf{\mathit{1}} \text{``}\subseteq\text{''} \mathbf{\mathit{2}}} = \mathsf{Dir}_{\mathbb{U}} \ .$$

4.5 Enriched systems

4.5.a. Let $\mathcal{J}$ be a category and V a $\mathcal{J}$-tensor system. A V-*enriched* $\mathcal{J}$*-system* $\mathbb{A}$ is made up of:

- for all point J of $\mathcal{J}$, a set $\mathrm{Cpt}(\mathbb{A}, J)$ *component of* $\mathbb{A}$ *at* J (we may write $\mathbb{A}_J = \mathrm{Cpt}(\mathbb{A}, J)$),
- for all arrow $j : J \to J'$ of $\mathcal{J}$, a map *exponentiation at* j:

$$\exp(\mathbb{A}, j) : \mathbb{A}_J \times \mathbb{A}_{J'} \to \mathtt{Pt}(\mathsf{V}_j)\,,$$

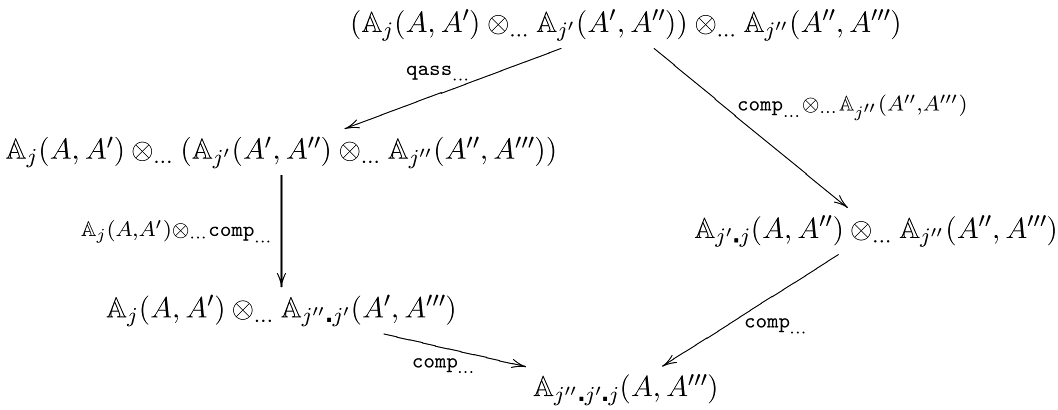
  (we may write $\mathbb{A}_j(-,-) = \exp(\mathbb{A}, j)$, but we must be careful not to confuse the *map* $\mathbb{A}_{\mathrm{id}(J)}(-,-) : \mathbb{A}_J \times \mathbb{A}_J \to \mathtt{Pt}(\mathsf{V}_J)$ with the *set* $\mathbb{A}_J$),
- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive (hence composable) arrows of $\mathcal{J}$, all element A of $\mathbb{A}_J$, all element A' of $\mathbb{A}_{J'}$ and all element A'' of $\mathbb{A}_{J''}$, a *composition at* (A, j, A', j', A''), *i.e.* an arrow of $\mathsf{V}_{j'\centerdot j}$:

$$\mathtt{comp}(\mathbb{A}, (A, j, A', j', A'')) : \mathbb{A}_j(A, A') \otimes_{j,j'} \mathbb{A}_{j'}(A', A'') \to \mathbb{A}_{j'\centerdot j}(A, A'')$$

  (we may write $\mathtt{comp}_{A,j,A',j',A''} = \mathtt{comp}(\mathbb{A}, (A, j, A', j', A''))$).

Now, assume that V is left-to-right quasi-associative. Then, we say that:

- $\mathbb{A}$ is *(strictly) associative (parallel to* V*)* if, for all triple $(j : J \to J', j' : J' \to J'', j'' : J'' \to J''')$ of consecutive arrows of $\mathcal{J}$, all element A of $\mathbb{A}_J$, all element A' of $\mathbb{A}_{J'}$, all element A'' of $\mathbb{A}_{J''}$, and all element A''' of $\mathbb{A}_{J'''}$, the following diagram (of $\mathsf{V}_{j''\centerdot j'\centerdot j}$) is commutative:



Similarly, assume that V is quasi-unitary. Then, we say that:

- $\mathbb{A}$ is *(strictly) unitary (parallel to* V*)* and its *family of units* is:

$$(\, u_{J,A}(\mathbb{A}) : U_J(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J)}(A, A)\,)_{J \in \mathtt{Pt}(\mathcal{J}), A \in \mathbb{A}_J}$$

  if:

  - for all point J of $\mathcal{J}$ and all element A of $\mathbb{A}_J$, the arrow:

$$u_{J,A}(\mathbb{A}) : U_J(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J)}(A, A)$$

    is an arrow of V_J ,

- for all arrow $j : J \to J'$ (with domain J) of $\mathcal{J}$ and all element A' of $\mathbb{A}_{J'}$, the following diagram (of V_j) is commutative:

$$\begin{array}{ccc} \mathbb{A}_j(A, A') & \xrightarrow{\texttt{qunit}_{\dots}} & U_J(\mathsf{V}) \otimes_{\dots} \mathbb{A}_j(A, A') \\ {\scriptstyle \mathrm{id}(\mathbb{A}_j(A,A'))}\downarrow & & \downarrow{\scriptstyle u_{J,A}(\mathbb{A}) \otimes_{\dots} \mathbb{A}_j(A,A')} \\ \mathbb{A}_j(A, A') & \xleftarrow[\texttt{comp}_{\dots}]{} & \mathbb{A}_{\mathrm{id}(J)}(A, A) \otimes_{\dots} \mathbb{A}_j(A, A') \end{array}$$

- for all arrow $j : J' \to J$ (with codomain J) of $\mathcal{J}$ and all element A' of $\mathbb{A}_{J'}$, the following diagram (of V_j) is commutative:

$$\begin{array}{ccc} \mathbb{A}_j(A', A) & \xrightarrow{\texttt{qunit}_{\dots}} & \mathbb{A}_j(A', A) \otimes_{\dots} U_J(\mathsf{V}) \\ {\scriptstyle \mathrm{id}(\mathbb{A}_j(A',A))}\downarrow & & \downarrow{\scriptstyle \mathbb{A}_j(A',A) \otimes_{\dots} u_{J,A}(\mathbb{A})} \\ \mathbb{A}_j(A', A) & \xleftarrow[\texttt{comp}_{\dots}]{} & \mathbb{A}_j(A', A) \otimes_{\dots} \mathbb{A}_{\mathrm{id}(J)}(A, A) \end{array}$$

Finally, assume that V is quasi-categorical (a fortiori, categorical). Then, we say that:

- $\mathbb{A}$ is *(strictly) categorical (parallel to* V*)* if:
  - $\mathbb{A}$ is (strictly) associative,
  - $\mathbb{A}$ is (strictly) unitary.

4.5.b. Let $\mathcal{J}$ be a category, V a $\mathcal{J}$-tensor system, $\mathbb{A}$ and $\mathbb{B}$ two V-enriched $\mathcal{J}$-systems. A V-*enriched* $\mathcal{J}$*-morphism* $\Gamma : \mathbb{A} \to \mathbb{B}$ is made up of:

- for all point J of $\mathcal{J}$, a map $\Gamma_J : \mathbb{A}_J \to \mathbb{B}_J$,
- for all arrow $j : J \to J'$ of $\mathcal{J}$, all element A of $\mathbb{A}_J$ and all element A' of $\mathbb{A}_{J'}$, an arrow of V_j :
$$\Gamma_j(A, A') : \mathbb{A}_j(A, A') \to \mathbb{B}_j(\Gamma_J(A), \Gamma_{J'}(A'))$$
(if $J = J'$ and $j = \mathrm{id}(J)$, we must be careful not to confuse the *arrow of* V_j :
$$\Gamma_{\mathrm{id}(J)}(A, A') : \mathbb{A}_{\mathrm{id}(J)}(A, A') \to \mathbb{B}_{\mathrm{id}(J)}(\Gamma_J(A), \Gamma_J(A'))$$
with the *map* $\Gamma_J : \mathbb{A}_J \to \mathbb{B}_J$),

with the following property:

- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive arrows of $\mathcal{J}$, all element A of $\mathbb{A}_J$, all element A' of $\mathbb{A}_{J'}$ and all element A'' of $\mathbb{A}_{J''}$, the following diagram (of $\mathsf{V}_{j'.j}$) is commutative:

$$
\begin{array}{ccc}
\mathbb{A}_j(A, A') \otimes_{\dots} \mathbb{A}_{j'}(A', A'') & \xrightarrow{\texttt{comp}_{\dots}} & \mathbb{A}_{j' \centerdot j}(A, A'') \\
\Big\downarrow {\scriptstyle \Gamma_j(A,A') \otimes_{\dots} \Gamma_{j'}(A',A'')} & & \Big\downarrow {\scriptstyle \Gamma_{j' \centerdot j}(A,A'')} \\
\mathbb{B}_j(\Gamma_J(A), \Gamma_{J'}(A')) \otimes_{\dots} \mathbb{B}_{j'}(\Gamma_{J'}(A'), \Gamma_{J''}(A'')) & \xrightarrow{\texttt{comp}_{\dots}} & \mathbb{B}_{j' \centerdot j}(\Gamma_J(A), \Gamma_{J''}(A''))
\end{array}
$$

Now, assume that $\mathbb{A}$ and $\mathbb{B}$ are unitary. Then, we say that:

- $\Gamma : \mathbb{A} \to \mathbb{B}$ is *unitary* if:
  - for all point J of $\mathcal{J}$ and all element A of $\mathbb{A}_J$, the following diagram (of V_j) is commutative:

$$
\begin{array}{ccc}
 & & \mathbb{A}_{\mathrm{id}(J)}(A, A) \\
 & \nearrow {\scriptstyle u_{J,A}(\mathbb{A})} & \\
U_J(\mathsf{V}) & & \Big\downarrow {\scriptstyle \Gamma_{\mathrm{id}(J)}(A,A)} \\
 & \searrow {\scriptstyle u_{J,\Gamma_J(A)}(\mathbb{B})} & \\
 & & \mathbb{B}_J(\Gamma_J(A), \Gamma_J(A))
\end{array}
$$

Finally, assume that $\mathbb{A}$ and $\mathbb{B}$ are categorical. Then, we say that:

- $\Gamma : \mathbb{A} \to \mathbb{B}$ is *categorical* if:
  - $\Gamma : \mathbb{A} \to \mathbb{B}$ is unitary,

(indeed, it is reasonable to consider that a morphism of associative enriched systems is "necessarily associative").

