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Calcul des covariants affines des systèmes différentiels

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COMPUTATION OF AFFINE COVARIANTS OF DIFFERENTIAL SYSTEMS

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Abstract. This paper deals with affine covariants of autonomous differential systems. We give a constructive method for computing them. This method allows in particular to deduce a (minimal) system of generators of the algebra of affine covariants from one of centro-afffine invariants. In the case of planar quadratic differential systems, we give a minimal system of the algebra of center-affine covariants and using the previous method we construct a minimal system of generators of the algebra of affine covariants. Computations are made with Maple. All algorithms constitute the package SIB.
Key words. Nonlinear differential systems, transformations of differential systems, classical invariant theory, classical groups, covariants.
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1. Motivations

Let $\mathcal{C}$ be the real or complex field, V the $\mathcal{C}$-vector space $\mathcal{C}^n$ and $\mathcal{A}(n,m)$ the set of autonomous differential systems

$$\frac{dx^j}{dt} = \sum_{k=0}^{m}\sum_{\alpha_1=1}^{n}\cdots\sum_{\alpha_k=1}^{n} a^j_{\alpha_1\alpha_2\ldots\alpha_k} x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_k},\ j\in\{1,\ldots,n\} \tag{1.1}$$

where $x = {}^T(x^1, x^2, \ldots, x^n) \in V$ is represented with superscript indices and coefficients $a^j_{\alpha_1\alpha_2\ldots\alpha_k}$ and "time" t belong to $\mathcal{C}$. The right hand of (1.1) is an n - tuple of polynomials of degree at most m.

This paper is motivated by different applications of classical invariant theory in the study of differential systems (1.1) [10, 14] : for instance, the characterisation of each group orbit (normal forms, ...) or the description of particular trajectories (singularities, ...).

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Many works [9, 10, 11] are devoted to invariants of differential systems with respect to the group of invertible matrices called the center-affine group. Systems of generators of invariants are constructed and used in qualitative study of differential systems (number and nature of critical points, first and particular algebraic integrals, symmetry axes of vector fields, ...). From the point of view of symbolic computation, this approach is important and fruitful because it allows us to express geometric properties of trajectories with the help of algebraic or semi-algebraic relations which depend on the coefficients of given systems. Furthermore covariants may be used to solve some problems that arise with numerical methods.
The idea to use systematically the classical invariant theory in qualitative theory of differential systems is due to C.S. Sibirskii ([9, 10]).
Up to now, few mathematicians have used the center-affine group of transformations of the phase space V. Our contribution is to propose a systematic method to construct systems of generators of affine covariants from those of center-affine invariants.
This method is easy to implement; we have done it with Maple for planar quadratic systems.

Other motivation of this work could be related to invariant theory. It is well-known that if a group is reductive, the algebra of invariants of all its representations are finitely generated. Few people try to determine the upper bound of degrees of generators of a given algebra of invariants. In ([1]), the authors give a complete and interesting description of this problem. However, up to now, there is no constructive method that gives a good bound ([1]). Only some algebraic forms and families of matrices give good examples of exact upper bound of generators. The two-dimensional quadratic systems give another example.

2. Introduction and notations

The space of algebraic forms of degree k can be looked upon as the quotient space of $V^{*\otimes k}$ by the subspace generated by all elements

$$x_1 \otimes \ldots \otimes x_i \otimes x_{i+1} \ldots \otimes x_k - x_1 \otimes \cdots \otimes x_{i+1} \otimes x_i \cdots \otimes x_k.$$

We denote $\mathcal{S}_k$ this space. The homogeneous part of degree k of the polynomial vector field (1.1) is a n-tuple of elements of $\mathcal{S}_k$. It should be identified with the vector space $S_k \bigotimes V^*$ denoted by $\mathcal{S}_k^1$. For example, $\mathcal{S}_1^1$ is the linear part of (1.1). In tensorial language, $\mathcal{S}_k^1$ is the space of tensors once contravariant and k times covariant which are symmetric with respect to the subscript indices.

Consequently,

$$\mathcal{A}(n,m) \simeq S_0^1 \oplus S_1^1 \oplus \cdots \oplus S_m^1.$$

Let G be a linear group acting rationally on a finite-dimensional vector space $\mathcal{W}$, $GL(\mathcal{W})$ the group of automorphisms of $\mathcal{W}$ and

$$\rho \ : G \longmapsto GL(\mathcal{W})$$

the corresponding rational representation. Let $\mathcal{C}[\mathcal{W}]$ be the algebra of polynomials whose indeterminates are the coordinates of a generic vector of $\mathcal{W}$.

Definition 1. *A polynomial function $K \in \mathcal{C}[\mathcal{W}]$ is said to be a G - invariant of $\mathcal{W}$ if there exists a character λ of the group G such that*

$$\forall g \in G, \quad K \circ \rho(g) = \lambda(g).K.$$

Here, The character of the group G is a rational (commutative) morphism of group G into $\mathcal{C}_m$ where $\mathcal{C}_m$ is the multiplicative group of $\mathcal{C}$.
If $\lambda(g) \equiv 1$, then the invariant is said absolute. Otherwise, it is said relative.

In our situation, $\mathcal{W}$ should be $\mathcal{A}(n,m)$ or $\mathcal{A}(n,m) \times V$ and the group G one of the following classical groups :

1. $Gl(n)$: group of center-affine transformations or invertible matrices,
2. $T(n)$: group of translations,
3. $\mathit{Aff}(n) = T(n) \ltimes Gl(n)$: semi-direct group of affine transformations.

In this work, the notion of covariant is taken in the following precise sense :

Definition 2. *A G - covariant of $\mathcal{A}(n,m)$ is a G - invariant of $\mathcal{A}(n,m) \times V$.*

For example, $\det(x, Ax, A^2x, \ldots, A^{n-1}x)$ where A is the linear part of (1.1) is a relative $Gl(n)$ - covariant of $\mathcal{A}(n,m)$.
Through this paper, we have a great deal with tensors. For sake of reasons of simplification of expressions we shall use the Einstein summation convention (for instance, $a^j_{\alpha_1\alpha_2}a^{\alpha_1}_{j\alpha_2}$ stands for the same as $\displaystyle\sum_{j=1}^{n}\sum_{\alpha_1=1}^{n} a^j_{\alpha_1\alpha_2}a^{\alpha_1}_{j\alpha_2}$).
Using this notations, the system (1.1) becomes

$$\begin{aligned}\frac{dx^j}{dt} &= a^j + a^j_{\alpha_1}x^{\alpha_1} + a^j_{\alpha_1\alpha_2}x^{\alpha_1}x^{\alpha_2} + \ldots + a^j_{\alpha_1\alpha_2\ldots\alpha_m}x^{\alpha_1}x^{\alpha_2}\ldots x^{\alpha_m}, \\ & \quad j, \alpha_1, \alpha_2, \ldots, \alpha_m \in \{1, \ldots, n\}\end{aligned} \tag{2.2}$$

The transformations $(x^i) \mapsto (Q^i_j x^j)$, $(x^i) \mapsto (x^i - p^i)$, $(x^i) \mapsto (Q^i_j x^j - p^i)$ of the above groups $Gl(n)$, $T(n)$ and $Aff(n)$ transform each system (1.1) of $\mathcal{A}(n,m)$ into the system of $\mathcal{A}(n,m)$ defined by the formula :

$$[\rho_1(P)(a)]^j_{\alpha_1\alpha_2\ldots\alpha_k} = Q^j_i P^{\beta_1}_{\alpha_1} P^{\beta_2}_{\alpha_2} \ldots P^{\beta_k}_{\alpha_k} a^i_{\beta_1\beta_2\ldots\beta_k}, \tag{2.3}$$

$$[\rho_2(p)(a)]^j_{\alpha_1\alpha_2\ldots\alpha_k} = \sum_{i=0}^{m-k} \binom{k+i}{i} a^j_{\alpha_1\alpha_2\ldots\alpha_k\beta_1\beta_2\ldots\beta_i} p^{\beta_1} p^{\beta_2} \ldots p^{\beta_i}, \tag{2.4}$$

$$[\rho_3(P,p)(a)]^j_{\alpha_1\alpha_2\ldots\alpha_k} =$$

$$\sum_{i=0}^{m-k} \binom{k+i}{i} Q^j_i P^{\gamma_1}_{\alpha_1} P^{\gamma_2}_{\alpha_2} \ldots P^{\gamma_k}_{\alpha_k} a^j_{\gamma_1\gamma_2\ldots\gamma_k\beta_1\beta_2\ldots\beta_i} p^{\beta_1} p^{\beta_2} \ldots p^{\beta_i}, \tag{2.5}$$

where P is the inverse of the matrix Q (or the linear part of g) of the corresponding transformation and $\binom{k+i}{i}$ is the binomial coefficient $\frac{(k+i)!}{k!i!}$.
In definition 1, when G is one of the groups $Gl(n)$, $T(n)$, $Aff(n)$, it is well-known ([2, 10]) that the character λ is equal to $det(g)^{-\kappa}$ where the integer $-\kappa$ is called the weight of the invariant K. Clearly, every relative invariant of $Gl(n)$ is an absolute invariant of $Sl(n)$. The converse is also true (for homogeneous polynomials). This allows us to consider the set of $Gl(n)$ - covariants (respectively invariants) of $\mathcal{A}(n,m)$ as an $\mathcal{C}$ - algebra, denoted by $\mathcal{K}(n,m)$ (respectively $\mathcal{I}(n,m)$). Similarly, the set of $Aff(n)$ - covariants (respectively $Aff(n)$ - invariants) of $\mathcal{A}(n,m)$ is a $\mathcal{C}$ - algebra, denoted $\mathcal{Q}(n,m)$ (respectively $\mathcal{J}(n,m)$). It is obvious that

$$\mathcal{J}(n,m) \subset \mathcal{I}(n,m) \text{ and } \mathcal{Q}(n,m) \subset \mathcal{K}(n,m).$$

Let us return to the general case of a group G and a vector space $\mathcal{W}$. Denote by $\mathcal{C}[\mathcal{W}]^G$ the algebra of $Gl(n)$ - invariants of $\mathcal{W}$. Following V.L. Popov [7], the main problem of the classical theory of invariants is to describe explicitly the algebras $\mathcal{C}[\mathcal{W}]^G$. The idea of the description is as follows :

1. see whether $\mathcal{C}[\mathcal{W}]^G$ has a finite system of generators;

2. if it does, give a constructive method for finding a minimal system (ideally, find it exlicitly) of generators of $\mathcal{C}[\mathcal{W}]^G$.

