Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation
ESA - CNRS 6090

Note sur la conique des cols

Driss Boularas

Rapport de recherche n° 1999-03

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. 05 55 45 73 23 - Fax. 05 55 45 73 22 - laco@unilim.fr

http://www.unilim.fr/laco/

A NOTE ON THE SADDLE CONIC OF QUADRATIC PLANAR DIFFERENTIAL SYSTEM

D. Boularas[1]

Abstract. Using invariant theory we give some properties of the saddle conic of quadratic differential systems and deduce semi-algabraic conditions for existence of one, two or three saddle points (in terms of affine invariants).

Keywords : invariant theory, nonlinear differential systems, qualitative theory of ordinary differential systems, critical points, saddle points.

1. Motivations and introduction

Many works (see the report by J. W. Reyn [Reyn, 1992]) are devoted to qualitative analysis of planar quadratic differential systems

$$\begin{aligned}\frac{dx}{dt} &= a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2,\\ \frac{dy}{dt} &= b_{0,0} + b_{1,0}x + b_{0,1}y + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2,\end{aligned} \tag{1.1}$$

where coefficients $a_{i,j}$ and $b_{i,j}$ are real.

The complete qualitative study of such system means the determination of the partition of the phase plane into trajectories. Following [Leontovitch and Maier, 1937] , this partition is completely defined by the number and the nature of critical points, the separatrix structure and the location of closed trajectories (the famous 16th Hilbert problem).

This work deals with the first class of problems.

In [Baltag and Vulpe, 1991, Vulpe, 1991] the authors established a complete tableau of the number and multiplicity of the critical points in the plane, including those at infinity and giving the corresponding described conditions. These conditions are algebraic and semi-algebraic (equalities and inequalities). They are given in terms of center-affine invariants and covariants.

[1]LACO, Département de Mathématiques, Faculté des Sciences, Université de Limoges, 123, Avenue A. Thomas, 87060, Limoges, France

From the symbolic computation point of view, this approach is very interesting since it provides a simple algorithm that gives number and multiplicity of critical points without solving (with radicals) algebraic equations.

The research on the nature of critical points falls into two different categories. The first one concerns the famous center-focus problem. Based on early work of Dulac (1908) and Kapteyn (1912), it has seen a series of contributions (see [Sibirskii, 1976] for the evolution of this question). Finally the explicit (in terms of invariants and covariants) conditions for finding systems (1.1) with one or two centers for the system (1.1) are obtained in [Boularas, Sibirskii and Vulpe, 1991].
The second direction is centered around the question of coexistence of critical points of different types and is initiated by Berlinskii [Berlinskii, 1960]. He established in particular theorem (3) below. However he has not characterised the possible situations by algebraic or semi-algebraic conditions on the coefficients of (1.1).
It was P. Curtz to give the first sufficient conditions expressed in terms of coefficients of systems (1.1) for the existence of one, two or three saddle point.
The goal of this note is to express these conditions with affine invariants. This leads us to consider the determinant of the Jacobian of the vector field associated to the system (1.1) :

$$q(x,y) = \begin{vmatrix} a_{1,0} + 2a_{2,0}x + a_{1,1}y & a_{0,1} + a_{1,1}x + 2a_{0,2}y \\ b_{1,0} + 2b_{2,0}x + b_{1,1}y & a_{0,1} + b_{1,1}x + 2b_{0,2}y \end{vmatrix}.$$

It is clear that a critical point (x_0, y_0) is a saddle point if, and only if, $q(x_0, y_0) < 0$. We call the algebraic curve $q(x, y) = 0$ the saddle conic.

In this note, we establish some algebraic and geometric properties of the polynomial $q(x)$ and give affine conditions of existence of one, two or three saddle points for system (1.1).
All computations are made with Maple and the package **SIB** which contains minimal systems of generators of center-affine and affine covariants of systems (1.1).