4.5.c. Let $\mathcal{J}$ be a category, V and W two $\mathcal{J}$-tensor systems, $\Omega : \mathsf{V} \to \mathsf{W}$ a $\mathcal{J}$-tensor morphism and $\mathbb{A}$ a V-enriched $\mathcal{J}$-system. The *enriched system deduced from* $\mathbb{A}$ *by change of enrichment along* Ω:

$$_{\Omega}\mathbb{A}$$

is the obviously obtained W-enriched $\mathcal{J}$-system such that:

- for all point J of $\mathcal{J}$:
$$({}_{\Omega}\mathbb{A})_J = \mathbb{A}_J \,,$$
- for all arrow $j : J \to J'$ of $\mathcal{J}$, all element A of $({}_{\Omega}\mathbb{A})_J$ (*i.e.* of $\mathbb{A}_J$) and all element A' of $({}_{\Omega}\mathbb{A})_{J'}$ (*i.e.* of $\mathbb{A}_{J'}$):
$$({}_{\Omega}\mathbb{A})_j(A, A') = \Omega_j(\mathbb{A}_j(A, A')) \,,$$

- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive arrows of $\mathcal{J}$, all element A of $({}_\Omega\mathbb{A})_J$ (*i.e.* of $\mathbb{A}_J$), all element A' of $({}_\Omega\mathbb{A})_{J'}$ (*i.e.* of $\mathbb{A}_{J'}$) and all element A'' of $({}_\Omega\mathbb{A})_{J''}$ (*i.e.* of $\mathbb{A}_{J''}$), the arrow of $\mathsf{W}_{j'\centerdot j}$:

$$\texttt{comp}({}_\Omega\mathbb{A}, (A, j, A', j', A'')) : ({}_\Omega\mathbb{A})_j(A, A') \otimes_{j,j'} ({}_\Omega\mathbb{A})_{j'}(A', A'') \to ({}_\Omega\mathbb{A})_{j'\centerdot j}(A, A'')$$

$$=$$

$$\texttt{comp}({}_\Omega\mathbb{A}, (A, j, A', j', A'')) : \Omega_j(\mathbb{A}_j(A, A')) \otimes_{j,j'} \Omega_{j'}(\mathbb{A}_{j'}(A', A'')) \to \Omega_{j'\centerdot j}(\mathbb{A}_{j'\centerdot j}(A, A''))$$

  is the following composed arrow of $\mathsf{W}_{j'\centerdot j}$:

$$\begin{array}{c} \Omega_j(\mathbb{A}_j(A, A')) \otimes_{\ldots} \Omega_{j'}(\mathbb{A}_{j'}(A', A'')) \\ \downarrow \texttt{tens}_{\ldots} \\ \Omega_{j'\centerdot j}(\mathbb{A}_j(A, A') \otimes_{\ldots} \mathbb{A}_{j'}(A', A'')) \\ \downarrow \Omega_{j'\centerdot j}(\texttt{comp}_{\ldots}) \\ \Omega_{j'\centerdot j}(\mathbb{A}_{j'\centerdot j}(A, A'')) \end{array}$$

Now assume that V and W are left-to-right quasi-associative $\mathcal{J}$-tensor systems, that $\Omega : \mathsf{V} \to \mathsf{W}$ is a left-to-right quasi-associative $\mathcal{J}$-tensor morphism and that $\mathbb{A}$ is an associative V-enriched $\mathcal{J}$-system. Then it is clear that ${}_\Omega\mathbb{A}$ is an associative W-enriched $\mathcal{J}$-system.

Similarly, assume that V and W are quasi-unitary $\mathcal{J}$-tensor systems, that $\Omega : \mathsf{V} \to \mathsf{W}$ is a quasi-unitary $\mathcal{J}$-tensor morphism and that $\mathbb{A}$ is a unitary V-enriched $\mathcal{J}$-system with family of units:

$$(\, u_{J,A}(\mathbb{A}) : U_J(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J)}(A, A)\,)_{J \in \mathtt{Pt}(\mathcal{J}), A \in \mathbb{A}_J} \ .$$

Then it is clear that ${}_\Omega\mathbb{A}$ is a unitary W-enriched $\mathcal{J}$-system with family of units:

$$(\, u_{J,A}({}_\Omega\mathbb{A}) : U_J(\mathsf{W}) \to ({}_\Omega\mathbb{A})_{\mathrm{id}(J)}(A, A)\,)_{J \in \mathtt{Pt}(\mathcal{J}), A \in ({}_\Omega\mathbb{A})_J} \ ,$$

where:

- for all point J of $\mathcal{J}$ and all element A of $({}_\Omega\mathbb{A})_J$ (*i.e.* of $\mathbb{A}_J$), the arrow of W_J:

$$u_{J,A}({}_\Omega\mathbb{A}) : U_J(\mathsf{W}) \to ({}_\Omega\mathbb{A})_{\mathrm{id}(J)}(A, A)$$

$$=$$

$$u_{J,A}({}_\Omega\mathbb{A}) : U_J(\mathsf{W}) \to \Omega_J(\mathbb{A}_{\mathrm{id}(J)}(A, A))$$

  is the following composed arrow of W_J:

$$\begin{array}{c} U_J(\mathsf{W}) \\ \downarrow \texttt{qunittens}_{\ldots} \\ \Omega_J(U_J(\mathsf{V})) \\ \downarrow \Omega_J(u_{J,A}(\mathbb{A})) \\ \Omega_J(\mathbb{A}_{\mathrm{id}(J)}(A, A)) \end{array}$$

Finally, assume that V and W are quasi-categorical (a fortiori, categorical) $\mathcal{J}$-tensor systems, that $\Omega : \mathsf{V} \to \mathsf{W}$ is a quasi-categorical $\mathcal{J}$-tensor morphism and that $\mathbb{A}$ is a categorical V-enriched $\mathcal{J}$-system. Then it is clear that ${}_\Omega\mathbb{A}$ is a categorical W-enriched $\mathcal{J}$-system.

4.5.d. Let $\mathcal{J}$ and $\mathcal{K}$ be two categories, $\theta : \mathcal{J} \to \mathcal{K}$ a functor, W a $\mathcal{K}$-tensor system and $\mathbb{B}$ a W-enriched $\mathcal{K}$-system. The *enriched system deduced from* $\mathbb{B}$ *by change of indexation*:

$$\mathbb{B}_\theta$$

is the obviously obtained W_θ-enriched $\mathcal{J}$-system such that:

- for all point J of $\mathcal{J}$:
$$(\mathbb{B}_\theta)_J = \mathbb{B}_{\theta(J)} \,,$$
- for all arrow $j : J \to J'$ of $\mathcal{J}$:
$$(\mathbb{B}_\theta)_j(-,-) : (\mathbb{B}_\theta)_J \times (\mathbb{B}_\theta)_{J'} \to \mathtt{Pt}((\mathsf{W}_\theta)_j)$$
$$=$$
$$\mathbb{B}_{\theta(j)}(-,-) : \mathbb{B}_{\theta(J)} \times \mathbb{B}_{\theta(J')} \to \mathtt{Pt}(\mathsf{W}_{\theta(j)}) \,,$$
- for all pair $(j : J \to J', j' : J' \to J'')$ of consecutive arrows of $\mathcal{J}$, all element B of $(\mathbb{B}_\theta)_J$ (*i.e.* of $\mathbb{B}_{\theta(J)}$), all element A' of $(\mathbb{B}_\theta)_{J'}$ (*i.e.* of $\mathbb{B}_{\theta(J')}$) and all element A'' of $(\mathbb{B}_\theta)_{J''}$ (*i.e.* of $\mathbb{B}_{\theta(J'')}$):
$$\mathtt{comp}(\mathbb{B}_\theta, (B, j, B', j', B'')) : (\mathbb{B}_\theta)_j(B, B') \otimes_{\dots} (\mathbb{B}_\theta)_{j'}(B', B'') \to (\mathbb{B}_\theta)_{j'\centerdot j}(B, B'')$$
$$=$$
$$\mathtt{comp}_{B,\theta(j),B',\theta(j'),B''} : \mathbb{B}_{\theta(j)}(B, B') \otimes_{\dots} \mathbb{B}_{\theta(j')}(B', B'') \to \mathbb{B}_{\theta(j')\centerdot\theta(j)}(B, B'') \,.$$

Now assume that W is a left-to-right quasi-associative $\mathcal{K}$-tensor system and that $\mathbb{B}$ is an associative W-enriched $\mathcal{K}$-system. Then it is clear that $\mathbb{B}_\theta$ is an associative W_θ-enriched $\mathcal{J}$-system.

Similarly, assume that W is a quasi-unitary $\mathcal{K}$-tensor system and that $\mathbb{B}$ is a unitary W-enriched $\mathcal{K}$-system with family of units:

$$(\, u_{K,B}(\mathbb{B}) : U_K(\mathsf{W}) \to \mathbb{B}_{\mathrm{id}(K)}(B, B) \,)_{K \in \mathtt{Pt}(\mathcal{K}), B \in \mathbb{B}_K} \,.$$

Then it is clear that $\mathbb{B}_\theta$ is a unitary W_θ-enriched $\mathcal{J}$-system with family of units:

$$(\, u_{\theta(J),B}(\mathbb{B}) : U_{\theta(J)}(\mathsf{W}) \to \mathbb{B}_{\theta(\mathrm{id}(J))}(B, B) = (\mathbb{B}_\theta)_J(B, B) \,)_{J \in \mathtt{Pt}(\mathcal{J}), B \in (\mathbb{B}_\theta)_J} \,.$$

Finally, assume that W is a quasi-categorical (a fortiori, categorical) $\mathcal{K}$-tensor system and that $\mathbb{B}$ is a categorical W-enriched $\mathcal{K}$-system. Then it is clear that $\mathbb{B}_\theta$ is a categorical W_θ-enriched $\mathcal{J}$-system.