For the reductive groups G (like $Gl(n)$ [1], this fact is known from Hilbert) the first problem is positively solved. However this is not the case for the affine group.

The main results of this work consist in the following. After recalling some definitions and results about center-affine invariants (§ 3) we construct effectively an isomorphism between the algebras $\mathcal{I}(n,m)$ and $\mathcal{Q}(n,m)$ (§ 4). That means in particular that we are able to deduce a minimal system of generators of $\mathcal{Q}(n,m)$ from that one of $\mathcal{I}(n,m)$. This is the situation of $\mathcal{I}(2,2)$. Of course the affine covariants are polynomials in center-affine covariants. In section § 5 we achieve the construction of a minimal system of generators of $\mathcal{K}(2,2)$. In the end (§ 6) we give an effective computation with Maple of a minimal system of generators of $\mathcal{Q}(2,2)$.
The produced algorithms constitute the package SIB. This package contains the minimal systems of generators of $\mathcal{I}(n,m)$, $\mathcal{K}(n,m)$ and $\mathcal{Q}(n,m)$. It permits to express any center-affine or affine covariant with repect to the center-affine generators and computation for each multi-degree the corresponding syzygies.

The affine covariants may be used in the study of normal forms and the qualitative behaviour of trajectories of systems (2.2) whose properties are invariant w.r.t. the affine group.

3. Recalls about Center-affine invariants

In this section we are interested in center-affine invariants and covariants of $\mathcal{A}(n,m)$. As a $Gl(n)$-module, $\mathcal{A}(n,m)$ is a direct sum of the subspaces $S^1_k, k \in \{0,1,2,\ldots,m\}$.
The algebra $\mathcal{K}(n,m)$ is multigraded :

$$\mathcal{K}(n,m) = \bigoplus_{n_0,n_1,n_2,\ldots n_m,p\in\mathbb{N}} K(n_0,n_1,\ldots,n_m,p)$$

where $K(n_0,n_1,\ldots,n_m,p)$ is the subalgebra of multi-homogeneous covariants of multi-degree $(n_0,n_1,\ldots,n_m,p)$, i.e. homogeneous of degree n_k with respect to coordinates of $(a^j_{i_1,i_2,\ldots i_k})$ and of degree p with respect to coordinates of $x \in V$.
In the same way, we decompose $\mathcal{I}(n,m)$ as follows

$$\mathcal{I}(n,m) = \bigoplus_{n_0,n_1,n_2,\ldots n_m\in\mathbb{N}} I(n_0,n_1,\ldots,n_m).$$

Before giving the fundamental theorem of center-affine invariants, let us recall the definitions of two fundamental operations over tensors which are the contraction and the alternation.
In the following definitions, we identify each element of a vector space with its coordinates in a selected basis.

Definition 3. *A contraction over the tensorial space* $V^{\otimes p} \otimes V^{*\otimes q}$ *is the linear map*

$$\varphi \ : V^{\otimes p} \otimes V^{*\otimes q} \longmapsto V^{\otimes (p-1)} \otimes V^{*\otimes (q-1)}$$

defined by

$$\varphi(v)^{j_1..\cancel{j_l}..j_p}_{i_1..\cancel{i_k}..i_q} = \sum_{h=1}^{n} v^{j_1...j_{l-1}hj_{l+1}...j_p}_{i_1...i_{k-1}hi_{k+1}...i_q}.$$

If $p = q$, *a sequence of contractions*

$$V^{\otimes p} \otimes V^{*\otimes p} \overset{\varphi_1}{\longmapsto} V^{\otimes (p-1)} \otimes V^{*\otimes (p-1)} \overset{\varphi_2}{\longmapsto} \cdots \overset{\varphi_p}{\longmapsto} V \otimes V^* \overset{\varphi_{p+1}}{\longmapsto} \mathcal{C}[V^{\otimes p} \otimes V^{*\otimes q}]$$

is called a complete contraction.

For each pair of natural numbers (l, k) with $1 \leq l \leq p$, $1 \leq l' \leq q$ we get one contraction.

Definition 4. *A contravariant (respectively covariant) n-vector is an element of* $V^{\otimes n}$ *(respectively* $V^{*\otimes n}$*) whose coordinates* $(\varepsilon^{i_1 i_2 \cdots i_n})$ $(\varepsilon_{i_1 i_2 \cdots i_n})$ *are defined by*

$$\varepsilon^{i_1 i_2 \cdots i_n} = \varepsilon_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & \text{if } (i_1 i_2 \cdots i_n) \text{ is an even permutation of } (12 \cdots n) \\ -1 & \text{if } (i_1 i_2 \cdots i_n) \text{ is an odd permutation of } (12 \cdots n) \\ 0 & \text{otherwise} \end{cases}$$

Definition 5. *A contravariant (covariant) alternation over the tensorial space* $V^{\otimes p} \otimes V^{*\otimes q}$ *is the map*

$$\varphi \ : V^{\otimes p} \otimes V^{*\otimes q} \longmapsto V^{\otimes (p)} \otimes V^{*\otimes (q-n)}$$

$$(\varphi : V^{\otimes p} \otimes V^{*\otimes q} \longmapsto V^{\otimes (p-n)} \otimes V^{*\otimes (q)})$$

that is defined by

$$\varphi(v)^{j_1...j_p}_{i_1..\cancel{h_1}..\cancel{h_2}.....\cancel{h_n}...i_q} = \sum_{h_1=1}^{n} \cdots \sum_{h_n=1}^{n} v^{j_1......j_p}_{i_1...h_1...h_2......h_n...i_q} \varepsilon^{h_1 \cdots h_n}$$

$$\left(\varphi(v)^{j_1..\cancel{h_1}..\cancel{h_2}.....\cancel{h_n}...j_p}_{i_1...i_q} = \sum_{h_1=1}^{n} \cdots \sum_{h_n=1}^{n} v^{j_1...h_1...h_2......h_n...j_p}_{i_1......i_q} \varepsilon_{h_1 \cdots h_n} \right)$$

Now, we are able to give the fundamental theorem of the classical theory of invariants :

Theorem 1. ([3], p.188-189) *The expressions obtained with the help of succesive alternations and complete contraction over the tensorial products*

$$(S_0^1)^{\otimes n_0} \otimes (S_1^1)^{\otimes n_1} \otimes \cdots \otimes (S_m^1)^{\otimes n_m} \otimes V^{\otimes r} \subset V^{*\otimes p} \otimes V^{\otimes q}$$

with $p = (n_0 + n_1 + \cdots + n_m + r)$ *and* $q = (n_1 + 2n_2 + \cdots + mn_m)$ *form a system of generators of* $K(n_0, n_1, \ldots, n_m, r)$.
Such polynomials are called basic covariants.

Example :
For instance, if $n = m = 2$, the polynomials $a_p^\alpha a_{\alpha q}^\beta a_{\beta\gamma}^\gamma \varepsilon^{pq}$ and $a_\gamma^\alpha a_\delta^\beta a_{\alpha\beta}^\gamma x^\delta$ belong respectively to $I(0,1,2,0)$ and $K(0,2,1,1)$. We shall come back to this case in the third section.

In order to have a complete contraction, the exponents $(n_0, n_2, \ldots, n_m, r)$ have to verify the relation

$$n_0 + r - (n_2 + 2n_3 + \cdots (m-1)n_m) = sn \text{ with } s \text{ integer .}$$

The above theorem is known as the fundamental theorem of the classical theory of invariants [3]. It can be presented with the contractions and determinants (alternations) ([13, 2]) of contravariant and covariant vectors obtained from the tensors of S_k^1 and V by symbolic decomposition. It defines a process of construction of covariants of $\mathcal{A}(n,m)$, degree by degree and gives for each suitable choice of alternations and contractions an element of $\mathcal{C}[\mathcal{A}(n,m) \times V]$. To get a minimal system of generators we need to know an upper bound of degrees of these generators which we denote by $\beta(n,m)$. This bound has been calculated by V. Popov [8] and recently improved by H. Derksen [1]. From our point of view, it is still too large for concrete examples. Indeed, let $\sigma(2,2)$ be the smallest integer d with the following property : if $a \in \mathcal{A}(2,2)$ and 0 doesn't lie in the Zariski closure of the $GL(2,\mathcal{C})$-orbit of a, then there exists a non-constant homogeneous invariant I of degree $\leq d$ such that $I(a) \neq 0$. In the case of planar quadratic differential systems (the list of a minimal system of generators of $\mathcal{I}(2,2)$ is given in the section 5 and the system of generators of the ideal of sygyzies is given in [10]), $\sigma(2,2) = 6$ and $\dim(\mathcal{I}(2,2)) = 9$. Following the first author,

$$\beta(2,2) \leq 12.LCM(1,2,\ldots,\sigma(2,2)) = 12.60 = 720$$

where LCM is the least common multiple. Following the second author,

$$\beta(2,2) \leq \max(\sigma(2,2), \frac{3}{8}\dim(\mathcal{I}(2,2))\sigma(2,2)^2) = \frac{243}{2}.$$

We should not forget that the dimension of the linear space $\mathcal{A}(2,2)$ is 12 and so, computations over polynomials depending on 12 indeterminates become fastly complicated.
Truly, $\beta(2,2) = 7$.

Let us return to computation of center-affine covariants. Actually, it is sufficient to consider the basic covariants with either the contravariant n-vectors or the covariant n-vectors. Indeed, by following relations between the coordinates of n-vectors and those of Kronecker's symbols

$$\varepsilon^{i_1 i_2 \cdots i_n}\varepsilon_{j_1 j_2 \cdots j_n} = \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_n} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_n} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_n}_{j_1} & \delta^{i_n}_{j_2} & \cdots & \delta^{i_n}_{j_n} \end{vmatrix}, \tag{3.6}$$

any basic covariant can be reduced to sum of basic covariants that contain n-vectors of one kind.

Examples.