2. Review of invariants and covariants of differential systems

The planar quadratic differential systems with real coefficients form a $\mathbb{R}$-vector-space of dimension 12 (precisely, isomorphic to $\mathbb{R}^2 \oplus \mathbb{R}^2 \bigotimes (\mathbb{R}^2)^\star \oplus S_2 \bigotimes (\mathbb{R}^2)^\star$ where $(\mathbb{R}^2)^\star$ is the dual of $\mathbb{R}^2$ and S_2 the space of algebraic quadratic forms). Using Einstein notation, they can be written in the condensed form

$$\frac{dx^j}{dt} = a^j + a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \ (j, \alpha, \beta = 1, 2). \tag{2.2}$$

where $x = (x^1, x^2)^T \in \mathbb{R}^2$ (the letter T means transposed) and $a^j_\alpha x^\alpha = a^j_1 x^1 + a^j_2 x^2$, $a^j_{\alpha\beta} x^\alpha x^\beta = a^j_{11}(x^1)^2 + 2a^j_{12} x^1 x^2 + a^j_{22}(x^2)^2$. The Einstein notation will be adopted in the whole paper : we suppress the symbol $\sum$ (sum) in all contractions.
In addition let $Aff(2,\mathbb{R})$ be the group of affine transformations

$$x \longmapsto y = P^{-1}x - p \tag{2.3}$$

with

$$P = \begin{pmatrix} p^1_1 & p^1_2 \\ p^2_1 & p^2_2 \end{pmatrix}, \quad \det(P) \neq 0 \qquad \text{and} \qquad p = (p^1, p^2)^T.$$

It acts rationaly over $\mathcal{A}$ following the rational representation

$$\rho : G \longmapsto GL(\mathcal{A})$$

where $GL(\mathcal{A})$ is the group of automorphisms of $\mathcal{A}$. Putting $\rho(P,p)(a) = b$, this representation is defined by the formulae :

$$b^j = q^j_i (a^i + a^i_\alpha p^\alpha + a^i_{\alpha\beta} p^\alpha p^\beta),$$

$$b^j_\alpha = q^j_i p^\beta_\alpha (a^i_\beta + 2a^i_{\beta\gamma} p^\gamma),$$

$$b^j_{\alpha,\beta} = q^j_i p^\gamma_\alpha p^\delta_\beta a^i_{\gamma\delta},$$

where $Q = (q^j_i)$ is the inverse matrix of P.
Let us denote $\mathbb{R}[a, x]$ the algebra of polynomials whose indeterminates are components of a generic vector a of $\mathcal{A} \times \mathbb{R}^2$: $a^1, a^2, a^1_1, a^1_2, \ldots, a^1_{22}, a^2_{22}, x^1, x^2$. The representation of the group $Aff(2,\mathbb{R})$ on $GL(\mathcal{A} \times \mathbb{R}^2)$ is the direct sum of ρ and $Aff(2,\mathbb{R})$. It is denoted r.

Definition 1. *A polynomial function $K \in \mathbb{R}[a, x]$ is said to be a $Aff(2,\mathbb{R})$ - covariant of $\mathcal{A}$ if there exists a function $\lambda : Aff(2,\mathbb{R}) \to \mathbb{R}$ such that*

$$\forall g \in G, \quad (K \circ r)(g) = \lambda(g).K.$$

If $\lambda(g) \equiv 1$, then the invariant is said absolute. Otherwise, it is said relative.

An $Aff(2,\mathbb{R})$ - invariant is an $Aff(2,\mathbb{R})$ - covariant which does not depend on x.

It can be proved [Dieudonné and Carrel,1970] that the function λ is a character group of $Aff(2,\mathbb{R})$ and equal to $\det(P)^{-\kappa}$ where the integer κ is called the weight of the covariant (or invariant).
The above definitions hold for any subgroup of $Aff(2,\mathbb{R})$, in particular for the center-affine group denoted $Gl(2,\mathbb{R})$ (put in the affine group $p \equiv 0$) or the special

group denoted $Sl(2,\mathbb{R})$ ($\det(P) = 1$ and $p \equiv 0$).
The sets of $Sl(2,\mathbb{R})$-covariants or invariants and homogeneous $Gl(2,\mathbb{R})$-covariants (called also center-affine covariants) or $Gl(2,\mathbb{R})$-invariants (center-affine invariants) are the same. The algebras of $Sl(2,\mathbb{R})$-invariants and $Sl(2,\mathbb{R})$-covariants are finitely generated.
In [Boularas, 1999] a package denoted **SIB** is elaborated with Maple. It contains minimal systems of generators of the algebras of center-affine (denoted $J_1,\ldots,J_{36}$, $K_1,\ldots,K_{33}$) and affine covariants (denoted by $Q_1,\ldots,Q_{36}$). In appendix, we join the system of center-affine covariants (that of affine covariants is too long). Moreover this package allows to express any center-affine invariant (repectively affine covariant) in the above systems of generators and to compute affine invariants of a given degree : the affine invariants are regarded as linear relations of affine covariants that do not depend on vector $x = (x^1, x^2)^T$.