4.5.e. Let $\mathcal{J}$ be a category, J a point of $\mathcal{J}$, V a quasi-categorical $\mathcal{J}$-tensor system and $\mathbb{A}$ a categorical V-enriched $\mathcal{J}$-system. The *component category of* $\mathbb{A}$ *at* J:

$$\mathbb{A}^*_J$$

is the obviously obtained category such that:

- $\mathtt{Pt}(\mathbb{A}^*_J) = \mathbb{A}_J$,
- for all points A_1 and A_2 of $\mathbb{A}^*_J$ (*i.e.* all elements A_1 and A_2 of $\mathbb{A}_J$):

$$\mathbb{A}^*_J(\, A_1, A_2\,) = \mathsf{V}_J(\, U_J(\mathsf{V}), \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2)\,)$$

  which means that:

$$\mathrm{Hom}_{\mathbb{A}^*_J}(\, A_1, A_2\,) = \mathrm{Hom}_{\mathsf{V}_J}(\, U_J(\mathsf{V}), \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2)\,)\,,$$

- for all point A of $\mathbb{A}^*_J$ (*i.e.* all element A of $\mathbb{A}_J$):

$$\mathrm{id}(A) : A \Longrightarrow_{\mathbb{A}^*_J} A$$

$$=$$

$$u_{J,A}(\mathbb{A}) : U_J(\mathsf{V}) \rightarrow_{\mathsf{V}_J} \mathbb{A}_{\mathrm{id}(J)}(A, A)\,,$$

- for all points A_1, A_2 and A_3 of $\mathbb{A}^*_J$ (*i.e.* all elements A_1, A_2 and A_3 of $\mathbb{A}_J$) and for all arrows $a_1 : A_1 \rightarrow A_2$ and $a_2 : A_2 \rightarrow A_3$ of $\mathbb{A}^*_J$ (*i.e.* all arrows $a_1 : U_J(\mathsf{V}) \rightarrow \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2)$ and $a_2 : U_J(\mathsf{V}) \rightarrow \mathbb{A}_{\mathrm{id}(J)}(A_2, A_3)$ of V_J), the arrow:

$$a_2 \centerdot a_1 : A_1 \rightarrow_{\mathbb{A}^*_J} A_3$$

$$=$$

$$a_2 \centerdot a_1 : U_J(\mathsf{V}) \rightarrow_{\mathsf{V}_J} \mathbb{A}_{\mathrm{id}(J)}(A_1, A_3)$$

  is the following composed arrow of V_J:

$$\begin{array}{c} U_J(\mathsf{V}) \\ \big\downarrow \mathtt{qunit}_{\dots} \\ U_J(\mathsf{V}) \otimes_{\dots} U_J(\mathsf{V}) \\ \big\downarrow a_1 \otimes_{\dots} a_2 \\ \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2) \otimes_{\dots} \mathbb{A}_{\mathrm{id}(J)}(A_2, A_3) \\ \big\downarrow \mathtt{comp}_{\dots} \\ \mathbb{A}_{\mathrm{id}(J)}(A_2, A_3) \end{array}$$

Now let $j : J \rightarrow J'$ be an arrow of $\mathcal{J}$. The *exponentation at j functor*:

$$\mathbb{A}^*_j(-,-) : (\mathbb{A}^*_J)^{op} \times \mathbb{A}^*_{J'} \rightarrow \mathsf{V}_j$$

is the obviously obtained functor such that:

- for all point A of $\mathbb{A}^*_J$ (*i.e.* all element A of $\mathbb{A}_J$) and all point A' of $\mathbb{A}^*_{J'}$ (*i.e.* all element A' of $\mathbb{A}_{J'}$):

$$\mathbb{A}^*_j(A, A') = \mathbb{A}_j(A, A')\,,$$

- for all arrow $a : A_1 \to A_2$ of $\mathbb{A}^*_J$ (*i.e.* all arrow $a : U_J(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2)$ of V_J) and for all arrow $a' : A'_1 \to A'_2$ of $\mathbb{A}^*_{J'}$ (*i.e.* all arrow $a' : U_{J'}(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J')}(A'_1, A'_2)$ of $\mathsf{V}_{J'}$), the arrow of V_j:

$$\mathbb{A}^*_j(a, a') : \mathbb{A}^*_j(A_2, A'_1) \to \mathbb{A}^*_j(A_1, A'_2)$$

$$=$$

$$\mathbb{A}^*_j(a, a') : \mathbb{A}_j(A_2, A'_1) \to \mathbb{A}_j(A_1, A'_2)$$

is, for instance, the following composed arrow of V_j:

$$\begin{array}{c}
\mathbb{A}_j(A_2, A'_1) \\
\Big\downarrow \scriptstyle \mathtt{qunit}_{\dots} \\
\mathbb{A}_j(A_2, A'_1) \otimes_{\dots} U_{J'}(\mathsf{V}) \\
\Big\downarrow \scriptstyle \mathtt{qunit}_{\dots} \otimes_{\dots} U_{J'}(\mathsf{V}) \\
(U_J(\mathsf{V}) \otimes_{\dots} \mathbb{A}_j(A_2, A'_1)) \otimes_{\dots} U_{J'}(\mathsf{V}) \\
\Big\downarrow \scriptstyle (a \otimes_{\dots} \mathbb{A}_j(A_2, A'_1)) \otimes_{\dots} a' \\
(\mathbb{A}_{\mathrm{id}(J)}(A_1, A_2) \otimes_{\dots} \mathbb{A}_j(A_2, A'_1)) \otimes_{\dots} \mathbb{A}_{\mathrm{id}(J')}(A'_1, A'_2) \\
\Big\downarrow \scriptstyle \mathtt{comp}_{\dots} \otimes_{\dots} \mathbb{A}_{\mathrm{id}(J')}(A'_1, A'_2) \\
\mathbb{A}_j(A_1, A'_1) \otimes_{\dots} \mathbb{A}_{\mathrm{id}(J')}(A'_1, A'_2) \\
\Big\downarrow \scriptstyle \mathtt{comp}_{\dots} \\
\mathbb{A}_j(A_1, A'_2)
\end{array}$$

(note that, thanks to associativity, left-to-right quasi-associativity, quasi-unitarity and coherence, there is another way to get the arrow $\mathbb{A}^*_j(a, a')$: its construction is left to the reader).

4.5.f. Let $\mathcal{J}$ be a category, J a point of $\mathcal{J}$, V a quasi-categorical $\mathcal{J}$-tensor system, $\mathbb{A}$ and $\mathbb{B}$ two categorical V-enriched $\mathcal{J}$-systems and $\Gamma : \mathbb{A} \to \mathbb{B}$ a categorical V-enriched $\mathcal{J}$-morphism. The *component functor of* Γ *at* J:

$$\Gamma^*_J : \mathbb{A}^*_J \to \mathbb{B}^*_J$$

is the obviously obtained functor such that:

- for all point A of $\mathbb{A}^*_J$ (*i.e.* all element A of $\mathbb{A}_J$):

$$\Gamma^*_J(A) = \Gamma_J(A)\,,$$

- for all points A_1 and A_2 of $\mathbb{A}^*_J$ (*i.e.* all elements A_1 and A_2 of $\mathbb{A}_J$) and for all arrow $a : A_1 \to A_2$ of $\mathbb{A}^*_J$ (*i.e.* all arrow $a : U_J(\mathsf{V}) \to \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2)$ of V_J), the arrow:

$$\Gamma^*_J(a) : \Gamma^*_J(A_1) \to_{\mathbb{B}^*_J} \Gamma^*_J(A_2)$$

$$=$$

$$\Gamma^*_J(a) : U_J(\mathsf{V}) \to_{\mathsf{V}_J} \mathbb{B}_{\mathrm{id}(J)}(\Gamma^*_J(A_1), \Gamma^*_J(A_2))$$
$$=$$
$$\Gamma^*_J(a) : U_J(\mathsf{V}) \to_{\mathsf{V}_J} \mathbb{B}_{\mathrm{id}(J)}(\Gamma_J(A_1), \Gamma_J(A_2))$$

is the following composed arrow of V_J:

$$\begin{array}{c} U_J(\mathsf{V}) \\ \downarrow a \\ \mathbb{A}_{\mathrm{id}(J)}(A_1, A_2) \\ \downarrow \Gamma_{\mathrm{id}(J)}(A_1,A_2) \\ \mathbb{B}_{\mathrm{id}(J)}(\Gamma_J(A_1), \Gamma_J(A_2)) \end{array}$$

Now let $j : J \to J'$ be an arrow of $\mathcal{J}$. The *natural transformation component of* Γ *at* j:

$$\Gamma^*_j(-,-) : \mathbb{A}^*_j(-,-) \Rightarrow \mathbb{B}^*_j(-,-) \circ ((\Gamma^*_J)^{op} \times \Gamma^*_{J'}) : (\mathbb{A}^*_J)^{op} \times \mathbb{A}^*_{J'} \to \mathsf{V}_j$$

is the obviously obtained natural transformation such that:

- for all point A of $\mathbb{A}^*_J$ (*i.e.* all element A of $\mathbb{A}_J$) and all point A' of $\mathbb{A}^*_{J'}$ (*i.e.* all element A' of $\mathbb{A}_{J'}$):
$$\Gamma^*_j(A, A') : \mathbb{A}^*_j(A, A') \to \mathbb{B}^*_j(\Gamma^*_J(A), \Gamma^*_J(A'))$$
$$=$$
$$\Gamma_j(A, A') : \mathbb{A}_j(A, A') \to \mathbb{B}_j(\Gamma_J(A), \Gamma_J(A')) \; .$$