Suppose $m = 1$ and $n = 2$. Systems (2.2) become

$$\frac{dx}{dt} = a + Ax.$$

where a is an element of V and A, an 2×2 matrix.

1. The polynomial $A^p_r A^q_s \varepsilon_{pq}\varepsilon^{rs}$ (the half of the determinant of the matrix A) is nothing else that $A^p_p A^r_s - A^p_r A^r_p = (Trace(A))^2 - Trace(A^2)$.
2. The center-affine invariants of (3) containing the vector a and the matrix A are necessairly the multiples of $det(a, Aa)$.

Remark 1. *Taking into account the Einstein symbolic notation, we remark that a total contraction over* $V^{\otimes p} \otimes V^{*\otimes p}$ *is a polynomial function* $\varphi : V^{\otimes p} \otimes V^{*\otimes p} \longmapsto \mathcal{C}$ *with* $\varphi(v) = v^{j_1 \cdots j_p}_{j_{\sigma(1)} \cdots j_{\sigma(p)}}$ *where* σ *is a permutation of* $(1, 2, \ldots, p)$.

It follows from this remark that any basic center - affine covariant belonging to $K(n_0, n_1, \dots, n_m, p)$ can be represented by the form

$$a^{i_1}\cdots a^{i_{n_0}}\, a^{i_{n_0+1}}_{j_1}\cdots a^{i_{n_0+n_1}}_{j_{n_1}}\;\cdots\cdots a^{i_{l_{m-1}+1}}_{j_{k_{m-1}+1}\cdots j_{k_{m-1}+m}}\cdots a^{i_{l_{m-1}+n_m}}_{j_{k_m-m}\cdots j_{k_m}}$$
$$\times x^{i_{l_m+1}}\dots x^{i_{l_m+p}}\varepsilon^{i_{l_m+p+1}\cdots i_{l_m+p+n}}\dots\varepsilon^{i_{l_m+p+(n-1)q+1}\cdots i_{l_m+p+nq}} \qquad (3.7)$$

or by the form

$$a^{i_1}\cdots a^{i_{n_0}}\, a^{i_{n_0+1}}_{j_1}\cdots a^{i_{n_0+n_1}}_{j_{n_1}}\;\cdots\cdots a^{i_{l_{m-1}+1}}_{j_{k_{m-1}+1}\cdots j_{k_{m-1}+m}}\cdots a^{i_{l_{m-1}+n_m}}_{j_{k_m-m}\cdots j_{k_m}}$$
$$\times x^{i_{l_m+1}}\dots x^{i_{l_m+p}}\varepsilon_{j_{k_m+1}\cdots j_{k_m+n}}\cdots\varepsilon_{j_{k_m+p+(n-1)q'+1}\cdots j_{k_m+nq'}}. \qquad (3.8)$$

where $(j_1, \dots, j_{k_m})$ and $(j_1, \dots, j_{k_m+nq'})$ represent the all possible permutations of $(i_1, \dots, i_{l_m+p+nq})$ and $l_s = \sum_{r=0}^{s} n_r$, $k_s = \sum_{r=1}^{1} n_r$.

Of course, we have $l_m + p + nq = k_m + nq'$.

The writtings (3.7) and (3.8) are used to expand a center-affine basic covariant.

4. Affine covariants

In this section we show that the knowledge of a system of generators of the algebra of center-affine invariants $\mathcal{I}(n, m)$ induces that one of a system of generators the algebra of affine invariants. Actually, this correspondance is an isomorphism. To construct it we need the following lemma.

Lemma 1. *The group $T(n)$ defines a polynomial representation on the vector space $\mathcal{A}(n, m)$.*

Proof. It suffices to prove that

$$\forall p, q \in T(n),\ \rho_2(p+q) = \rho_2(p)\rho_2(q). \qquad (4.9)$$

By formula 2.4, we have
$[\rho_2(p+q)(a)]^j_{\alpha_1\alpha_2\dots\alpha_k} =$

$$\sum_{r=0}^{m-k}\sum_{s=0}^{r}\binom{k+r}{r}\binom{r}{s}\, a^j_{\alpha_1\alpha_2\dots\alpha_k\beta_1\beta_2\dots\beta_s\gamma_1\gamma_2\dots\gamma_{r-s}}p^{\beta_1}\dots p^{\beta_s}q_1^{\gamma}\dots q^{\gamma_{r-s}},$$

$[\rho_2(p)\rho_2(q)(a)]^j_{\alpha_1\alpha_2\dots\alpha_k} =$

$$\sum_{i=0}^{m-k}\binom{k+i}{i}\sum_{l=0}^{m-k-i}\binom{k+i+l}{l}\, a^j_{\alpha_1\alpha_2\dots\alpha_k\beta_1\beta_2\dots\beta_i\gamma_1\gamma_2\dots\gamma_l}p^{\beta_1}\dots p^{\beta_i}q^{\gamma_1}\dots q^{\gamma_l}.$$

We remark that the coefficients of algebraic form

$$a^{j}_{\alpha_1\alpha_2\dots\alpha_k\beta_1\beta_2\dots\beta_i\gamma_1\gamma_2\dots\gamma_l}p^{\beta_1}\dots p^{\beta_i}q^{\gamma_1}\dots q^{\gamma_l}$$

in both expressions are equal to $\dfrac{(k+i+l)!}{k!i!}$. Hence, the equality (4.9).

Recall that the actions of the groups $Gl(n)$ and $T(n)$ over V are respectively defined by $(P,x)\longmapsto P^{-1}x$ and $(p,x)\longmapsto x-p$.

Theorem 2. *A polynomial $Q\in\mathcal{C}[\mathcal{A}(n,m)\times V]$ is an $Aff(n)$-covariant if and only if it exists a $Gl(n)$-invariant $I\in\mathcal{I}(n,m)$ such that*

$$Q(a,x)=I(\rho_2(x)(a)),\ \forall\,(a,x)\in\mathcal{A}(n,m)\times V.$$

Proof. Let us consider a $Gl(n)$ - invariant I of $\mathcal{A}(n,m)$ and let Q be the polynomial $Q(a,x)=I(\rho_2(x)(a))$. We can suppose that I is a basic invariant. Following (3.8), (3.8) and theorem 1 the polynomial $Q(a,x)$ is a sum of basic center-affine covariants. It remains to prove that $Q(a,x)$ is a $T(n)$ - invariant. By lemma (1),

$$Q(\rho_2(p)(a),x-p)\ =\ I(\rho_2(x-p)\rho_2(p)(a))\ =\ Q(a,x).$$

Let us suppose now that $Q(a,x)$ is an $Aff(n)$-covariant and put $I(a)=Q(a,0)$. Then $I(a)$ is a $Gl(n)$-invariant. So, substituting $p=x$ in the relation

$$Q(\rho_2(p)(a),x-p)=Q(a,x),\forall p\in T(n),\ \forall(a,x)\in\mathcal{A}(n,m)\times\mathcal{C}^n,$$

we obtain

$$I(\rho_2(x)(a))=Q(\rho_2(x)(a),0)=Q(a,x).\forall\,x\in T(n),\ \forall\,a\in\mathcal{A}(n,m)\times\mathcal{C}^n,$$

Thus, the correspondance $I(a)\longleftrightarrow I(\rho_2(x)(a))=Q(a,x)$ gives an isomorphism which we denote Φ, between the $\mathcal{C}$ - algebras $\mathcal{I}(n,m)$ and $\mathcal{Q}(n,m)$. The affine covariant $\Phi(I)$, where $I\in\mathcal{I}(n,m)$, denoted $I(x)$ is called sometimes the "translated" of I.

In the next section, we compute effectively a minimal system of generators of $\mathcal{Q}(2,2)$.

Definition 6. *A basic affine covariant is an image by Φ of a basic center-affine invariant.*

Example : the polynomial

$$a^{\alpha}_{\alpha} + 2a^{\alpha}_{\alpha\beta}x^{\beta} + \cdots + \binom{m+1}{1} a^{\alpha}_{\alpha\beta_1\cdots\beta_{m-1}}x^{\beta_1}\cdots x^{\beta_{m-1}}$$

is a basic affine covariant. It corresponds to the divergence of the vector field defined by the system.

Remark 2. *Let I be a center-affine invariant of $\mathcal{A}(n,m)$. If it is multi-homogeneous of multi-degree $(d_0, d_1, \ldots, d_m)$ then the affine covariant $Q(a,x) = I(x.a)$ has the expansion*

$$Q(a,x) = I(a) + Q_1(a,x) + \cdots + Q_s(a,x)$$

where the polynomial $Q_i(a,x)$ is homogeneous of degree i with respect to coordinates x and $s = \sum_{j=0}^{m}(m-j)d_i$.

Corollary 1. *A family $\mathcal{F}$ of $Gl(n)$-invariants is algebraically dependant if and only if $\Phi(\mathcal{F})$ is algebraically dependant.*

It is clear that the previous theorem remains true if we substitute the group $GL(n)$ by any of its subgroups like the orthogonal one, $O(n)$. In the particular case where this subgroup contains only identity, we have :

Corollary 2. *The family of polynomials*

$$\left\{[\rho_2(x)(a)]^j_{\alpha_1\alpha_2\ldots\alpha_k}, 1 \le \alpha_1 \le \ldots \le \alpha_k \le n, k \in \{0,\ldots,m\}, j \in \{0,\ldots,n\}\right\}$$

forms a minimal polynomial system of generators of $T(n)$-covariants of $\mathcal{A}(n,m)$.