3. Algebraic Properties of the Saddle Conic

Let introduce for the systems (2.2) the quantities :

$$A_{00} = \begin{vmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{vmatrix},\ A_{i0} = \begin{vmatrix} a_{1i}^1 & a_2^1 \\ a_{1i}^2 & a_2^2 \end{vmatrix},\ A_{0i} = \begin{vmatrix} a_1^1 & a_{2i}^1 \\ a_1^2 & a_{2i}^2 \end{vmatrix},\ A_{ij} = \begin{vmatrix} a_{1i}^1 & a_{2j}^1 \\ a_{1i}^2 & a_{2j}^2 \end{vmatrix}.$$

The saddle conic of (2.2) has the expression (we represent here by x the previous vector (x,y)) :

$$q(x) = A_{00}+2(A_{10}+A_{01})x^1+2(A_{20}+A_{02})x^2+4[A_{11}(x^1)^2+(A_{12}+A_{21})x^1x^2+A_{22}(x^2)^2].$$

It contains all informations about the distribution of saddles and antisaddles in the phase plane.
Following [Boularas, 1999] the polynomial $q(x)$ is an affine *absolute* covariant :

$$q(x) = \frac{1}{2}(Q_1^2 - Q_2) = \frac{1}{2}[(J_1^2 - J_2) + 2(J_1K_1 - K_3) + 2(K_1^2 - K_7)],$$

where Q_1 and Q_2 are affine covariants. Let us consider its two discriminants :

$$4\delta_1 = \begin{vmatrix} 4A_{11} & 2(A_{12}+A_{21}) \\ 2(A_{12}+A_{21}) & 4A_{22} \end{vmatrix}, \tag{3.4}$$

$$2\delta_2 = \begin{vmatrix} 4A_{11} & 2(A_{12}+A_{21}) & A_{01}+A_{10} \\ 2(A_{12}+A_{21}) & 4A_{22} & A_{02}+A_{20} \\ A_{01}+A_{10} & A_{02}+A_{20} & A_{00} \end{vmatrix}. \tag{3.5}$$

With the help of package **SIB**, we obtain the affine invariants :

$$\delta_1 = 2J_7 - J_8 - J_9,$$

$$\delta_2 = 4J_1(J_{12} - J_{11}) - J_1^2(J_8 + J_9 - 2J_7) + 2J_2(J_9 - J_7) + 2(J_4 - J_5)(2J_3 - J_4 - J_5).$$

The total degree of δ_1 is 4 and that of δ_2 is 6.

In the following, all covariants will be regarded as elements of the ring $F[x^1, x^2]$ where F is the ring of polynomial functions over $\mathcal{A}$. A covariant is zero if and only if all its coefficients in x^1, x^2 are zero.

Lemma 1. **([Sibirskii, 1988],p.56)** *The quadratic homogeneous parts of the equations (2.2) have a common factor if, and only if $\delta_1 = 0$.*

Proof. The resultant of the polynomials $a_{11}^1(x^1)^2 + 2a_{12}^1x^1x^2 + a_{22}^1(x^2)^2$ and $a_{11}^2(x^1)^2 + 2a_{12}^2x^1x^2 + a_{22}^2(x^2)^2$ is equal to δ_1.

Lemma 2. *The differential system (2.2) can be reduced by a rotation into the form*

$$\begin{aligned} \frac{dx^1}{dt} &= a^1 + a_\alpha^1 x^\alpha, \\ \frac{dx^2}{dt} &= a^2 + a_\alpha^2 x^\alpha + a_{\alpha\beta}^2 x^\alpha x^\beta \end{aligned}$$

if, and only if $K_1^2 - K_7 = 0$.