4.6 The enriched system of $\mathbb{U}$-small compositive graphs

4.6.a. Let $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$ be three compositive graphs. Then:

$$-\underline{\circ}_{\underline{\Box}\mathcal{G},\mathcal{G}',\mathcal{G}''}- : \mathcal{F}unc(\mathcal{G}, \mathcal{G}') \,\underline{\Box}\, \mathcal{F}unc(\mathcal{G}', \mathcal{G}'') \to \mathcal{F}unc(\mathcal{G}, \mathcal{G}'')$$

is the obviously obtained functor such that:

- for all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{G}''$:
$$(-\underline{\circ}_{\underline{\Box}\mathcal{G},\mathcal{G}',\mathcal{G}''}-)([\phi, \phi']) = \phi' \circ \phi \, ,$$
- for all functors $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{G}''$ and all natural transformation $t : \phi_1 \Rightarrow \phi_2$:
$$(-\underline{\circ}_{\underline{\Box}\mathcal{G},\mathcal{G}',\mathcal{G}''}-)([t, \phi']) = \phi' \circ t \, ,$$
- for all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'_1, \phi'_2 : \mathcal{G}' \to \mathcal{G}''$ and all natural transformation $t' : \phi'_1 \Rightarrow \phi'_2$:
$$(-\underline{\circ}_{\underline{\Box}\mathcal{G},\mathcal{G}',\mathcal{G}''}-)([\phi, t']) = t' \circ \phi \; .$$

Similarly, let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs (in particular, $\mathcal{G}'$ may be a category) and $\mathcal{A}''$ a category. Then:

$$-\underline{\circ}_{\boxtimes\mathcal{G},\mathcal{G}',\mathcal{A}''}- : \mathcal{F}unc(\mathcal{G},\mathcal{G}')\,\underline{\boxtimes}\,\mathcal{F}unc(\mathcal{G}',\mathcal{A}'') \to \mathcal{F}unc(\mathcal{G},\mathcal{A}'')$$

is the obviously obtained functor such that:

- for all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{A}''$:

$$(-\underline{\circ}_{\boxtimes\mathcal{G},\mathcal{G}',\mathcal{A}''}-)([\phi, \phi']) = \phi' \circ \phi\,,$$

- for all functors $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ and $\phi' : \mathcal{G}' \to \mathcal{A}''$ and all natural transformation $t : \phi_1 \Rightarrow \phi_2$:

$$(-\underline{\circ}_{\boxtimes\mathcal{G},\mathcal{G}',\mathcal{A}''}-)([t, \phi']) = \phi' \circ t\,,$$

- for all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'_1, \phi'_2 : \mathcal{G}' \to \mathcal{A}''$ and all natural transformation $t' : \phi'_1 \Rightarrow \phi'_2$:

$$(-\underline{\circ}_{\boxtimes\mathcal{G},\mathcal{G}',\mathcal{A}''}-)([\phi, t']) = t' \circ \phi\,,$$

- for all functors $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ and $\phi'_1, \phi'_2 : \mathcal{G}' \to \mathcal{A}''$ and all natural transformations $t : \phi_1 \Rightarrow \phi_2$ and $t' : \phi'_1 \Rightarrow \phi'_2$:

$$(-\underline{\circ}_{\boxtimes\mathcal{G},\mathcal{G}',\mathcal{A}''}-)([t, t']) = t' \circ t\,.$$

Finally, let $\mathcal{A}$ and $\mathcal{A}'$ and $\mathcal{A}''$ be three categories. Then:

$$-\underline{\circ}_{\otimes\mathcal{A},\mathcal{A}',\mathcal{A}''}- : \mathcal{F}unc(\mathcal{A},\mathcal{A}')\,\underline{\otimes}\,\mathcal{F}unc(\mathcal{A}',\mathcal{A}'') \to \mathcal{F}unc(\mathcal{A},\mathcal{A}'')$$

is the obviously obtained functor such that:

- for all functors $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi' : \mathcal{A}' \to \mathcal{A}''$:

$$(-\underline{\circ}_{\otimes\mathcal{A},\mathcal{A}',\mathcal{A}''}-)([\phi, \phi']) = \phi' \circ \phi\,,$$

- for all functors $\phi_1, \phi_2 : \mathcal{A} \to \mathcal{A}'$ and $\phi'_1, \phi'_2 : \mathcal{A}' \to \mathcal{A}''$ and all natural transformations $t : \phi_1 \Rightarrow \phi_2$ and $t' : \phi'_1 \Rightarrow \phi'_2$:

$$(-\underline{\circ}_{\otimes\mathcal{A},\mathcal{A}',\mathcal{A}''}-)([t, t']) = t' \circ t\,.$$

4.6.b. Let $\mathcal{G}$ be a compositive graph. Then:

$$\underline{\texttt{uid}}(\mathcal{G})_\emptyset : \boldsymbol{1}_\emptyset \to \mathcal{F}unc(\mathcal{G},\mathcal{G})$$

is the functor which maps the point $\mathit{0}$ to the functor $\mathrm{id}(\mathcal{G}) : \mathcal{G} \to \mathcal{G}$ (which is indeed a point of $\mathcal{F}unc(\mathcal{G},\mathcal{G})$).

Let $\mathcal{A}$ be a category. Then:

$$\underline{\texttt{uid}}(\mathcal{A}) : \boldsymbol{1} \to \mathcal{F}unc(\mathcal{A},\mathcal{A})$$

is the functor (from the category $\boldsymbol{1}$ towards $\mathcal{F}unc(\mathcal{A},\mathcal{A})$, which is also a category) which maps the point $\mathit{0}$ to the functor $\mathrm{id}(\mathcal{A}) : \mathcal{A} \to \mathcal{A}$ (hence it maps the arrow $\mathrm{id}(\mathit{0})$ to the natural transformation $\mathrm{id}(\mathrm{id}(\mathcal{A})) : \mathrm{id}(\mathcal{A}) \Rightarrow \mathrm{id}(\mathcal{A}) : \mathcal{A} \to \mathcal{A}$, which is indeed an arrow of $\mathcal{F}unc(\mathcal{A},\mathcal{A})$).

4.6.c. Let $\mathbb{U}$ be a universe. The *enriched system of* $\mathbb{U}$*-small compositive graphs*:

$$\mathbb{C}\mathrm{omp}_{\mathbb{U}}$$

is the obviously obtained categorical $\mathsf{Comp}_{\mathbb{U}}$-enriched $\boldsymbol{1}$-system such that:

- $(\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{0} = \mathtt{Pt}(\mathcal{C}omp_{\mathbb{U}})$ is the set of $\mathbb{U}$-small compositive graphs,
- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:

$$(\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{G}, \mathcal{G}') = \mathcal{F}unc(\mathcal{G}, \mathcal{G}')\,,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$:

$$\begin{array}{ccc} (\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{G}, \mathcal{G}') \otimes_{\mathrm{id}(0),\mathrm{id}(0)} (\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{G}', \mathcal{G}'') & & \mathcal{F}unc(\mathcal{G}, \mathcal{G}') \,\underline{\Box}\, \mathcal{F}unc(\mathcal{G}', \mathcal{G}'') \\ \Big\downarrow \scriptstyle{\mathtt{comp}_{\mathcal{G},\mathrm{id}(0),\mathcal{G}',\mathrm{id}(0),\mathcal{G}''}} & = & \Big\downarrow \scriptstyle{-\underline{\circ}\,\underline{\Box}_{\mathcal{G},\mathcal{G}',\mathcal{G}''}-} \\ (\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{G}, \mathcal{G}'') & & \mathcal{F}unc(\mathcal{G}, \mathcal{G}'') \end{array}$$

Then:

- the family:

$$(\,\underline{\mathtt{uid}}(\mathcal{G})_{\emptyset} : \boldsymbol{1}_{\emptyset} \to \mathcal{F}unc(\mathcal{G}, \mathcal{G})\,)_{\mathcal{G} \in \mathtt{Pt}(\mathcal{C}omp_{\mathbb{U}})}$$

  is its family of units,
- the category $(\mathbb{C}\mathrm{omp}_{\mathbb{U}})^{*}_{0}$ (component of the enriched system $\mathbb{C}\mathrm{omp}_{\mathbb{U}}$ at 0) is canonically isomorphic, then identified, to the category $\mathcal{C}omp_{\mathbb{U}}$ of $\mathbb{U}$-small compositive graphs,
- the functor $(\mathbb{C}\mathrm{omp}_{\mathbb{U}})^{*}_{\mathrm{id}(0)} : (\mathcal{C}omp_{\mathbb{U}})^{op} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$ is the functor such that:
  - for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:

$$(\mathbb{C}\mathrm{omp}_{\mathbb{U}})^{*}_{\mathrm{id}(0)}(\mathcal{G}, \mathcal{G}') = \mathcal{F}unc(\mathcal{G}, \mathcal{G}')\,,$$

  - for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$, $\mathcal{G}'$, $\mathcal{G}''$ and $\mathcal{G}'''$ and all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'' : \mathcal{G}'' \to \mathcal{G}'''$:

$$(\mathbb{C}\mathrm{omp}_{\mathbb{U}})^{*}_{\mathrm{id}(0)}(\phi, \phi'') = \mathcal{F}unc(\phi, \phi'')\,.$$

Similarly, the *enriched system of* $\mathbb{U}$*-small categories*:

$$\mathbb{C}\mathrm{at}_{\mathbb{U}}$$

is the obviously obtained categorical $\mathsf{Cat}_{\mathbb{U}}$-enriched $\boldsymbol{1}$-system such that:

- $(\mathbb{C}\mathrm{at}_{\mathbb{U}})_{0} = \mathtt{Pt}(\mathcal{C}at_{\mathbb{U}})$ is the set of $\mathbb{U}$-small categories,