5. The case of planar quadratic differentiel systems

Many works are devoted to the study of center-affine invariants of systems

$$\frac{dx^j}{dt} = a^j + a^j_{\alpha}x^{\alpha} + a^j_{\alpha\beta}x^{\alpha}x^{\beta} \ (j,\alpha,\beta = 1,2) \tag{5.10}$$

They are based on the classical approach, called Aronhold's symbolic method ([3]) and resumed in [10]. In particular, a minimal system of generators of $\mathcal{I}(2,2)$ was obtained :

Minimal system of generators of $\mathcal{I}(2,2)$

$J_1 = a^\alpha_\alpha,\quad J_2 = a^\alpha_\beta a^\beta_\alpha,\quad J_3 = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\varepsilon^{pq},\quad J_4 = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\gamma}\varepsilon^{pq},$
$J_5 = a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\varepsilon^{pq},\ J_6 = a^\alpha_p a^\beta_\gamma a^\gamma_{\alpha q} a^\delta_{\beta\delta}\varepsilon^{pq},\ J_7 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\beta s} a^\delta_{\gamma\delta}\varepsilon^{pq}\varepsilon^{rs},$

$J_8 = a^\alpha_{pr} a^\beta_{\alpha q} a^\gamma_{\delta s} a^\delta_{\beta\gamma}\varepsilon^{pq}\varepsilon^{rs},\ J_9 = a^\alpha_{pr} a^\beta_{\beta q} a^\gamma_{\gamma s} a^\delta_{\alpha\delta}\varepsilon^{pq}\varepsilon^{rs},\ J_{10} = a^\alpha_p a^\beta_\delta a^\gamma_\mu a^\delta_{\alpha q} a^\mu_{\beta\gamma}\varepsilon^{pq},$

$J_{11} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\gamma} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},\ J_{12} = a^\alpha_p a^\beta_{qr} a^\gamma_{\beta s} a^\delta_{\alpha\delta} a^\mu_{\gamma\mu}\varepsilon^{pq}\varepsilon^{rs},$

$J_{13} = a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\varepsilon^{pq}\varepsilon^{rs},\ J_{14} = a^\alpha_p a^\beta_r a^\gamma_{\alpha q} a^\delta_{\beta s} a^\mu_{\gamma\delta} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},$

$J_{15} = a^\alpha_{pr} a^\beta_{qk} a^\gamma_{\alpha s} a^\delta_{\delta l} a^\mu_{\beta\gamma} a^\nu_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs}\varepsilon^{kl},\ J_{16} = a^\alpha_p a^\beta_r a^\gamma_\delta a^\delta_{\alpha q} a^\mu_{\beta s} a^\nu_{\gamma\tau} a^\tau_{\mu\nu}\varepsilon^{pq}\varepsilon^{rs},$

$J_{17} = a^\alpha_{\alpha\beta} a^\beta,\quad J_{18} = a^p_\alpha a^\alpha a^q\varepsilon_{pq},\quad J_{19} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma,\quad J_{20} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma,$

$J_{21} = a^p_{\alpha\beta} a^\alpha a^\beta a^q\varepsilon_{pq},\ J_{22} = a^\alpha_{\alpha\beta} a^\beta_{\gamma\delta} a^\gamma a^\delta,\ J_{23} = a^\alpha_{\beta\gamma} a^\beta_{\alpha\delta} a^\gamma a^\delta,$

$J_{24} = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta} a^\delta,\ J_{25} = a^\alpha_{\alpha p} a^\beta_{\gamma q} a^\gamma_{\beta\delta} a^\delta\varepsilon^{pq},\quad J_{26} = a^\alpha_{\alpha p} a^\beta_{\delta q} a^\gamma_{\beta\gamma} a^\delta\varepsilon^{pq},$

$J_{27} = a^p_\alpha a^\alpha_{\beta\gamma} a^\beta a^\gamma a^q\varepsilon_{pq},\ J_{28} = a^\alpha_\beta a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu,\ J_{29} = a^\alpha_\gamma a^\beta_{\alpha\beta} a^\gamma_{\delta\mu} a^\delta a^\mu,$

$J_{30} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\delta} a^\delta_{\gamma\mu} a^\mu\varepsilon^{pq},\qquad J_{31} = a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\mu} a^\delta_{\gamma\delta} a^\mu\varepsilon^{pq},$

$J_{32} = a^\alpha_p a^\beta_{\beta q} a^\gamma_{\alpha\mu} a^\delta_{\gamma\delta} a^\mu\varepsilon^{pq},\qquad J_{33} = a^\alpha_{\beta\nu} a^\beta_{\alpha\gamma} a^\gamma_{\delta\mu} a^\delta a^\mu a^\nu,$

$J_{34} = a^\alpha_{\mu p} a^\beta_{\alpha q} a^\gamma_{\beta\nu} a^\delta_{\gamma\delta} a^\mu a^\nu\varepsilon^{pq},\qquad J_{35} = a^\alpha_p a^\beta_\nu a^\gamma_{\alpha q} a^\delta_{\beta\mu} a^\mu_{\gamma\delta} a^\nu\varepsilon^{pq},$

$J_{36} = a^\alpha_{pr} a^\beta_{\nu q} a^\gamma_{\alpha s} a^\delta_{\beta\gamma} a^\mu_{\delta\mu} a^\nu\varepsilon^{pq}\varepsilon^{rs}.$

With the help of tranvectants it was established by N. Vulpe [11] that any element of $\mathcal{I}(2,2)$ is a polynomial of $J_1,\ldots,J_{36}$. In [9], it was given a minimal system of generators of the ideal of the syzygies of $\mathcal{I}(2,2)$.

Minimal system of generators of $\mathcal{K}(2,2)$

Now, let us show that we can deduce from this family of center-affine invariants a minimal system of generators of $\mathcal{K}(2,2)$. Obviously, the proposed method can be applied for any degree m.

It is clear that if we substitute a^j by x^j in the invariants $J_{17},J_{18},\ldots,J_{36}$, we obtain center-affine covariants that we note respectiveley $K_1,K_2,\ldots,K_{20}$. Actually, we can generalize this procedure of substitution to get all other covariants.

Introduce the new center-affine covariant

$$K_{21} = \det(a^j, x^j) = a^1x^2 - a^2x^1.$$

Consider a basic center-affine invariant $I(a^i, a^i_j, \ldots)$ whose notation is (3.7) or (3.8). This is a function of a^i and other coordinates $a^j_{\alpha_1\alpha_2,\ldots,\alpha_k}$. There are different ways to replace p times the vector (a^j) by (x^j). For example, if $I = a^p_{\alpha\beta}a^q a^\alpha a^\beta \varepsilon_{pq}$ and $p = 1$, the substitution may give $a^p_{\alpha\beta}x^q a^\alpha a^\beta \varepsilon_{pq}$ or $a^p_{\alpha\beta}a^q a^\alpha x^\beta \varepsilon_{pq}$. We shall show that, in general, these two covariants are the same modulo K_{21}.
Let us denote $K'(a^i, x^i, \ldots)$ and $K''(a^i, x^i, \ldots)$ two basic center-affine homogeneous covariants of multi- degree $(n_0 - p, n_1, \ldots, n_m, p)$ obtained from the tensorial notation (3.7) (or (3.8) of I by replacing "p" $a = (a^j)$ by "p" $x = (x^j)$. Then

$$K'(a^i, a^i, \ldots) - K''(a^i, a^i, \ldots) = I(a^i, \ldots) - I(a^i, \ldots) = 0.$$

Because of multi-homogeneity with respect to coordinates a^j and x^j, we deduce from this fact that the difference $K'(a^i, x^i, \ldots) - K''(a^i, x^i, \ldots)$ contains as a factor the covariant $a^1x^2 - a^2x^1 = K_{21}$. Consequently, we have the following result

Lemma 2. *Let I be a multi-homogeneous basic center-affine invariant of multi-degree $(n_0, n_1, \ldots, n_m)$ and p some natural number such that $p \leq n_0$. For any two center-affine covariants K', K'' obtained from the invariant I by replacing in its tensorial notation "p" vectors (a^j) by "p" vectors $x = (x^j)$, the difference $K' - K''$ is a multiple of K_{21}.*

That is to say that each center-invariant $J_1, \ldots, J_{36}$ of multi-degree $(n_0, n_1, n_2, 0)$ gives rise, for all $p \in \{1, \ldots, n_1 - 1\}$ to one and only one covariant of multi-degree $(n_0 - p, n_1, n_2, p)$ that cannot be expressed as a polynomial of covariants of lower total degree.

Applying this lemma to the family $\{J_i,\ i = 1 \cdots 36\}$ we obtain a polynomialy independant familly of covariants $\mathcal{K}(2,2)$:

$K_1 = a^\alpha_{\alpha\beta}x^\beta$, $K_2 = a^p_\alpha x^\alpha x^q \varepsilon_{pq}$, $K_3 = a^\alpha_\beta a^\beta_{\alpha\gamma}x^\gamma$, $K_4 = a^\alpha_\gamma a^\beta_{\alpha\beta}x^\gamma$,
$K_5 = a^p_{\alpha\beta}x^\alpha x^\beta x^q \varepsilon_{pq}$, $K_6 = a^\alpha_{\alpha\beta}a^\beta_{\gamma\delta}x^\gamma x^\delta$, $K_7 = a^\alpha_{\beta\gamma}a^\beta_{\alpha\delta}x^\gamma x^\delta$,

$K_8 = a^\alpha_\gamma a^\beta_\delta a^\gamma_{\alpha\beta}x^\delta$, $K_9 = a^\alpha_{\alpha p}a^\beta_{\gamma q}a^\gamma_{\beta\delta}x^\delta \varepsilon^{pq}$, $K_{10} = a^\alpha_{\alpha p}a^\beta_{\delta q}a^\gamma_{\beta\gamma}x^\delta \varepsilon^{pq}$,