Proof. The necessary condition is trivial. Suppose that $K_1^2 - K_7 = 0$. That means that

$$\begin{vmatrix} a_{11}^1 & a_{12}^1 \\ a_{11}^2 & a_{12}^2 \end{vmatrix} (x^1)^2 + \begin{vmatrix} a_{11}^1 & a_{22}^1 \\ a_{11}^2 & a_{22}^2 \end{vmatrix} x^1x^2 + \begin{vmatrix} a_{12}^1 & a_{22}^1 \\ a_{12}^2 & a_{22}^2 \end{vmatrix} (x^2)^2 = 0.$$

Consequently, there exists two real constants k_1 and k_2 such that $k_1^2 + k_2^2 = 1$ and $k_1a_{\alpha\beta}^1x^\alpha x^\beta + k_2a_{\alpha\beta}^2x^\alpha x^\beta = 0$.
The rotation $X^1 := k_1x^1 + k_2x^2$, $X^2 := -k_2x^1 + k_1x^2$ leads the initial system to the searched form.

It follows from this lemma

Proposition 1. *If the system (2.2) has four isolated critical points (real or complex), then $K_1^2 - K_7 \neq 0$.*

4. Geometric Properties of the Saddle Conic

Suppose that the system (2.2) has four isolated critical points. By [Coppel, 1966], any three of these points are never into the same straight. Then it is possible to find an affine transformation of the plane, denoted Φ such that the points $(0,0)^T = O$, $(0,1)^T = A$, $(1,0) = B$, and $(c,d) = D$ become critical for the transformed system :

$$\frac{dy^i}{dt} = b^i + b^i_\alpha y^\alpha + b^i_{\alpha\beta} y^\alpha y^\beta \ (i, \alpha, \beta = 1, 2) \tag{4.6}$$

whose coefficients verify the relations :

$$\begin{gathered} b^i = 0, b^i_1 = -b^i_{11},\, b^i_2 = -b^i_{22}, \quad (i = 1, 2) \\ b^i_{11} c(c-1)\, 2b^i_{12} cd + b^i_{22} d(d-1) = 0, \quad (i = 1, 2). \end{gathered} \tag{4.7}$$

Moreover $cd \neq 0$ and $c + d - 1 \neq 0$.
Let B_{ij}, $(i, j = 1, 2)$ be the transformed quantities of A_{ij} and $\tilde{\delta}_1, \tilde{\delta}_2$ the expressions of δ_1 and δ_2 where the A_{ij} are replaced by B_{ij}.
Since the affine invariants δ_1 and δ_2 are relative and of weight 2, we have :

$$\tilde{\delta}_1 = \Delta^{-2}\delta_1 \qquad \text{and} \qquad \tilde{\delta}_2 = \Delta^{-2}\delta_2.$$

where Δ is the determinant of the linear part of Φ.

Remark 1. *The signs of the affine invariants δ_1 and δ_2 do not change under affine transformation of the plane.*

We arrive to an interesting geometrical fact :

Lemma 3. *The quadrilateral whose vertices are the four isolated singular points of the quadratic system (2.2) is convex (resp. not convex) if and only if $\delta_1 = 2J_7 - J_8 - J_9 < 0$ (resp. $\delta_1 > 0$).*

Proof . Note that the quadrilateral is not convex if and only if one vertix lies in the triangle formed by other vertices. For systems (4.6 -4.7), the vertices O, A, B being fixed, le quadrilateral is convex if and only if $(c+d-1)cd > 0$. Taking into account the relations (4.7) we obtain

$$\tilde{\delta}_1 = -2\frac{(c+d-1)(b^1_{11}b^2_{22} - b^2_{11}b^1_{22})^2}{cd}.$$

Moreover, $B_{12} = b^1_{11}b^2_{22} - b^2_{11}b^1_{22} \neq 0$, because $K_1^2 - K_7 \neq 0$. This achieves the proof.