- for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

$$(\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{A},\mathcal{A}') = \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}')\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{A},\mathcal{A}') \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} (\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow \mathtt{comp}_{\mathcal{A},\mathrm{id}(\mathit{0}),\mathcal{A}',\mathrm{id}(\mathit{0}),\mathcal{A}''} \\ (\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{A},\mathcal{A}'') \end{array} = \begin{array}{c} \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}') \,\underline{\otimes}\, \mathcal{F}\!\mathit{unc}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow -\underline{\circ}_{\underline{\otimes}\,\mathcal{A},\mathcal{A}',\mathcal{A}''}- \\ \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}'') \end{array}$$

Then:

- the family:

$$(\,\underline{\mathtt{uid}}(\mathcal{A}) : \boldsymbol{1} \to \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A})\,)_{\mathcal{A}\in\mathtt{Pt}(\mathcal{C}at_{\mathbb{U}})}$$

  is its family of units,

- the category $(\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathit{0}}$ (component of the enriched system $\mathbb{C}\mathrm{at}_{\mathbb{U}}$ at $\mathit{0}$) is canonically isomorphic, then identified, to the category $\mathcal{C}at_{\mathbb{U}}$ of $\mathbb{U}$-small categories,

- the functor $(\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{C}at_{\mathbb{U}})^{op} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}$ is the functor such that:

  - for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

  $$(\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})}(\mathcal{A},\mathcal{A}') = \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}')\,,$$

  - for all $\mathbb{U}$-small categories $\mathcal{A}$, $\mathcal{A}'$, $\mathcal{A}''$ and $\mathcal{A}'''$ and all functors $\phi : \mathcal{A} \to \mathcal{A}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$:

  $$(\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})}(\phi,\phi'') = \mathcal{F}\!\mathit{unc}(\phi,\phi'')\,.$$

Up to a change of enrichment, there is of course a categorical $\mathsf{Comp}_{\mathbb{U}}$-enriched $\boldsymbol{1}$-morphism of *canonical injection*:

$$\mathbb{C}\mathrm{at}_{\mathbb{U}} \subseteq \mathbb{C}\mathrm{omp}_{\mathbb{U}} : \quad {}_{\mathsf{Cat}_{\mathbb{U}}\subseteq\mathsf{Comp}_{\mathbb{U}}}\mathbb{C}\mathrm{at}_{\mathbb{U}} \to \mathbb{C}\mathrm{omp}_{\mathbb{U}}\,.$$

Finally, the *enriched system of* $\mathbb{U}$*-small directed graphs*:

$$\mathbb{D}\mathrm{ir}_{\mathbb{U}}$$

is the obviously obtained categorical $\mathsf{Dir}_{\mathbb{U}}$-enriched $\boldsymbol{1}$-system such that:

- $(\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathit{0}} = \mathtt{Pt}(\mathcal{D}ir_{\mathbb{U}})$ is the set of $\mathbb{U}$-small directed graphs,

- for all $\mathbb{U}$-small directed graphs $\mathcal{R}$ and $\mathcal{R}'$:

$$(\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{R},\mathcal{R}') = \mathcal{F}\!\mathit{unc}(\mathcal{R},\mathcal{R}')\,,$$

- for all $\mathbb{U}$-small directed graphs $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$:

$$\begin{array}{ccc} (\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{R},\mathcal{R}') \otimes_{\mathrm{id}(0),\mathrm{id}(0)} (\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{R}',\mathcal{R}'') & & \mathcal{F}unc(\mathcal{R},\mathcal{R}') \,\square\, \mathcal{F}unc(\mathcal{R}',\mathcal{R}'') \\ \Big\downarrow \mathtt{comp}_{\mathcal{R},\mathrm{id}(0),\mathcal{R}',\mathrm{id}(0),\mathcal{R}''} & = & \Big\downarrow -\underline{\circ}\underline{\square}_{\mathcal{R},\mathcal{R}',\mathcal{R}''}- \\ (\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{R},\mathcal{R}'') & & \mathcal{F}unc(\mathcal{R},\mathcal{R}'') \end{array}$$

Then:

- the family:

$$(\,\underline{\mathtt{uid}}(\mathcal{R})_{\emptyset} : \mathbf{1}_{\emptyset} \to \mathcal{F}unc(\mathcal{R},\mathcal{R})\,)_{\mathcal{R}\in\mathtt{Pt}(\mathcal{D}ir_{\mathbb{U}})}$$

  is its family of units,

- the category $(\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{0}$ (component of the enriched system $\mathbb{D}\mathrm{ir}_{\mathbb{U}}$ at 0) is canonically isomorphic, then identified, to the category $\mathcal{D}ir_{\mathbb{U}}$ of $\mathbb{U}$-small directed graphs,

- the functor $(\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(0)} : (\mathcal{D}ir_{\mathbb{U}})^{op} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{D}ir_{\mathbb{U}}$ is the functor such that:

  - for all $\mathbb{U}$-small directed graphs $\mathcal{R}$ and $\mathcal{R}'$:

$$(\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(0)}(\mathcal{R},\mathcal{R}') = \mathcal{F}unc(\mathcal{R},\mathcal{R}')\,,$$

  - for all $\mathbb{U}$-small directed graphs $\mathcal{R}$, $\mathcal{R}'$, $\mathcal{R}''$ and $\mathcal{R}'''$ and all functors $\phi : \mathcal{R} \to \mathcal{R}'$ and $\phi'' : \mathcal{R}'' \to \mathcal{R}'''$:

$$(\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(0)}(\phi,\phi'') = \mathcal{F}unc(\phi,\phi'')\ .$$

Up to a change of enrichment, there is of course a categorical $\mathsf{Comp}_{\mathbb{U}}$-enriched $\mathbf{1}$-morphism of *canonical injection*:

$$\mathbb{D}\mathrm{ir}_{\mathbb{U}} \subseteq \mathbb{C}\mathrm{omp}_{\mathbb{U}} : \quad {}_{\mathsf{Dir}_{\mathbb{U}}\subseteq\mathsf{Comp}_{\mathbb{U}}}\mathbb{D}\mathrm{ir}_{\mathbb{U}} \to \mathbb{C}\mathrm{omp}_{\mathbb{U}}\ .$$

4.6.d. Let $\mathbb{U}$ be a universe. The *enriched system of* $\mathbb{U}$*-small compositive graphs and* $\mathbb{U}$*-small categories*:

$$\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}}$$

is the obviously obtained categorical $\mathsf{CompCat}_{\mathbb{U}}$-enriched $\mathbf{2}$-system such that:

- $(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{0} = \mathtt{Pt}(\mathcal{C}omp_{\mathbb{U}})$ is the set of $\mathbb{U}$-small compositive graphs,
- $(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{1} = \mathtt{Pt}(\mathcal{C}at_{\mathbb{U}})$ is the set of $\mathbb{U}$-small categories,
- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:

$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(0)}(\mathcal{G},\mathcal{G}') = \mathcal{F}unc(\mathcal{G},\mathcal{G}')\,,$$

- for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small category $\mathcal{A}'$:

$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{(0,1)}(\mathcal{G},\mathcal{A}') = \mathcal{F}unc(\mathcal{G},\mathcal{A}')\,,$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$ and $\mathcal{A}'$:

$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}') = \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}')\,,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}') \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G}',\mathcal{G}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},\mathrm{id}(\mathit{0}),\mathcal{G}',\mathrm{id}(\mathit{0}),\mathcal{G}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}'') \end{array} = \begin{array}{c} \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{G}') \,\underline{\square}\, \mathcal{F}\!\mathit{unc}(\mathcal{G}',\mathcal{G}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\square}_{\mathcal{G},\mathcal{G}',\mathcal{G}''}-} \\ \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{G}'') \end{array}$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$ and all $\mathbb{U}$-small category $\mathcal{A}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}') \otimes_{\mathrm{id}(\mathit{0}),(\mathit{0},\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},\mathrm{id}(\mathit{0}),\mathcal{G}',(\mathit{0},\mathit{1}),\mathcal{A}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{A}'') \end{array} = \begin{array}{c} \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{G}') \,\underline{\boxtimes}\, \mathcal{F}\!\mathit{unc}(\mathcal{G}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\boxtimes}_{\mathcal{G},\mathcal{G}',\mathcal{A}''}-} \\ \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{A}'') \end{array}$$

- for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small categories $\mathcal{A}'$ and $\mathcal{A}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{A}') \otimes_{(\mathit{0},\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},(\mathit{0},\mathit{1}),\mathcal{A}',\mathrm{id}(\mathit{1}),\mathcal{A}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{A}'') \end{array} = \begin{array}{c} \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{A}') \,\underline{\boxtimes}\, \mathcal{F}\!\mathit{unc}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\boxtimes}_{\mathcal{G},\mathcal{A}',\mathcal{A}''}-} \\ \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{A}'') \end{array}$$

- for all $\mathbb{U}$-small categories $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}') \otimes_{\mathrm{id}(\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{A},\mathrm{id}(\mathit{1}),\mathcal{A}',\mathrm{id}(\mathit{1}),\mathcal{A}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{A},\mathcal{A}'') \end{array} = \begin{array}{c} \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}') \,\underline{\otimes}\, \mathcal{F}\!\mathit{unc}(\mathcal{A}',\mathcal{A}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\otimes}_{\mathcal{A},\mathcal{A}',\mathcal{A}''}-} \\ \mathcal{F}\!\mathit{unc}(\mathcal{A},\mathcal{A}'') \end{array}$$

Then:

- the family:

$$(\,\underline{\mathtt{uid}}(\mathcal{G})_{\emptyset} : \boldsymbol{\mathit{1}}_{\emptyset} \to \mathcal{F}\!\mathit{unc}(\mathcal{G},\mathcal{G})\,)_{\mathcal{G}\in\mathtt{Pt}(\mathcal{C}omp_{\mathbb{U}})}$$

  is its family of units of index the point $\mathit{0}$,

- the family:
$$(\,\underline{\mathtt{uid}}(\mathcal{A}) : \mathbf{\mathit{1}} \to \mathcal{F}unc(\mathcal{A},\mathcal{A})\,)_{\mathcal{A}\in\mathtt{Pt}(\mathcal{C}at_{\mathbb{U}})}$$
is its family of units of index the point *1*,