$K_{11} = a^p_\alpha a^\alpha_{\beta\gamma}x^\beta x^\gamma x^q \varepsilon_{pq}$, $K_{12} = a^\alpha_\beta a^\beta_{\alpha\gamma}a^\gamma_{\delta\mu}x^\delta x^\mu$, $K_{13} = a^\alpha_\gamma a^\beta_{\alpha\beta}a^\gamma_{\delta\mu}x^\delta x^\mu$,

$$K_{14} = a^{\alpha}_{p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\delta} a^{\delta}_{\gamma\mu} a^{\mu} \varepsilon^{pq}, \qquad K_{15} = a^{\alpha}_{p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\mu} a^{\delta}_{\gamma\delta} x^{\mu} \varepsilon^{pq},$$

$$K_{16} = a^{\alpha}_{p} a^{\beta}_{\beta q} a^{\gamma}_{\alpha\mu} a^{\delta}_{\gamma\delta} a^{\mu} \varepsilon^{pq}, \qquad K_{17} = a^{\alpha}_{\beta\nu} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\delta\mu} x^{\delta} x^{\mu} x^{\nu},$$

$$K_{18} = a^{\alpha}_{\mu p} a^{\beta}_{\alpha q} a^{\gamma}_{\beta\nu} a^{\delta}_{\gamma\delta} x^{\mu} x^{\nu} \varepsilon^{pq}, \qquad K_{19} = a^{\alpha}_{p} a^{\beta}_{\nu} a^{\gamma}_{\alpha q} a^{\delta}_{\beta\mu} a^{\mu}_{\gamma\delta} x^{\nu} \varepsilon^{pq},$$

$$K_{20} = a^{\alpha}_{pr} a^{\beta}_{\nu q} a^{\gamma}_{\alpha s} a^{\delta}_{\beta\gamma} a^{\mu}_{\delta\mu} x^{\nu} \varepsilon^{pq} \varepsilon^{rs}, \;\; K_{21} = a^{p} x^{q} \varepsilon_{pq}, \quad K_{22} = a^{p}_{\alpha} a^{\alpha} x^{q} \varepsilon_{pq},$$

$$K_{23} = a^{\alpha} a^{\beta} a^{p}_{\alpha\beta} x^{q} \varepsilon_{pq}, \;\; K_{24} = a^{\alpha} a^{\beta}_{\alpha\beta} a^{\gamma}_{\gamma\delta} x^{\delta}, \;\; K_{25} = a^{\alpha} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\beta\delta} x^{\delta},$$

$$K_{26} = a^{\alpha} a^{\beta} a^{p}_{\gamma} a^{\gamma}_{\alpha\beta} x^{q} \varepsilon_{pq}, \;\; K_{27} = a^{\alpha} a^{\beta}_{\gamma} a^{\gamma}_{\beta\delta} a^{\delta}_{\alpha\mu} x^{\mu} \leq \alpha_2, \;\; K_{28} = a^{\alpha} a^{\beta}_{\delta} a^{\gamma}_{\beta\gamma} a^{\delta}_{\alpha\mu} x^{\mu},$$

$$K_{29} = a^{\alpha} a^{\beta} a^{\gamma}_{\delta\nu} a^{\delta}_{\gamma\mu} a^{\mu}_{\alpha\beta} x^{\nu}, \;\; K_{30} = a^{\alpha} a^{\beta}_{\alpha p} a^{\gamma}_{\beta q} a^{\delta}_{\gamma\nu} a^{\mu}_{\delta\mu} x^{\nu} \varepsilon^{pq}, \qquad K_{31} = a^{p} a^{q}_{\alpha\beta} x^{\alpha} x^{\beta} \varepsilon_{pq},$$

$$K_{32} = a^{p} a^{q}_{\alpha} a^{\alpha}_{\beta\gamma} x^{\beta} x^{\gamma} \varepsilon_{pq}, \;\; K_{33} = a^{\alpha} a^{\beta}_{\alpha\gamma} a^{\gamma}_{\beta\delta} a^{\delta}_{\mu\nu} x^{\mu} x^{\nu},$$

Theorem 3. *The set of covariants*

$$\{J_i;\, i = 1, \ldots, 36\} \bigcup \{K_j;\, j = 1, \ldots, 33\}$$

forms a minimal system of generators of the ring $\mathcal{K}(2,2)$.

Proof. We have seen that this family is algebraically independant (consequence of the previous lemma and the procedure of construction of the covariants $\{K_i;\, i = 1, \ldots, 33\}$). It is easy to show that it generates the algebra $\mathcal{K}(2,2)$. Let K be a center-affine covariant of multi-degre (n_0, n_1, n_2, p). Following theorem 1, we can suppose that $K = K(x^i, a^i, \ldots)$ is a basic covariant i.e. it is of the form (3.7) or (3.8). As the family $\{J_1, J_2, \ldots, J_{36}\}$ generates the algebra of center-affine invariants $\mathcal{I}(2,2)$, the invariant $I = K(a^i, a^i, \ldots)$ is a polynomal expression of these invariants : $I = \mathcal{P}(J_1, J_2, \ldots, J_{36}) = \lambda_1 I_1 + \lambda_2 I_2 + \ldots \lambda_l I_l$ where λ_i are constant coefficients and I_j are basic invariants of multi-degree $(n_0 + p, n_1, n_2, 0)$. Each term of I is a product of invariants J_i. After substituting p (a^i) by p (x^i), we obtain a center-affine covariant of multi-degree (n_0, n_1, n_2, p). Of course, we can choose this substitution in such a way that the obtained covariant is a product of $K_1, K_2, \ldots, K_{33}$. Let us denote this covariant K' It has the same multidegree that K. The difference $K - K'$ is a multiple of K_{21}. Let $\tilde{K}$ be the quotient of $K - K'$ by K_{21}. Its multi-degree is $(n_0 - 1, n_1, n_2, p - 1)$. We repeat this procedure ... At the end, we obtain a polynomial expression of K with respect to $J_1, J_2, \ldots, J_{36}, K_1, K_2, \ldots, K_{33}$.

Remark 3. *In the literrature, we often find quadratic systems without free terms :*

$$\frac{dx^j}{dt} = a^j_{\alpha} x^{\alpha} + a^j_{\alpha\beta} x^{\alpha} x^{\beta} \;\; (j, \alpha, \beta = 1, 2). \tag{5.11}$$

The family

$$\{J_i;\, i = 1,\ldots,16\}\bigcup\{K_j;\, j = 1,\ldots,20\}$$

is a minimal system of covariants of these systems. This explains the indexation of covariants suggested in the previous theorem.

Remark 4. *Concerning the affine invariants, it appears that they are nothing more than the polynomial expressions of affine covariants which do not depend over the vector x. They form a subalgebra of $\mathcal{Q}(n,m)$.*

Examples

$$\begin{array}{rcll} J_1(x) & = & a^\alpha_\alpha + x^\alpha a^\beta_{\alpha\beta} & = J_1 + 2K_1, \\ J_3(x) & = & a^\alpha_p a^\beta_{\alpha q} a^\gamma_{\beta\gamma}\epsilon^{pq} + 2a^\alpha_{p\delta} a^\beta_{\beta q} a^\gamma_{\beta\gamma} x^\delta \epsilon^{pq} & = J_3 + 2(K_9 - K_{10}), \\ J_5(x) & = & a^\alpha_p a^\beta_{\gamma q} a^\gamma_{\alpha\beta}\epsilon^{pq} + 2a^\alpha_{p\delta} a^\beta_{\gamma q} a^\gamma_{\alpha\beta} x^\delta \epsilon^{pq} & = J_5 + 2(K_9 - K_{10}), \\ & & \ldots & \\ J_{13}(x) & = & a^\alpha_p a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu}\epsilon^{pq}\epsilon^{rs} + & \\ & & 2a^\alpha_{pu} a^\beta_{qr} a^\gamma_{\gamma s} a^\delta_{\alpha\beta} a^\mu_{\delta\mu} x^u \epsilon^{pq}\epsilon^{rs} & = J_{13} - 2J_7K_1. \end{array} \tag{5.12}$$

Taking into account the relation $J_7(x) = J_7$, it is obvious that the polynomials $J_3 - J_5$ and $J_1J_7 + J_{13}$ are affine invariants.

6. Effective computation of a system of generators of $\mathcal{Q}(2,2)$

In this section we deal with the differential systems (5.10). The goal is to compute effectively a system of generators of $\mathcal{Q}(2,2)$ from that one of $\mathcal{I}(2,2)$ (see the previous section).
The algorithms are implanted in Maple 5.4. Some of algorithms presented here can be generalised to any dimension of differential systems.

6.1. Computing in $\mathcal{K}(2,2)$. The algebra of center-affine covariants of quadratic differential systems (5.10), $\mathcal{K}(2,2)$ is a multigraded algebra :

$$\mathcal{K}(2,2) = \bigoplus_{l\in\mathbb{N}} \bigoplus_{n_0+n_1+n_2+p=l} K(n_0,n_1,n_2,p)$$

where $K(n_0, n_1, n_2, p)$ is the subalgebra of multi-homogeneous covariants of multi-degree (n_0, n_1, n_2, p), i.e. homogeneous of degree n_k with respect to cordinates of $(a^j_{i_1,i_2,\dots i_k})$ and of degree p with respect to coordinates x^j.
As it is proved previously, the family of covariants

$$\mathcal{F} = \{J_1, \dots, J_{36}, K_1, \dots, K_{33}\}$$

is a minimal system of generators of $\mathcal{K}(2,2)$. The main task which arises here is to express any center-affine covariant (invariant) in this basis. For this, we need to introduce the notion of type of basic center-affine covariant : this is the sextuple (p, q, r, s, τ, v) where (p, q, r, s) is its multi-degree with repect to coordinates $(a^i, a^i_j, a^i_{j,k}, x^i)$, τ is the number of contravariant 2-vectors ε^{jk} and, v, following the remark (1), is the permutation of upper indices of the covariant.

Algorithm 1.1 : expansion of a basic covariant of type (p, q, r, s, τ, v).

We take into account three facts :

1. the coordinates (a^i_{jk}) are symmetric with respect to subscripts,
2. if $n = 2$, $\varepsilon^{ij} = \varepsilon_{ij} = j - i$
3. the expansion of a basic covariant is obtained by successive contractions.

The procedure is called `devcov`.

Algorithm 1.2 : decomposition of the multi-degree m w.r.t. the list of multi-degrees of $\mathcal{F}$. This list, denoted t, has 49 elements :

t := [[0, 1, 0, 0], [1, 0, 1, 0], [0, 2, 0, 0], [2, 1, 0, 0], [1, 1, 1, 0], [0, 1, 2, 0], [3, 0, 1, 0], [2, 0, 2, 0], [1, 2, 1, 0], [1, 0, 3, 0], [0, 2, 2, 0], [0, 0, 4, 0], [3, 1, 1, 0], [2, 1, 2, 0], [1, 3, 1, 0], [0, 3, 2, 0], [0, 1, 4, 0], [3, 0, 3, 0], [2, 0, 4, 0], [1, 2, 3, 0], [1, 0, 5, 0], [0, 2, 4, 0], [0, 0, 6, 0], [0, 3, 4, 0], [1, 0, 0, 1], [0, 0, 1, 1], [1, 1, 0, 1], [0, 1, 1, 1], [2, 0, 1, 1], [1, 0, 2, 1], [0, 2, 1, 1], [0, 0, 3, 1], [2, 1, 1, 1], [1, 1, 2, 1], [0, 1, 3, 1], [2, 0, 3, 1], [1, 0, 4, 1], [0, 2, 3, 1], [0, 0, 5, 1], [0, 1, 0, 2], [1, 0, 1, 2], [0, 0, 2, 2], [1, 1, 1, 2], [0, 1, 2, 2], [1, 0, 3, 2], [0, 0, 4, 2], [0, 0, 1, 3], [0, 1, 1, 3], [0, 0, 3, 3]].