A saddle point is an elementary critical point whose corresponding linearised system admits real eigenvalues of opposite signs. Its geometrical index is equal to -1. All others elementary critical points (nodes, center and focus) of geometrical index $+1$ are called anti-saddles.
To know whether a given critical point x_0 is a saddle or not we have to compute the determinant of the linearised part around the considered point i.e., $q(x_0)$: x_0 is a saddle if and only if $q(x_0) < 0$. In the case of four isolated critical points we get the following result

Theorem 1. *Suppose that there are four real critical points. If the quadrilateral with vertices at the points is convex then two opposite critical points are saddles and the other two are antisaddles. But if the quadilateral is not convex then either the three exterior vertices are saddles and the interior antisaddle or the exterior vertices are antisaddles and the interior vertex a saddle.*

This theorem is etablishaed the first time by Berlinski [Berlinskii, 1960].
Proof [Coppel, 1966] . After substitution x_0 by critical points O, A, B and D in (4.6 - 4.7), we obtain :

$$q(0) = B_{12}, \quad q(A) = -\frac{(c+d-1)B_{12}}{d},$$

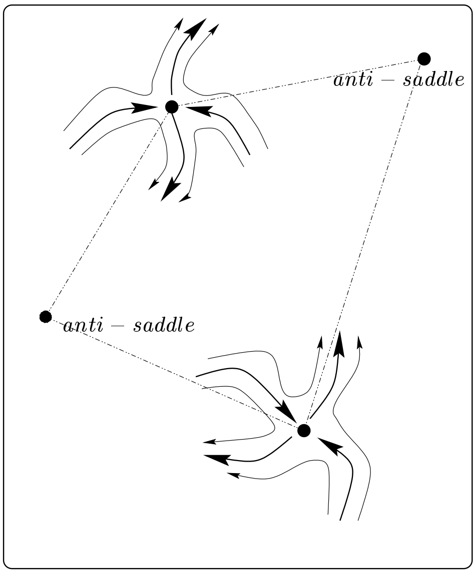
$$q(B) = -\frac{(c+d-1)B_{12}}{c}, \quad q(D) = (c+d-1)B_{12}.$$

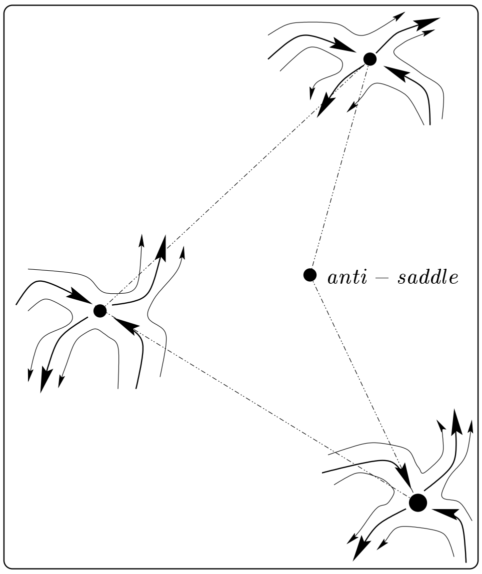
Consequently,

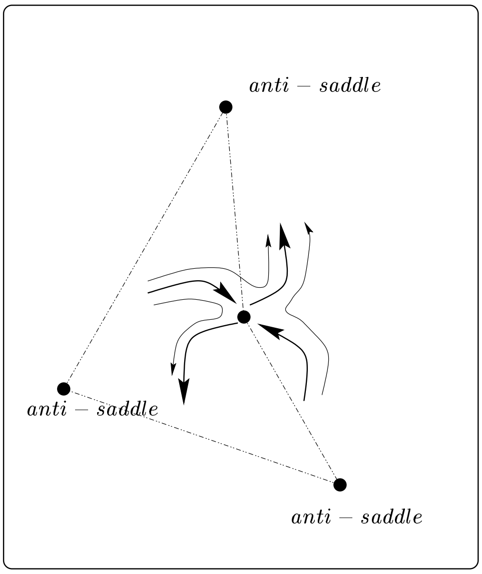
$$q(0)q(A)q(B)q(D) = \frac{(c+d-1)^3 B_{12}^4}{cd}.$$

If $\tilde{\delta}_1 < 0$, the quadrilateral OABD is convex and $q(0)q(A)q(B)q(D) > 0$. There are three possibilities : 0, two or four saddles.