- the category $(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathit{0}}$ (component of the enriched system $\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}}$ at *0*) is canonically isomorphic, then identified, to the category $\mathcal{C}omp_{\mathbb{U}}$ of $\mathbb{U}$-small compositive graphs,

- the category $(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathit{1}}$ (component of the enriched system $\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}}$ at *1*) is canonically isomorphic, then identified, to the category $\mathcal{C}at_{\mathbb{U}}$ of $\mathbb{U}$-small categories,

- the functor $(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})} : (\mathcal{C}omp_{\mathbb{U}})^{op} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$ is the functor such that:
  - for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small category $\mathcal{A}'$:
  $$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{A}') = \mathcal{F}unc(\mathcal{G},\mathcal{A}')\,,$$
  - for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$, all $\mathbb{U}$-small categories $\mathcal{A}''$ and $\mathcal{A}'''$ and all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'' : \mathcal{A}'' \to \mathcal{A}'''$:
  $$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})}(\phi,\phi'') = \mathcal{F}unc(\phi,\phi'')\,,$$

- in addition:
$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} = (\mathbb{C}\mathrm{omp}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{C}omp_{\mathbb{U}})^{op} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$$
and:
$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{1})} = (\mathbb{C}\mathrm{at}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{C}at_{\mathbb{U}})^{op} \times \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}\,.$$

Up to a change of indexation, there is of course a categorical $(\mathsf{Comp}_{\mathbb{U}})_{\mathbf{\mathit{2}}|\mathbf{\mathit{1}}}$-enriched $\mathbf{\mathit{2}}$-morphism of *canonical injection*:
$$\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}} \subseteq \mathbb{C}\mathrm{omp}_{\mathbb{U}} : \mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}} \to (\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathbf{\mathit{2}}|\mathbf{\mathit{1}}}\,.$$
In addition, up to other changes of indexation:
$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathbf{\mathit{1}}\subseteq\mathbf{\mathit{2}}} = \mathbb{C}\mathrm{omp}_{\mathbb{U}}$$
and:
$$(\mathbb{C}\mathrm{omp}\mathbb{C}\mathrm{at}_{\mathbb{U}})_{\mathbf{\mathit{1}}\text{“}\subseteq\text{”}\mathbf{\mathit{2}}} = \mathbb{C}\mathrm{at}_{\mathbb{U}}\,.$$

Similarly, the *enriched system of* $\mathbb{U}$*-small compositive graphs and* $\mathbb{U}$*-small directed graphs*:
$$\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}}$$
is the obviously obtained categorical $\mathsf{CompDir}_{\mathbb{U}}$-enriched $\mathbf{\mathit{2}}$-system such that:

- $(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathit{0}} = \mathtt{Pt}(\mathcal{C}omp_{\mathbb{U}})$ is the set of $\mathbb{U}$-small compositive graphs,
- $(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathit{1}} = \mathtt{Pt}(\mathcal{D}ir_{\mathbb{U}})$ is the set of $\mathbb{U}$-small directed graphs,
- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$:
$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}') = \mathcal{F}unc(\mathcal{G},\mathcal{G}')\,,$$

- for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small directed graph $\mathcal{R}'$:

$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{R}') = \mathcal{F}unc(\mathcal{G},\mathcal{R}')\,,$$

- for all $\mathbb{U}$-small directed graphs $\mathcal{R}$ and $\mathcal{R}'$:

$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{R},\mathcal{R}') = \mathcal{F}unc(\mathcal{R},\mathcal{R}')\,,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$, $\mathcal{G}'$ and $\mathcal{G}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}') \otimes_{\mathrm{id}(\mathit{0}),\mathrm{id}(\mathit{0})} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G}',\mathcal{G}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},\mathrm{id}(\mathit{0}),\mathcal{G}',\mathrm{id}(\mathit{0}),\mathcal{G}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}'') \end{array} = \begin{array}{c} \mathcal{F}unc(\mathcal{G},\mathcal{G}')\,\underline{\square}\,\mathcal{F}unc(\mathcal{G}',\mathcal{G}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\square}_{\mathcal{G},\mathcal{G}',\mathcal{G}''}-} \\ \mathcal{F}unc(\mathcal{G},\mathcal{G}'') \end{array}$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$ and all $\mathbb{U}$-small directed graph $\mathcal{R}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{0})}(\mathcal{G},\mathcal{G}') \otimes_{\mathrm{id}(\mathit{0}),(\mathit{0},\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G}',\mathcal{R}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},\mathrm{id}(\mathit{0}),\mathcal{G}',(\mathit{0},\mathit{1}),\mathcal{R}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{R}'') \end{array} = \begin{array}{c} \mathcal{F}unc(\mathcal{G},\mathcal{G}')\,\underline{\square}\,\mathcal{F}unc(\mathcal{G}',\mathcal{R}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\square}_{\mathcal{G},\mathcal{G}',\mathcal{R}''}-} \\ \mathcal{F}unc(\mathcal{G},\mathcal{R}'') \end{array}$$

- for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small directed graphs $\mathcal{R}'$ and $\mathcal{R}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{R}') \otimes_{(\mathit{0},\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{R}',\mathcal{R}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{G},(\mathit{0},\mathit{1}),\mathcal{R}',\mathrm{id}(\mathit{1}),\mathcal{R}''}} \\ (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{R}'') \end{array} = \begin{array}{c} \mathcal{F}unc(\mathcal{G},\mathcal{R}')\,\underline{\square}\,\mathcal{F}unc(\mathcal{R}',\mathcal{R}'') \\ \Big\downarrow {\scriptstyle -\underline{\circ}\,\underline{\square}_{\mathcal{G},\mathcal{R}',\mathcal{R}''}-} \\ \mathcal{F}unc(\mathcal{G},\mathcal{R}'') \end{array}$$

- for all $\mathbb{U}$-small directed graphs $\mathcal{R}$, $\mathcal{R}'$ and $\mathcal{R}''$:

$$\begin{array}{c} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{R},\mathcal{R}') \otimes_{\mathrm{id}(\mathit{1}),\mathrm{id}(\mathit{1})} (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{R}',\mathcal{R}'') \\ \Big\downarrow {\scriptstyle \mathtt{comp}_{\mathcal{R},\mathrm{id}(\mathit{1}),\mathcal{R}',\mathrm{id}(\mathit{1}),\mathcal{R}''}} \\ \mathcal{F}unc(\mathcal{R},\mathcal{R}')\,\underline{\square}\,\mathcal{F}unc(\mathcal{R}',\mathcal{R}'') \end{array} = \begin{array}{c} -\underline{\circ}\,\underline{\square}_{\mathcal{R},\mathcal{R}',\mathcal{R}''}- \\ \Big\downarrow {\scriptstyle (\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathrm{id}(\mathit{1})}(\mathcal{R},\mathcal{R}'')} \\ \mathcal{F}unc(\mathcal{R},\mathcal{R}'') \end{array}$$

Then:

- the family:
$$(\,\underline{\texttt{uid}}(\mathcal{G})_{\emptyset} : \mathit{1}_{\emptyset} \to \mathcal{F}unc(\mathcal{G},\mathcal{G})\,)_{\mathcal{G}\in\texttt{Pt}(\mathcal{C}omp_{\mathbb{U}})}$$
is its family of units of index the point *0*,
- the family:
$$(\,\underline{\texttt{uid}}(\mathcal{R})_{\emptyset} : \mathit{1}_{\emptyset} \to \mathcal{F}unc(\mathcal{R},\mathcal{R})\,)_{\mathcal{R}\in\texttt{Pt}(\mathcal{D}ir_{\mathbb{U}})}$$
is its family of units of index the point *1*,
- the category $(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathit{0}}$ (component of the enriched system $\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}}$ at *0*) is canonically isomorphic, then identified, to the category $\mathcal{C}omp_{\mathbb{U}}$ of $\mathbb{U}$-small compositive graphs,
- the category $(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathit{1}}$ (component of the enriched system $\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}}$ at *1*) is canonically isomorphic, then identified, to the category $\mathcal{D}ir_{\mathbb{U}}$ of $\mathbb{U}$-small directed graphs.
- the functor $(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})} : (\mathcal{C}omp_{\mathbb{U}})^{op} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$ is the functor such that:
  - for all $\mathbb{U}$-small compositive graph $\mathcal{G}$ and all $\mathbb{U}$-small directed graph $\mathcal{R}'$:
  $$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})}(\mathcal{G},\mathcal{R}') = \mathcal{F}unc(\mathcal{G},\mathcal{R}')\,,$$
  - for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$, all $\mathbb{U}$-small directed graphs $\mathcal{R}''$ and $\mathcal{R}'''$ and all functors $\phi : \mathcal{G} \to \mathcal{G}'$ and $\phi'' : \mathcal{R}'' \to \mathcal{R}'''$:
  $$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{(\mathit{0},\mathit{1})}(\phi,\phi'') = \mathcal{F}unc(\phi,\phi'')\,,$$
- in addition:
$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} = (\mathbb{C}\mathrm{omp}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{C}omp_{\mathbb{U}})^{op} \times \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$$
and:
$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{1})} = (\mathbb{D}\mathrm{ir}_{\mathbb{U}})^*_{\mathrm{id}(\mathit{0})} : (\mathcal{D}ir_{\mathbb{U}})^{op} \times \mathcal{D}ir_{\mathbb{U}} \to \mathcal{D}ir_{\mathbb{U}}\,.$$

Up to a change of indexation, there is of course a categorical $(\mathsf{Comp}_{\mathbb{U}})_{\mathit{2}|\mathit{1}}$-enriched *2*-morphism of *canonical injection*:

$$\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}} \subseteq \mathbb{C}\mathrm{omp}_{\mathbb{U}} : \mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}} \to (\mathbb{C}\mathrm{omp}_{\mathbb{U}})_{\mathit{2}|\mathit{1}}\,.$$