The decomposition of the multi-degree m following the list t is given by the procedure `reduire` which is presented in the annex 2. Let $[p, q, r, s]$ the multi-degree of center-affine covariant. This procedure gives all possible decompositions $[p, q, r, s] = \sum_1^{49} \lambda_i t[i]$.

Algoritm 1.3 : construction of a generating family of $K(p,q,r,s)$ (as a linear space), starting from $\mathcal{F}$.
With the help of the previous algorithm, we get the different decompositions of the 4-tuple $K(p,q,r,s)$ w.r.t. the family $\mathcal{F}$. As we know that the dimension of spaces $K(t[1])$, $K(t[2])$, $K(t[3])$, $K(t[4])$, $K(t[7])$, $K(t[9])$, $K(t[11])$, $K(t[13])$, $K(t[16])$, $K(t[18])$, $K(t[19])$, $K(t[20])$, $K(t[21])$, $K(t[22])$, $K(t[23])$, $K(t[24])$, $K(t[25])$, $K(t[26])$, $K(t[27])$, $K(t[29])$, $K(t[31])$, $K(t[33])$, $K'(t[36])$, $K'(t[37])$ is one, that one of $K(t[5])$, $K(t[8])$, $K(t[10])$, $K(t[14])$, $K(t[28])$, $K(t[30])$, $K(t[32])$, $K(t[34])$, $K(t[42])$, $K(t[44])$ is two and dimension of spaces $K(t[6])$, $K(t[12])$, $K(t[15])$, $K(t[17])$, $K(t[35])$ is three, for each term of the decomposition $[p,q,r,s] = \sum_1^{49} \lambda_i t[i]$ we take the linear basis of $K(t[i])$ and so, we form the generating family of the linear space $K(p,q,r,s)$ (w.r.t. to the family $\mathcal{F}$). For example, for $K(0,2,4,0)$, the algorithm **1.3** gives us w.r.t. t, the decompositions

$$\begin{aligned}[0,2,4,0] &= 2[0,1,0,0] + [0,0,4,0] = [0,1,0,0] + [0,1,4,0] \\ &= [0,2,0,0] + [0,0,4,0] = 2[0,1,2,0] = [0,2,2,0].\end{aligned}$$

For the first decomposition, we have $J[1]^2J[7]$, $J[1]^2J[8]$, $J[1]^2J[9]$, for the second, $J[1]J[11]$, $J[1]J[12]$, $J[1]J[13]$, for the third, $J[2]J[7]$, $J[2]J[8]$, $J[2]J[9]$, for the fourth, $J[3]^2$, $J[3]J[4]$, $J[3]J[5]$, $J[4]^2$, $J[4]J[5]$, $J[5]^2$ and for the last, $J[14]$. These 16 covariants (here, invariants) form a generating family of $K(0,2,4,0)$. This algorithm corresponds to the procedure `dcpcov`.

Algoritm 1.4 : expression of any covariant in the basis $\mathcal{F}$.
This task is reduced to solving of a linear system. Before to do it, we can operate main simplifications. According to [1, 12] any center-affine covariant K of any tensorial representation of the group $GL(2,\mathcal{C})$ can be written as an homogeneous polynomial in $\mathcal{C}[\mathcal{A}(2,2)][x^1,x^2]$ of degree p :

$$K = A_{11\cdots 11}(x^1)^p + A_{11\cdots 12}(x^1)^{p-1}x^2 + \cdots + A_{22\cdots 22}(x^2)^p.$$

With the help of some operator, the author shows that K, as a polynomial in $[x^1,x^2]$, vanishes if and only if the leading coefficient $A_{11\cdots 11}$ (called elsewhere the semi-invariant), vanishes. That means that the leading coefficient define uniquely the whole covariant. This fact is also true for any homogeneous relation between covariants.
This algorithm corresponds to the procedure `ExpCovAff`.

Note that this procedure gives also the syszygies of multidegree (p,q,r,s).

6.2. Computing of a minimal system of generators of $\mathcal{Q}(2,2)$. The goal of this subsection is to compute a minimal system of generators of $\mathcal{Q}(2,2)$ using the center-affine covariants $\mathcal{I}(2,2)$ which is given in the previous subsection. We have noted it $\mathcal{F}$. Following the theorem (2), the image $\mathcal{G} = \phi(\mathcal{F})$ is a minimal system of generators of $\mathcal{Q}(2,2)$. To have their expanded expressions, it suffices to make the substitutions

$$a^i \leftarrow a^i + a^i_\alpha x^\alpha + a^i_{\alpha\beta} x^\alpha x^\beta,$$

$$a^i_j \leftarrow a^i_j + 2a^i_{j\alpha} x^\alpha,$$

$$a^i_{jk} \leftarrow a^i_{jk}$$

in $\mathcal{F}$. The minimal system of generators of $\mathcal{Q}(2,2)$ is then known.
It remains to express these covariants in $\mathcal{F}$. This problem is the same that the determination of their multi-homogeneous parts.
Let J be a center-affine invariant. We decompose the corresponding affine covariant $\Phi(J)$ into multi-homogeneous polynomials

$$\Phi(J) = L_1 + \cdots + L_k.$$

The different steps for computing $\Phi(J)$ are the following :

1. determination of the multi-degrees of $\Phi(J)$ (algoritm 2.1) w.r.t. the list of variables

   $varH :=$
   $[\{A_1, A_2\}, \{A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}\}, \{A_{1,11}, A_{1,12}, A_{2,11}, A_{2,12}, A_{1,22}, A_{2,22}\}, \{x_1, x_2\}]$

2. determination of the multi-homogeneous part L_i of $\Phi(J)$ corresponding to a giving multi-degree (algoritm 2.2, algoritm 2.3).

3. expression of each multi-homogeneous part L_i of $\Phi(J)$ w.r.t. $\mathcal{F}$.

Algoritm 2.1 : Multi-degrees of affine covariants.
Let J be center-affine invariant. Its type is $(p, q, r, 0)$. Each affine covariant of planar quadratic systems is a sum of (p', q', r', s') - multi-homogeneous center-affine covariants. For finding these multi-degrees, it suffices (algorithm 2.1) to expand the univariate polynomial

$$pp := (a + b \times x + c \times x^2)^p \times (b + 2 \times c \times x)^q \times c^r$$

and to take the list of elements

$$(degree(pp, a), degree(pp, b), degree(pp, c), degree(pp, x)).$$

This procedure is called `DegHom`.

Algoritm 2.2 : this algorithm tests whether a monomial P has a multi-degree $n = (n_1, n_2, n_3, n_4)$ w.r.t. the list of the variables *varH*. This procedure is called `DegMonHom`.

Algoritm 2.3 : let P be an affine covariant. Following the remark (2), P is a sum of multi-homogeneous center-affine covariants. The procedure given by this algorithm (called `PartHom`) consists in the determination of its multi-homogeneous parts.

Algoritm 2.4 : this algorithm (that uses the previous ones) gives the expression of any basic affine covariant relatively the familyF. The corresponding procedure is called `DevCovAff`.

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7. Annexes

7.1. Algorithms (Maple 5.5). All algorithms are implemented in Maple 5.5. They form the package denominated `SIB`.

7.1.1 Computing in $\mathcal{K}(2,2)$

Algorithm 1.1 : expansion of a basic covariant of type (p,q,r,s,τ,v).

$devcov := \mathbf{proc}(p,q,r,s,\tau,v)$

$\quad\mathbf{local}\ \alpha,\ j,\ I1,\ I2,\ I3,\ Xx,\ Eps1,\ Eps2,\ J,\ JJ;\quad \mathbf{global}\ A,\ x;$

$\quad I1 := 1;\quad I2 := 1;\quad I3 := 1;\quad Xx := 1;\ Eps1 := 1;\quad Eps2 := 1;$

$\quad \alpha := \mathrm{vector}(p+q+r+s+2\times\tau);$

$\quad \mathbf{for}\ j\ \mathbf{to}\ p\ \mathbf{do}\quad I1 := I1\times A_{\alpha_j}\ \mathbf{od};$

$\quad \mathbf{for}\ j\ \mathbf{to}\ q\ \mathbf{do}\quad I2 := I2\times A_{\alpha_{p+j},\,\alpha_{v_j}}\ \mathbf{od};$

$\quad \mathbf{for}\ j\ \mathbf{to}\ r\ \mathbf{do}$

$\quad I3 := I3\times A_{\alpha_{p+q+j},10\times\min(\alpha_{v_{q+2\times j-1}},\,\alpha_{v_{q+2\times j}})+\max(\alpha_{v_{q+2\times j-1}},\,\alpha_{v_{q+2\times j}})}\quad \mathbf{od};$

$\quad \mathbf{for}\ j\ \mathbf{to}\ s\ \mathbf{do}\ Xx := Xx\times x_{\alpha_{p+q+r+j}}\ \mathbf{od};$

$\quad \mathbf{for}\ j\ \mathbf{to}\ \tau\ \mathbf{do}\ Eps1 := Eps1\times(\alpha_{p+q+r+s+2\times j}-\alpha_{p+q+r+s+2\times j-1})\ \mathbf{od};$

$\quad \mathbf{for}\ j\ \mathbf{to}\ 1/2\times p+1/2\times s-1/2\times r+\tau\,\mathbf{do}$

$\qquad Eps2 := Eps2\times(\alpha_{v_{q+2\times r+2\times j}}-\alpha_{v_{q+2\times r+2\times j-1}})\quad \mathbf{od};$

$\quad J_0 := Eps1\times Eps2\times I1\times I2\times I3\times Xx;$

$\quad \mathbf{for}\ j\ \mathbf{to}\ p+q+r+s+2\times\tau\ \mathbf{do}$

$\quad J_j := \mathrm{subs}(\alpha_j=1,\ J_{j-1})+\mathrm{subs}(\alpha_j=2,\ J_{j-1})\ \mathbf{od};$