The first and third cases cannot hold because $q(0)$ and $q(D)$ have the same sign positive (resp. negative) sign. Thus, $c + d > 1$ (resp. $c + d < 1$). Taking into account the inequality $\frac{(c+d-1)}{cd} > 0$ necessairily $cd > 0$ (resp. $cd < 0$) and $q(A)$ and $q(B)$ must be of negative (rep. opposite) sign. If $\tilde{\delta}_1 > 0$, then the quadrilateral OABD is not convex and $q(0)q(A)q(B)q(D) < 0$. That implies that there exists one or three saddle points.







Using the same tool (Poincaré's index of vectors fields around critical points) it was proposed another simplified proof of this theorem in [Sestier, 1980].
Actually, the second discriminant of the saddle conic may distinguish the cases of one and three saddle points.

Theorem 2. *Suppose that the differential system (2.2) admits four real isolated critical points. Then it has*

- *one saddle point if and only if* $\delta_1 > 0$ *and* $\delta_2 < 0$,

- *two saddle points if and only if* $\delta_1 < 0$,
- *three saddle points if and only if* $\delta_1 > 0$ *and* $\delta_2 > 0$.

Proof. For the systems 4.6, 4.7, we have

$$\tilde{\delta}_2 = \frac{(c+d-1)(c+d)(c-1)(d-1)(b^1_{11}b^2_{22} - b^2_{11}b^1_{22})^3}{c^2d^2}.$$

Suppose that $\tilde{\delta}_1 > 0$ i.e., $\dfrac{(c+d-1)}{cd} < 0$ and 0 is a saddle point. If $q(D)$ is of negative sign, then $c+d-1 > 0$ and c and d of opposite sign. Thus one of the points A or B is saddle. Without loss of generality, we can suppose that A is a saddle point. Then $c < 0$, $d > 1$ and $c+d > 1 > 0$. Necessarily, $\tilde{\delta}_2 > 0$.
Suppose that $q(D)$ is of positive sign, then $c+d-1 < 0$ and c and d have the same sign. If c and d are of negative sign, then there is three antisaddle points and $\tilde{\delta}_2 < 0$. If c and d are of positive sign, then A and B are of type saddle and $0 < c < 1$, $0 < d < 1$. There are three saddle points and $\tilde{\delta}_2 > 0$.

This result is partially obtained in [**?**].

5. Appendix

5.1. Minimal system of generators of the algebra of center-affine invariants.

$$J_1 = a^\alpha_\alpha, \quad J_2 = a^\alpha_\beta a^\beta_\alpha, \quad J_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq}, \quad J_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq},$$

$$J_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq}, \; J_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta}\varepsilon^{pq}, \; J_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\beta s} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},$$

$$J_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs}, \; J_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs}, \; J_{10} = a^\alpha_p a^\beta_\delta a^\gamma_\mu a^\delta_{\alpha q} a^\mu_{\beta\gamma}\varepsilon^{pq},$$

$$J_{11} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs}, \; J_{12} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\delta} a^\mu_{\gamma\mu}\varepsilon^{pq}\varepsilon^{rs},$$

$$J_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs}, \; J_{14} = a^\alpha_p a^\beta_r a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},$$

$$J_{15} = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{\alpha s} a^\delta_{\delta l} a^\mu_{\beta\gamma} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl}, \; J_{16} = a^\alpha_p a^\beta_r a^\gamma_\delta a^\delta_{\alpha q} a^\mu_{\beta s} a^\nu_{\gamma\tau} a^\tau_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},$$

$$J_{17} = a^\alpha_{\alpha\beta} a^\beta, \quad J_{18} = a^p_\alpha a^\alpha a^q \varepsilon_{pq}, \quad J_{19} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma, \quad J_{20} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma,$$

$$J_{21} = a^p_{\alpha\beta} a^\alpha a^\beta a^q \varepsilon_{pq}, \; J_{22} = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} a^\gamma a^\delta, \; J_{23} = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} a^\gamma a^\delta,$$

$$J_{24} = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} a^\delta, \; J_{25} = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} a^\delta \varepsilon^{pq}, \quad J_{26} = a^\alpha_{\alpha p} a^\beta_{\delta q} a^\gamma_{\beta\gamma} a^\delta \varepsilon^{pq},$$