In addition, up to other changes of indexation:

$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathit{1}\subseteq\mathit{2}} = \mathbb{C}\mathrm{omp}_{\mathbb{U}}$$

and:

$$(\mathbb{C}\mathrm{omp}\mathbb{D}\mathrm{ir}_{\mathbb{U}})_{\mathit{1}\,\text{“}\subseteq\text{”}\,\mathit{2}} = \mathbb{D}\mathrm{ir}_{\mathbb{U}}\,.$$

5 Generated categories

5.1 Categories of paths

5.1.a. Let $\mathcal{R}$ be a directed graph. The *category of paths of* $\mathcal{R}$:

$$\mathrm{Path}(\mathcal{R})$$

is the obviously obtained category such that:

- its points are the points of $\mathcal{R}$,
- its arrows are:
  - the *paths of length 0 at the points of* $\mathcal{R}$, *i.e.* the:

    $$[R, R] : R \to R$$

    where R is a point of $\mathcal{R}$,
  - for all $m \geq 1$, the *paths of length* m *between points of* $\mathcal{R}$, *i.e.* the:

    $$[R_1, r_1, \ldots, r_j, \ldots, r_m, R_{m+1}] : R_1 \to R_{m+1}$$

    where R_1 and R_{m+1} are two points of $\mathcal{R}$ and $r_1 : R_1 \to R_2$, $\ldots$,$r_j : R_j \to R_{j+1}$, $\ldots$, $r_m : R_m \to R_{m+1}$ are (consecutive) arrows of $\mathcal{R}$,
- for all point R of $\mathrm{Path}(\mathcal{R})$ (*i.e.* of $\mathcal{R}$):

  $$\mathtt{selid}(\mathrm{Path}(\mathcal{R}))(R) = [R, R]\,,$$

- for all points R_1, R_2 and R_3 of $\mathcal{R}$, all integers $m \geq 1$ and $n \geq 1$, all path:

  $$[R_1 = R_{1,1}, r_{1,1}, \ldots, r_{1,m}, R_{1,m+1} = R_2] : R_1 \to R_2\,,$$

  of length m, and all path:

  $$[R_2 = R_{2,1}, r_{2,1}, \ldots, r_{2,n}, R_{2,n+1} = R_3] : R_2 \to R_3\,,$$

  of length n:

  $$\begin{aligned} &\mathtt{comp}(\mathrm{Path}(\mathcal{R}))([R_1, r_{1,1}, \ldots, r_{1,m}, R_2], [R_2, r_{2,1}, \ldots, r_{2,n}, R_3]) \\ &= \ [R_1, r_{1,1}, \ldots, r_{1,m}, r_{2,1}, \ldots, r_{2,n}, R_3] : R_1 \to R_3\,. \end{aligned}$$

Then:

$$\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}) : \mathcal{R} \to \mathrm{Path}(\mathcal{R})$$

denotes the (“canonical inclusion”) functor *identifying the arrows of* $\mathcal{R}$ *with the paths of length 1*, *i.e.* the functor such that:

- $(\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}))(R) = R$ for all point R of $\mathcal{R}$,
- $(\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}))(r) = [R_1, r, R_2]$ for all arrow $r : R_1 \to R_2$ of $\mathcal{R}$.

Now it is clear that:

- if $\mathbb{U}$ is a universe and if $\mathcal{R}$ is a $\mathbb{U}$-small directed graph, then $\mathrm{Path}(\mathcal{R})$ is a $\mathbb{U}$-small category.

5.1.b. Let us now prove the following result.

Proposition 1 *Let $\mathcal{R}$ be a directed graph and $\mathcal{A}$ a category. Then the functor:*

$$\mathcal{F}unc(\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}), \mathcal{A}) : \mathcal{F}unc(\mathrm{Path}(\mathcal{R}), \mathcal{A}) \to \mathcal{F}unc(\mathcal{R}, \mathcal{A})$$

is an isomorphism.

Proof. If $\phi : \mathcal{R} \to \mathcal{A}$ is a functor, then its *prolongation* to $\mathrm{Path}(\mathcal{R})$:

$$\texttt{prol}(\phi) = (\mathcal{F}unc(\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}), \mathcal{A}))^{-1}(\phi) : \mathrm{Path}(\mathcal{R}) \to \mathcal{A}$$

is the obviously obtained functor such that:

- for all point R of $\mathcal{R}$:
$$\texttt{prol}(\phi)(R) = \phi(R) \, ,$$
- for all point R of $\mathcal{R}$:
$$\texttt{prol}(\phi)([R, R]) = \mathrm{id}(\phi(R)) \, ,$$
- for all integer $m \geq 1$, all points R_1 and R_{m+1} and all path:
$$[R_1, r_1, \ldots, r_m, R_{m+1}] : R_1 \to R_{m+1} \, ,$$
  of length m, of $\mathcal{R}$:
$$\texttt{prol}(\phi)([R_1, r_1, \ldots, r_j, \ldots, r_m, R_{m+1}]) = \phi(r_m) \bullet \ldots \bullet \phi(r_j) \bullet \ldots \bullet \phi(r_1) \; .$$

Similarly, let $\phi_1, \phi_2 : \mathcal{R} \to \mathcal{A}$ be two functors and $t : \phi_1 \Rightarrow \phi_2$ a natural transformation. Then clearly the *prolongation* of t:

$$\texttt{prol}(t) = (\mathcal{F}unc(\mathcal{R} \subseteq \mathrm{Path}(\mathcal{R}), \mathcal{A}))^{-1}(t) : \texttt{prol}(\phi_1) \Rightarrow \texttt{prol}(\phi_2) : \mathrm{Path}(\mathcal{R}) \to \mathcal{A}$$

is the obviously obtained natural transformation such that:

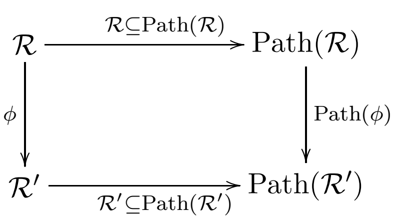
- for all point R of $\mathcal{R}$:
$$\texttt{prol}(t)(R) = t(R) \; .$$

End of proof.

Hence, it follows from proposition 1 that, if $\mathcal{R}$ and $\mathcal{R}'$ are two directed graphs and if $\phi : \mathcal{R} \to \mathcal{R}'$ is a functor, there is a unique functor (prolonging ϕ both to $\mathrm{Path}(\mathcal{R})$ and to $\mathrm{Path}(\mathcal{R}')$):

$$\mathrm{Path}(\phi) = \texttt{prol}((\mathcal{R}' \subseteq \mathrm{Path}(\mathcal{R}')) \circ \phi) : \mathrm{Path}(\mathcal{R}) \to \mathrm{Path}(\mathcal{R}')$$

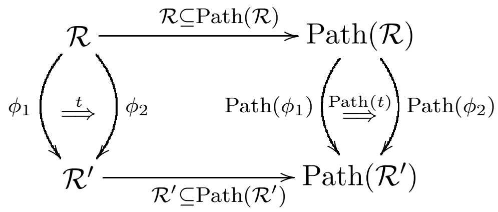
such that the following diagram (of functors) is commutative:



Similarly, let $\mathcal{R}$ and $\mathcal{R}'$ be two directed graphs, $\phi_1, \phi_2 : \mathcal{R} \to \mathcal{R}'$ two functors and $t : \phi_1 \Rightarrow \phi_2$ a natural transformation. Then there is a unique natural transformation (prolonging t both to $\mathrm{Path}(\mathcal{R})$ and to $\mathrm{Path}(\mathcal{R}')$):

$$\mathrm{Path}(t) = \mathtt{prol}((\mathcal{R}' \subseteq \mathrm{Path}(\mathcal{R}')) \circ t) : \mathrm{Path}(\phi_1) \Rightarrow \mathrm{Path}(\phi_2) : \mathrm{Path}(\mathcal{R}) \to \mathrm{Path}(\mathcal{R}')$$

such that the following diagram (of functors and natural transformations) is commutative:



5.1.c. Let $\mathbb{U}$ be a universe. Then the *category-of-paths functor*:

$$\mathrm{Path}_{\mathbb{U}}(-) : \mathcal{Dir}_{\mathbb{U}} \to \mathcal{Cat}_{\mathbb{U}}$$

is the obviously obtained functor such that:

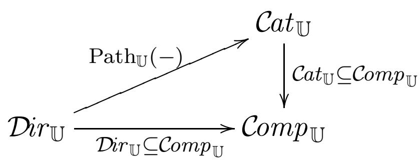
- for all $\mathbb{U}$-small directed graph $\mathcal{R}$:

$$\mathrm{Path}_{\mathbb{U}}(-)(\mathcal{R}) = \mathrm{Path}(\mathcal{R}) \,,$$

- for all $\mathbb{U}$-small directed graphs $\mathcal{R}$ and $\mathcal{R}'$ and all functor $\phi : \mathcal{R} \to \mathcal{R}'$:

$$\mathrm{Path}_{\mathbb{U}}(-)(\phi) = \mathrm{Path}(\phi) \;.$$

Clearly, it follows from proposition 1 that, in the following diagram (of functors):



the functor $\mathrm{Path}_{\mathbb{U}}(-) : \mathcal{Dir}_{\mathbb{U}} \to \mathcal{Cat}_{\mathbb{U}}$ is a $(\mathcal{Dir}_{\mathbb{U}} \subseteq \mathcal{Comp}_{\mathbb{U}})$-left adjoint of the functor $\mathcal{Cat}_{\mathbb{U}} \subseteq \mathcal{Comp}_{\mathbb{U}} : \mathcal{Cat}_{\mathbb{U}} \to \mathcal{Comp}_{\mathbb{U}}$ ("compatible with the enrichments").