$\quad JJ := \mathrm{sort}(\mathrm{eval}(J_{p+q+r+s+2\times\tau}),[A_1,\ A_2,\ A_{1,1},\ A_{1,2},\ A_{2,1},\ A_{2,2},$

$\quad A_{1,11},\ A_{1,12},\ A_{1,22},\ A_{2,11},\ A_{2,12},\ A_{2,22},\ x_1,\ x_2]);$

$\quad \mathrm{collect}(JJ,\ [x_1,\ x_2])$

$\quad\mathbf{end}$

Algorithm 1.2 : decomposition of a multi-degree following the list t.

intquo := **proc**(*i1*, *i2*)
  if $i2 = 0$ **then** RETURN(∞) **else** RETURN(iquo(*i1*, *i2*)) **fi end**
reduire1 := **proc**(m, t, M, e)
  local i, k; **global** T;
  for k **to** max(seq(m_i, $i = 1..\text{nops}(m)$)) **do**
  if $m - k \times t = [0 \$ \text{nops}(m)]$ **then** RETURN($M + k \times T_e$) **else** *next* **fi**
  od;
  RETURN(*NULL*)
  end
reduire := **proc**(m, t, M, e)
  local i, j; **global** T;
  if $\{\text{op}(m)\} = \{0\}$ **then** RETURN(M) **fi** ;
  if $\text{nops}(t) \neq 1$ **then**
  i := min(seq(intquo(m_j, t_{1j}), $j = 1..\text{nops}(m)$)) ;
  seq(reduire($m - j \times t_1$, subsop($1 = NULL$, t), $M + j \times T_e$, $e + 1$),
  $j = 0..i$)
  else RETURN(reduire1(m, t_1, M, e))
  fi
  end

Algorithm 1.3 : it constructs a basis of the linear space $K(p, q, r, s)$, starting from $\mathcal{F}$.

$dcpcov$:= **proc**(m)
local $g, i, j, k, l, H, HH, J, R, S$; **global** M, T, U, t;
$S := \{\}$; $R := \mathrm{subs}(M = 0, \{\mathrm{reduire}(m, t, M, 1)\})$;
$J := \mathrm{vector}(\mathrm{nops}(R))$;
for l **to** $\mathrm{nops}(R)$ **do**
$H := \{1\}$; $J_l := \mathrm{map}(op, \mathrm{indets}(R_l))$;
for k **to** $\mathrm{nops}(J_l)$ **do**
if $\mathrm{has}(\{11, 16, 13, 41, 43, 45, 46, 47, 48, 49, 1, 2, 3, 4, 7, 9, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 29, 31, 33, 36, 37, 38, 39, 40\}, J_{lk})$ **then**
$H := \{\mathrm{seq}(H_j \times {U_{J_{lk},1}}^{\mathrm{coeff}(R_l, T_{J_{lk}})}, j = 1..\mathrm{nops}(H))\}$; **fi**;
if $\mathrm{has}(\{10, 14, 42, 44, 5, 8, 28, 30, 32, 34\}, J_{lk})$ **then**
$HH := \{\}$;
$HH := \{\mathrm{seq}({U_{J_{lk},1}}^i \times {U_{J_{lk},2}}^{(\mathrm{coeff}(R_l, T_{J_{lk}})-i)}, i = 0..\mathrm{coeff}(R_l, T_{J_{lk}}))\}$;
$H := \{\mathrm{seq}(\mathrm{seq}(H_j \times HH_i, j = 1..\mathrm{nops}(H)), i = 1..\mathrm{nops}(HH))\}$
fi;
if $\mathrm{has}(\{15, 12, 6, 17, 35\}, J_{lk})$ **then**
$HH := \{\}$;
for i **from** 0 **to** $\mathrm{coeff}(R_l, T_{J_{lk}})$ **do for** g **from** 0 **to** $\mathrm{coeff}(R_l, T_{J_{lk}})$ **do**
if $\mathrm{coeff}(R_l, T_{J_{lk}}) - i - g < 0$ **then** *next*
else $HH := HH$ union $\{{U_{J_{lk},1}}^i \times {U_{J_{lk},2}}^g \times {U_{J_{lk},3}}^{(\mathrm{coeff}(R_l, T_{J_{lk}})-i-g)}\}$
fi; **od**; **od**;
$H := \{\mathrm{seq}(\mathrm{seq}(H_j \times HH_i, j = 1..\mathrm{nops}(H)), i = 1..\mathrm{nops}(HH))\}$
fi; **od**;
$S := S$ union $\mathrm{eval}(H, 1)$
od; S; **end**

Algoritm 1.4 : expression of any covariant in the basis $\mathcal{F}$.

$ExpCovAff$:= **proc**(cv, CV)
local S, Ss, SS, SV, CVV, i, $varT$, $delpol$, sys, $sosol$, ind, α;
global V, A, x, U, K;
SS := dcpcov(cv) ; S := [seq(SS_i, $i = 1$..nops(SS))] ;
SV := subs($U = V$, S) ; SV := subs($x_1 = 1$, $x_2 = 0$, eval(SV)) ;
CVV := subs($x_1 = 1$, $x_2 = 0$, eval(CV)) ;
$varT$:=
$[A_1, A_2, A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}, A_{1,11}, A_{1,12}, A_{1,22}, A_{2,11}, A_{2,12}, A_{2,22}]$;
ind := {seq(α_i, $i = 1$..nops(SV))} ;
$delpol$:=
collect(add($\alpha_i \times SV_i$, $i = 1$..nops(SV)) $-$ CVV, $varT$, $distributed$) ;
sys := {coeffs($delpol$, $varT$)} ; $sosol$:= solve(sys, ind) ;
Ss := add(subs($sosol$, α_i) $\times S_i$, $i = 1$..nops(S)) ;
RETURN(subs(seq($\alpha_j = 0$, $j = 0$..nops(ind)), Ss)); **end**

7.1.2 Computing of affine covariants

Algorithm 2.1 : list of multi-degree of terms of an affine covariant.

$DegHom$:= **proc**(p, q, r, s)
local l, pp, i, H;
H := {} ; pp := expand($(a + b \times x + c \times x^2)^p \times (b + 2 \times c \times x)^q \times c^r$) ;
if type(pp, $monomial$) = $true$ **then** H := {[p, q, r, s]}**else**
l := convert(pp, $list$) ;
for i **to** nops(l) **do**
H := H union {[degree(l_i, a), degree(l_i, b), degree(l_i, c), degree(l_i, x)]}
od; H; **fi**
end

Algorithm 2.2 : we test if the monomial P has the multidegree n w.r.t. $varH$.

```
DegMonHom := proc(P, varH, n)
    if varH = [] then RETURN(true) fi ;
    if degree(P, varH_1) = n_1 then
    if nops(varH) = 1 then RETURN(true)
    else RETURN(DegMonHom(P, subsop(1 = NULL, varH),
    subsop(1 = NULL, n)))
    fi
    else RETURN(false); fi
    end
```

Algorithm 2.3 : determination of the (n_1, n_2, n_3, n_4) multi-homogeneous part of a given affine covariant P w.r.t. $varH$.

```
PartHom := proc(P, varH, n)
    local   PP, L, varTx, i;
    varTx := [seq(op(varH_i), i = 1..nops(varH))] ;
    PP := collect(P, varTx, distributed) ;
    L := select(DegMonHom, convert(PP, list), varH, n) ;
    convert(L, ` + `)
    end
```

Algorithm 2.4 : expression of a given affine covariant w.r.t. to F.

```
DevCovAff := proc(v, CT)
    local  C, VV, Vv, H, vv, i, j, varT;
    C := subs(A = B, CT) ;
    VV := collect(subs(B_1 = A_1 + A_{1,1} × x_1 + A_{1,2} × x_2 + A_{1,11} × x_1^2
    + 2 × A_{1,12} × x_1 × x_2 + A_{1,22} × x_2^2, B_2 = A_2 + A_{2,1} × x_1 + A_{2,2} × x_2
    + A_{2,11} × x_1^2 + 2 × A_{2,12} × x_1 × x_2 + A_{2,22} × x_2^2,
    B_{1,1} = A_{1,1} + 2 × A_{1,11} × x_1 + 2 × A_{1,12} × x_2,
    B_{1,2} = A_{1,2} + 2 × A_{1,12} × x_1 + 2 × A_{1,22} × x_2,
    B_{2,1} = A_{2,1} + 2 × A_{2,11} × x_1 + 2 × A_{2,12} × x_2,
    B_{2,2} = A_{2,2} + 2 × A_{2,12} × x_1 + 2 × A_{2,22} × x_2, B_{1,11} = A_{1,11},
    B_{1,12} = A_{1,12}, B_{1,22} = A_{1,22}, B_{2,11} = A_{2,11}, B_{2,12} = A_{2,12},
    B_{2,22} = A_{2,22}, C), [x_1, x_2], distributed);
    H := DegHom(v_1, v_2, v_3, v_4) ;
    for i to nops(H) do
    vv_i := subs(x_1 = 1, x_2 = 0, PartHom(VV, varH, H_i)) ;
    Vv_i := ExpCovAff(H_i, vv_i) ;
    Vv_i := subs(seq(α_j = 0, j = 0..nops(ind)), Vv_i) ;
    for j to nops(ind) do Vv_i := subs(α_j = 0, Vv_i) od ;
    Vv_i := subs(U_{1,1} = J_1, U_{2,1} = J_17, U_{3,1} = J_2, U_{4,1} = J_18, U_{5,1} = J_19,
    U_{5,2} = J_20, U_{6,1} = J_3, U_{6,2} = J_4, U_{6,3} = J_5, U_{7,1} = J_21, U_{8,1} = J_22,
    U_{8,2} = J_23, U_{9,1} = J_24, U_{10,1} = J_25, U_{10,2} = J_26, U_{11,1} = J_6, U_{12,1} = J_7,
    U_{12,2} = J_8, U_{12,3} = J_9, U_{13,1} = J_27, U_{14,1} = J_28, U_{14,2} = J_29, U_{15,1} = J_30,
    U_{15,2} = J_31, U_{15,3} = J_32, U_{16,1} = J_10, U_{17,1} = J_11, U_{17,2} = J_12,
    U_{17,3} = J_13, U_{18,1} = J_33, U_{19,1} = J_34, U_{20,1} = J_35, U_{21,1} = J_36, U_{22,1} = J_14,
    U_{23,1} = J_15, U_{24,1} = J_16, U_{25,1} = K_21, U_{26,1} = K_1, U_{27,1} = K_22,
    U_{28,1} = K_3, U_{28,2} = K_4, U_{29,1} = K_23, U_{30,1} = K_24, U_{30,2} = K_25,
    U_{31,1} = K_8, U_{32,1} = K_9, U_{32,2} = K_10, U_{33,1} = K_26, U_{34,1} = K_27,
    U_{34,2} = K_28, U_{35,1} = K_14, U_{35,2} = K_15, U_{35,3} = K_16, U_{36,1} = K_29,
    U_{37,1} = K_30, U_{38,1} = K_19, U_{39,1} = K_20, U_{40,1} = K_2, U_{41,1} = K_31,
    U_{42,1} = K_6, U_{42,2} = K_7, U_{43,1} = K_32, U_{44,1} = K_12, U_{44,2} = K_13,
    U_{45,1} = K_33, U_{46,1} = K_18, U_{47,1} = K_5, U_{48,1} = K_11, U_{49,1} = K_17,
    Vv_i); od; collect(add(Vv_i, i = 1..nops(H)), [seq(K_i, i = 0..33)]); end
```