$$J_{27}=a^p_\alpha a^\alpha_{\beta\gamma}a^\beta a^\gamma a^q\varepsilon_{pq},\ J_{28}=a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}a^\delta a^\mu,\ J_{29}=a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}a^\delta a^\mu,$$
$$J_{30}=a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\delta}a^\delta_{\gamma\mu}a^\mu\varepsilon^{pq},\qquad J_{31}=a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\mu}a^\delta_{\gamma\delta}a^\mu\varepsilon^{pq},$$
$$J_{32}=a^\alpha_p a^\beta_{\beta q}a^\gamma_{\alpha\mu}a^\delta_{\gamma\delta}a^\mu\varepsilon^{pq},\qquad J_{33}=a^\alpha_{\beta\nu}a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}a^\delta a^\mu a^\nu,$$
$$J_{34}=a^\alpha_{\mu p}a^\beta_{\alpha q}a^\gamma_{\beta\nu}a^\delta_{\gamma\delta}a^\mu a^\nu\varepsilon^{pq},\qquad J_{35}=a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q}a^\delta_{\beta\mu}a^\mu_{\gamma\delta}a^\nu\varepsilon^{pq},$$
$$J_{36}=a^\alpha_{pr}a^\beta_{\nu q}a^\gamma_{\alpha s}a^\delta_{\beta\gamma}a^\mu_{\delta\mu}a^\nu\varepsilon^{pq}\varepsilon^{rs}.$$

5.2. Minimal system of generators of the algebra of center-affine covariants. It is formed by the minimal system of genrators of the algebra of center-affine invariants and the following covariants :

$$K_1=a^\alpha_{\alpha\beta}x^\beta,\quad K_2=a^p_\alpha x^\alpha x^q\varepsilon_{pq},\quad K_3=a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma,\quad K_4=a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma,$$
$$K_5=a^p_{\alpha\beta}x^\alpha x^\beta x^q\varepsilon_{pq},\ K_6=a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta,\ K_7=a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta,$$
$$K_8=a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta}x^\delta,\ K_9=a^\alpha_{\alpha p}a^\beta_{\gamma q}a^\gamma_{\beta\delta}x^\delta\varepsilon^{pq},\quad K_{10}=a^\alpha_{\alpha p}a^\beta_{\delta q}a^\gamma_{\beta\gamma}x^\delta\varepsilon^{pq},$$
$$K_{11}=a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q\varepsilon_{pq},\ K_{12}=a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu,\ K_{13}=a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu,$$
$$K_{14}=a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\delta}a^\delta_{\gamma\mu}a^\mu\varepsilon^{pq},\qquad K_{15}=a^\alpha_p a^\beta_{\alpha q}a^\gamma_{\beta\mu}a^\delta_{\gamma\delta}x^\mu\varepsilon^{pq},$$
$$K_{16}=a^\alpha_p a^\beta_{\beta q}a^\gamma_{\alpha\mu}a^\delta_{\gamma\delta}a^\mu\varepsilon^{pq},\qquad K_{17}=a^\alpha_{\beta\nu}a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu x^\nu,$$
$$K_{18}=a^\alpha_{\mu p}a^\beta_{\alpha q}a^\gamma_{\beta\nu}a^\delta_{\gamma\delta}x^\mu x^\nu\varepsilon^{pq},\qquad K_{19}=a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q}a^\delta_{\beta\mu}a^\mu_{\gamma\delta}x^\nu\varepsilon^{pq},$$
$$K_{20}=a^\alpha_{pr}a^\beta_{\nu q}a^\gamma_{\alpha s}a^\delta_{\beta\gamma}a^\mu_{\delta\mu}x^\nu\varepsilon^{pq}\varepsilon^{rs},\quad K_{21}=a^p x^q\varepsilon_{pq},\quad K_{22}=a^p_\alpha a^\alpha x^q\varepsilon_{pq},$$
$$K_{23}=a^\alpha a^\beta a^p_{\alpha\beta}x^q\varepsilon_{pq},\ K_{24}=a^\alpha a^{beta}_{\alpha\beta}a^\gamma_{\gamma\delta}x^\delta,\ K_{25}=a^\alpha a^\beta_{\alpha\gamma}a^\gamma_{\beta\delta}x^\delta,$$
$$K_{26}=a^\alpha a^\beta a^p_\gamma a^\gamma_{\alpha\beta}x^q\varepsilon_{pq},\ K_{27}=a^\alpha a^\beta_\gamma a^\gamma_{\beta\delta}a^\delta_{\alpha\mu}x^\mu,\ K_{28}=a^\alpha a^\beta_\delta a^\gamma_{\beta\gamma}a^\delta_{\alpha\mu}x^\mu,$$
$$K_{29}=a^\alpha a^\beta a^\gamma_{\delta\nu}a^\delta_{\gamma\mu}a^\mu_{\alpha\beta}x^\nu,\ K_{30}=a^\alpha a^\beta_{\alpha p}a^\gamma_{\beta q}a^\delta_{\gamma\nu}a^\mu_{\delta\mu}x^\nu\varepsilon^{pq},$$
$$K_{31}=a^p a^q_{\alpha\beta}x^\alpha x^\beta\varepsilon_{pq},\ K_{32}=a^p a^q_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma\varepsilon_{pq},\ K_{33}=a^\alpha a^\beta_{\alpha\gamma}a^\gamma_{\beta\delta}a^\delta_{\mu\nu}x^\mu x^\nu,$$