5.2 Categories of classes of paths

5.2.a. Let $\mathcal{G}$ be a compositive graph. The *category of paths of* $\mathcal{G}$ is the category:

$$\mathrm{Path}(\mathcal{G}) = \mathrm{Path}(\mathtt{subl}(\mathcal{G}))$$

of paths of the directed graph *sublying* $\mathcal{G}$. In addition:

- $\mathrm{Rel}(\mathcal{G})$ is the relation on $\mathrm{Path}(\mathcal{G})$ such that:

- for all point G of $\mathcal{G}$ and all identity arrow $g : G \Longrightarrow G$ of $\mathcal{G}$:

$$[G, g, G] \operatorname{Rel}(\mathcal{G}) \; [G, G] \, ,$$

- for all points G_1, G_2 and G_3 of $\mathcal{G}$ and all composable pair $(g_1 : G_1 \to G_2, g_2 : G_2 \to G_3)$ of $\mathcal{G}$:

$$[G_1, g_2 \centerdot g_1, G_3] \operatorname{Rel}(\mathcal{G}) \; [G_1, g_1, g_2, G_3] \, ,$$

- $\operatorname{Cong}(\mathcal{G})$ is the *congruence*, *i.e.* the equivalence relation compatible with the structure of category, on $\operatorname{Path}(\mathcal{G})$, generated by the relation $\operatorname{Rel}(\mathcal{G})$,

Then the *category of classes of paths of* $\mathcal{G}$:

$$\operatorname{ClPath}(\mathcal{G})$$

is the quotient category of the category $\operatorname{Path}(\mathcal{G})$ of paths of $\mathcal{G}$ by the congruence $\operatorname{Cong}(\mathcal{G})$.

Now, there is a *quotient* functor (for paths of length 1):

$$\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G}) : \mathcal{G} \to \operatorname{ClPath}(\mathcal{G}) \; .$$

It is clear that:

- $\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G})$ is "the identity on points",
- $\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G})$ is "a surjection on arrows", which means that:
  - for all point G of $\mathcal{G}$ (*i.e.* of $\operatorname{ClPath}(\mathcal{G})$):

$$\texttt{selid}(\operatorname{ClPath}(\mathcal{G}))(G) = \operatorname{id}(G) = (\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G}))(\, [G, G] \,) = \overline{[G, G]} \, ,$$

  - for all points G and H of $\mathcal{G}$ (*i.e.* of $\operatorname{ClPath}(\mathcal{G})$) and all arrow $c : G \to H$ of $\operatorname{ClPath}(\mathcal{G})$ which is not an identity arrow of $\operatorname{ClPath}(\mathcal{G})$, there is an integer $m \geq 1$ and a path:

$$[G{=}G_1, g_1, \ldots, g_m, G_{m+1}{=}H] : G \to H \, ,$$

  of length m, of $\mathcal{G}$, such that:

$$c = (\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G}))(\, [G, g_1, \ldots, g_m, H] \,) = \overline{[G, g_1, \ldots, g_m, H]} \; .$$

Let $\mathcal{A}$ be a category. Then of course $\mathcal{A} \mid \operatorname{ClPath}(\mathcal{A}) : \mathcal{A} \to \operatorname{ClPath}(\mathcal{A})$ is an isomorphism, which allows us to identify $\mathcal{A}$ and $\operatorname{ClPath}(\mathcal{A})$.

Let $\mathbb{U}$ be a universe and $\mathcal{G}$ a $\mathbb{U}$-small compositive graph. Then clearly $\operatorname{ClPath}(\mathcal{G})$ is a $\mathbb{U}$-small category.

5.2.b. Let us now prove the following result.

Proposition 2 *Let $\mathcal{G}$ be a compositive graph and $\mathcal{A}$ a category. Then the functor:*

$$\mathcal{F}unc(\mathcal{G} \mid \operatorname{ClPath}(\mathcal{G}), \mathcal{A}) : \mathcal{F}unc(\operatorname{ClPath}(\mathcal{G}), \mathcal{A}) \to \mathcal{F}unc(\mathcal{G}, \mathcal{A})$$

is an isomorphism.

Proof. If $\phi : \mathcal{G} \to \mathcal{A}$ is a functor, then its *expansion* to $\mathrm{ClPath}(\mathcal{G})$:

$$\mathtt{exp}(\phi) = (\mathcal{F}unc(\mathcal{G} \mid \mathrm{ClPath}(\mathcal{G}), \mathcal{A}))^{-1}(\phi) : \mathrm{ClPath}(\mathcal{G}) \to \mathcal{A}$$

is the obviously obtained functor such that:

- for all point G of $\mathcal{G}$:

$$\mathtt{exp}(\phi)(G) = \phi(G)\,,$$

- for all point G of $\mathcal{G}$:

$$\mathtt{exp}(\phi)(\overline{[G, G]}) = \mathrm{id}(\phi(G))\,,$$

- for all integer $m \geq 1$, all points G_1 and G_{m+1} and all path:

$$[G_1, g_1, \ldots, g_j, \ldots, g_m, G_{m+1}] : G_1 \to G_{m+1}\,,$$

  of length m, of $\mathcal{G}$:

$$\mathtt{exp}(\phi)(\overline{[G_1, g_1, \ldots, g_j, \ldots, g_m, G_{m+1}]}) = \phi(g_m) \centerdot \ldots \centerdot \phi(g_j) \centerdot \ldots \centerdot \phi(g_1)\,.$$

Similarly, let $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{A}$ be two functors and $t : \phi_1 \Rightarrow \phi_2$ a natural transformation. Then clearly the *expansion* of t:

$$\mathtt{exp}(t) = (\mathcal{F}unc(\mathcal{G} \mid \mathrm{ClPath}(\mathcal{G}), \mathcal{A}))^{-1}(t) : \mathtt{exp}(\phi_1) \Rightarrow \mathtt{exp}(\phi_2) : \mathrm{Path}(\mathcal{G}) \to \mathcal{A}$$

is the obviously obtained natural transformation such that:

- for all point G of $\mathcal{G}$:

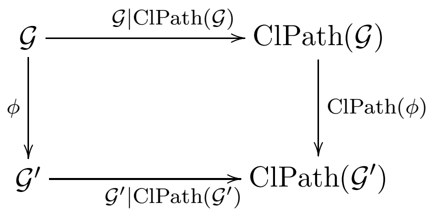
$$\mathtt{exp}(t)(G) = t(G)\,.$$

End of proof.

Hence, it follows from proposition 2 that, if $\mathcal{G}$ and $\mathcal{G}'$ are two compositive graphs and if $\phi : \mathcal{G} \to \mathcal{G}'$ is a functor, there is a unique functor (expanding ϕ both to $\mathrm{ClPath}(\mathcal{G})$ and to $\mathrm{ClPath}(\mathcal{G}')$):

$$\mathrm{ClPath}(\phi) = \mathtt{exp}((\mathcal{G}' \mid \mathrm{ClPath}(\mathcal{G}')) \circ \phi) : \mathrm{ClPath}(\mathcal{G}) \to \mathrm{ClPath}(\mathcal{G}')$$

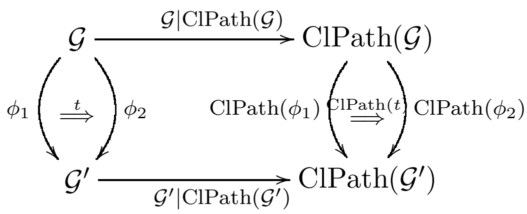
such that the following diagram (of functors) is commutative:



Similarly, let $\mathcal{G}$ and $\mathcal{G}'$ be two compositive graphs, $\phi_1, \phi_2 : \mathcal{G} \to \mathcal{G}'$ two functors and $t : \phi_1 \Rightarrow \phi_2$ a natural transformation. Then there is a unique natural transformation (expanding t both to $\mathrm{ClPath}(\mathcal{G})$ and to $\mathrm{ClPath}(\mathcal{G}')$):

$$\mathrm{ClPath}(t) = \mathtt{exp}((\mathcal{G}' \mid \mathrm{ClPath}(\mathcal{G}')) \circ t) : \mathrm{ClPath}(\phi_1) \Rightarrow \mathrm{ClPath}(\phi_2) : \mathrm{ClPath}(\mathcal{G}) \to \mathrm{ClPath}(\mathcal{G}')$$

such that the following diagram (of functors and natural transformations) is commutative:



5.2.c. Let $\mathbb{U}$ be a universe. Then the *category-of-classes-of-paths functor*:

$$\mathrm{ClPath}_{\mathbb{U}}(-) : \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}$$

is the obviously obtained functor such that:

- for all $\mathbb{U}$-small compositive graph $\mathcal{G}$:

$$\mathrm{ClPath}_{\mathbb{U}}(-)(\mathcal{G}) = \mathrm{ClPath}(\mathcal{G}) \; ,$$

- for all $\mathbb{U}$-small compositive graphs $\mathcal{G}$ and $\mathcal{G}'$ and all functor $\phi : \mathcal{G} \to \mathcal{G}'$:

$$\mathrm{ClPath}_{\mathbb{U}}(-)(\phi) = \mathrm{ClPath}(\phi) \; .$$

Clearly, it follows from proposition 2 that, in the following diagram (of functors):

$$\begin{array}{c} \mathcal{C}at_{\mathbb{U}} \\ {\scriptstyle \mathrm{ClPath}_{\mathbb{U}}(-)} \uparrow \;\; \downarrow {\scriptstyle \mathcal{C}at_{\mathbb{U}} \subseteq \mathcal{C}omp_{\mathbb{U}}} \\ \mathcal{C}omp_{\mathbb{U}} \end{array}$$

the functor $\mathrm{ClPath}_{\mathbb{U}}(-) : \mathcal{C}omp_{\mathbb{U}} \to \mathcal{C}at_{\mathbb{U}}$ is a left adjoint of the functor $\mathcal{C}at_{\mathbb{U}} \subseteq \mathcal{C}omp_{\mathbb{U}} : \mathcal{C}at_{\mathbb{U}} \to \mathcal{C}omp_{\mathbb{U}}$ ("compatible with the enrichments").

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