References

[Leontovitch and Maier, 1937] Leontovitch E. A. Maier A. G. (1937), About trajectories which define the qualitative structure of the partition of the sphere into trajectories, *Doklady AN SSSR, 14 (1937), 251 - 254.*

[Berlinskii, 1960] Berlinskii A. N. (1960), On the behaviour of integral curves of some differential equations. *Izvestija Vyssh. Uchebn Zaved. Matematika, 2, 3-18.*

[Dieudonné and Carrel,1970] Dieudonné J. , Carrell J. (1970), Invariant theory, old and new, *Advances in Math. 4 (1970), 1-80.*

[Sibirskii, 1976] Sibirskii C. S. (1976) , Algebraic invariants of differential equations and matrices, *Schtiintsa, Kishinev, Moldova* (in russian).

[Sibirskii, 1988] Sibirskii C. S. (1988), Introduction to the algebraic theory of invariants of differential equations, *Nonlinear Science, Theory and Applications, Manchester University Press, 1988*

[Boularas, Sibirskii and Vulpe, 1991] Boularas D., Vulpe N. I., Sibirskii C. S., The problem of a center in the large for a general quadratic system, *Soviet Math. Dokl., 41, N°2, 287-290, Translation by AMS of Dokl. Akad. Nauk SSSR, 311, N°4.*

[Boularas, 1999] Boularas D. (1999) , Covariants of differential systems, *submitted*

[Coppel, 1966] Coppel W.A (1966), A survey of quadratic systems, *Journal of Differential Equations 2 , 293-304.*

[Curtz, 1980] Curtz P., Stabilité locale des systèmes quadratiques, *Ann.Scient.Ec. Norm.Sup. (Paris) T. 13, N° 4 , 293-302.*

[Curtz, 1981] Curtz P., Nœuds des systmes quadratiques, *C.R.Acad.Sc.Paris, Srie I, t.293, (Nov. 1981), 521-524.*

[Reyn, 1991] Reyn J.W. (1991) , Classes of quadratic systems of differential equations in the plane, Nakai Series in *Pure, Applied mathematics and Theoretical Physics, vol 4, Dynamical Systems, Procedings of the special program Of Nakai Institue of Mathematics, Tianjin, China, September 1990-June 1991.*

[Reyn, 1992] Reyn J.W. (1992), A bibliography of the theory of quadratic systems of differential equations in the plane, second edition, *Report 92-17 (1992), Delft University of Technology, The Netherlands.*

[Sestier, 1980] Sestier A., Note on a theorem of Berlinskii, *Proceedings of the AMS, Volume 78, N^o 3 , 358-360.*

[Baltag and Vulpe, 1991] Baltag L. , Vulpe N. I. (1991), Number and multiplicity of critical points of a differential system. *Izv.Akad.Nauk Respub.Moldova, Buletinul. Matematica] 1992, no. 3, 3–8, 93*

[Vulpe, 1991] Vulpe N. I. (1991), Number and multiplicity of critical points at infinity of quadratic differential system. *Differential Equations, 27 (1991), n 4*