7.2. Minimal system of generators of the algebra $\mathcal{Q}(2,2)$.

Q_1	$J_1 + 2\,K_1$
Q_2	$J_2 + 4\,K_3 + 4\,K_7,$
Q_3	$J_3 - 2\,K_{10} + 2\,K_9,$
Q_4	$J_4 - 2\,K_{10},$
Q_5	$J_5 - 2\,K_{10} + 2\,K_9,$
Q_6	$(J_5 - 2\,J_3 + J_4)\,K_1 + J_6 - 2\,K_{16} + 4\,K_{15} - 2\,J_1\,K_{10} + J_1\,K_9 + 4\,K_{18},$
Q_7	$J_7,$
Q_8	$J_8,$
Q_9	$J_9,$
Q_{10}	$(2\,J_1\,J_5 + 8\,K_{18} + 2\,J_1\,K_{10} - J_1\,J_3)\,K_1 + (2\,J_9 - 6\,J_8 + 2\,J_7)\,K_2 + (J_4 - 2\,J_3 + 5\,J_5 - 4\,K_{10})\,K_3 + (J_4 - 7\,J_5)\,K_4 - 4\,J_8\,K_5 + (-8\,K_9 + 4\,K_{10} - 2\,J_4 - 4\,J_5 + 4\,J_3)\,K_6 + (-2\,J_3 + 8\,J_5 + 4\,K_9)\,K_7 + (-J_1^2 + 3\,J_2)\,K_9 + (-J_1^2 - J_2)\,K_{10} + J_{10} + 6\,K_{19} - 3\,J_1\,K_{14} + 4\,J_1\,K_{18} + 4\,J_1\,K_{15} - 2\,J_1\,K_{16},$
Q_{11}	$J_{11} - 2\,K_{20},$
Q_{12}	$(J_8 - J_7)\,K_1 + J_{12} - 2\,K_{20},$
Q_{13}	$J_{13} - 2\,J_7\,K_1,$
Q_{14}	$(J_1\,J_9 - J_1\,J_8 - 2\,J_{12})\,K_1 + (-2\,J_9 - 2\,J_7)\,K_3 + 2\,J_8\,K_4 + (2\,J_7 + 2\,J_8)\,K_6 - 4\,J_7\,K_7 + (4\,J_3 - 2\,J_5)\,K_9 + J_{14} - 2\,J_4\,K_{10},$
Q_{15}	$J_{15},$
Q_{16}	$-4\,J_7\,K_1^3 + (10\,J_{12} - 4\,K_{20})\,K_1^2 + (\frac{3}{2}\,J_1\,J_{12} - J_3^2 - 16\,K_9\,K_{10} - 8\,J_9\,K_7 + 20\,J_7\,K_4 + 8\,J_8\,K_6 - \frac{3}{2}\,J_5^2 + \frac{1}{2}\,J_2\,J_8 - 8\,J_8\,K_3 - \frac{5}{2}\,J_1\,J_{11} + \frac{3}{2}\,J_4\,J_5)K_1 + (-8\,J_{12} + 4\,K_{20} - \frac{5}{2}\,J_1\,J_8 + J_{13} + 7\,J_{11})\,K_3 + (\frac{1}{2}\,J_1\,J_8 - 5\,J_{11} + 5\,J_{12} - J_{13})\,K_4 - 8\,J_{15}\,K_5 + (8\,J_{13} + 4\,J_{12} + 4\,K_{20} - 4\,J_{11})\,K_6 + (-16\,K_{20} - J_1\,J_8 + J_1\,J_9 - 10\,J_{12} + 8\,J_{11} - 8\,J_{13})\,K_7 + (3\,J_8 - 3\,J_7)\,K_8 + 6\,J_1\,K_9^2 + (-24\,K_{16} + \frac{3}{2}\,J_1\,J_4 + \frac{1}{2}\,J_1\,J_3 - J_1\,J_5 - 12\,K_{18} + 8\,K_{14} - 28\,J_1\,K_{10})\,K_9 + 20\,J_1\,K_{10}^2 + (20\,K_{16} + \frac{1}{2}\,J_1\,J_5 - 8\,K_{14} + 4\,K_{15} - \frac{3}{2}\,J_1\,J_3 - \frac{3}{2}\,J_1\,J_4 + 24\,K_{18})\,K_{10} + (6\,J_8 - 2\,J_9)\,K_{12} + (-2\,J_8 - 10\,J_7)\,K_{13} + (3\,J_3 + 3\,J_5 - 5\,J_4)\,K_{14} + 5\,J_4\,K_{15} + (J_5 - 4\,J_3)\,K_{16} + (4\,J_8 + 16\,J_9)\,K_{17} + 4\,J_5\,K_{18} + (-J_2 + \frac{1}{2}\,J_1^2)\,K_{20} + J_{16},$
Q_{17}	$J_{17} + K_4 + K_6,$
Q_{18}	$(-\frac{3}{2}\,K_2 - 2\,K_{21} - K_5)\,K_1^2 + (-2\,J_1\,K_5 - 2\,J_1\,K_2 - 4\,K_{22} + K_{11} + 2\,K_{31})\,K_1 + (\frac{1}{2}\,J_2 + 2\,J_{17} + 2\,K_3 + \frac{3}{2}\,K_7 - \frac{1}{2}\,J_1^2)\,K_2 + (K_5 + 4\,K_{21})\,K_3 - 2\,K_{21}\,K_4 + (-J_1^2 + K_7)\,K_5 + 2\,K_{21}\,K_7 + J_1\,K_{11} + (J_2 + 2\,J_{17})\,K_{21} + J_{18} - 2\,K_{23} - 2\,K_{32} - J_1\,K_{22} + 3\,J_1\,K_{31},$

(continued on the next page)

Q_{19}	$J_{19} + 2\,K_{25} + K_8 - J_1\,K_1^2 + J_1\,K_7 + 2\,K_1\,K_4 - 2\,K_{13} + 3\,K_{12} + 2\,K_{17}$,
Q_{20}	$-K_1^3 - 2\,J_1\,K_1^2 + (\frac{1}{2}\,J_2 - \frac{1}{2}\,J_1^2 + 2\,K_3 + K_7 + 2\,K_4 + 2\,K_6)\,K_1 + J_{20} + 2\,K_{24} + J_1\,K_4 - K_{13} + 2\,J_1\,K_6$,
Q_{21}	$(-\frac{3}{2}\,J_1\,K_5 - \frac{3}{2}\,J_1\,K_2)\,K_1^2 + ((-\frac{1}{2}\,J_1^2 - 3\,J_{17} - \frac{1}{2}\,J_2 - \frac{3}{2}\,K_6)\,K_2 + (\frac{3}{2}\,K_5 - 3\,K_{21})\,K_4 + (-\frac{3}{2}\,J_{17} - K_6 - 2\,J_1^2)\,K_5 - \frac{3}{2}\,K_{21}\,K_6 + 2\,J_1\,K_{11} + (-3\,J_{17} - 3\,J_2)\,K_{21} + 2\,J_{18} + 2\,J_1\,K_{31} - 2\,K_{32})K_1 + (-3\,J_1\,J_{17} + \frac{3}{2}\,J_1\,K_7 - J_1\,K_4 + 3\,K_{12} - 3\,K_{13} + K_8 + 3\,J_{20} + 6\,K_{25} + J_1\,K_3 + 6\,K_{25} + J_1\,K_3 - 2\,J_1\,K_6)\,K_2 + 3\,K_{22}\,K_3 + (2\,K_{31} + J_1\,K_{21} + 2\,K_{11} - 4\,K_{22})\,K_4(\frac{3}{2}\,K_{25} - 3\,K_{13} + 3\,K_{12} + K_{17} - J_1\,J_2 - J_1\,K_6)\,K_5 + (K_{31} - 2\,J_1\,K_{21} + K_{11})\,K_6(\frac{3}{2}\,K_{11} - 3\,K_{22} + 3\,J_1\,K_{21})\,K_7 + 3\,K_{21}\,K_8 + (\frac{1}{2}\,J_2 + \frac{1}{2}\,J_1^2)\,K_{11} + 2\,K_{21}\,K_{13} + (2\,J_1\,J_{17} - 2\,J_{20} + 3\,K_{25})\,K_{21} - 3\,J_1\,K_{23} + 3\,K_{26} + (2\,J_{17} + \frac{3}{2}\,J_2 + \frac{3}{2}\,J_1^2)\,K_{31} + J_{21} + \frac{3}{2}\,K_{21}\,K_{17} - 3\,J_1\,K_{32}$,
Q_{22}	$-J_1\,K_1^3 + (-J_{17} - J_1^2 + K_3 - \frac{1}{2}\,K_6)\,K_1^2 + (2\,J_{19} + K_{25} + J_1\,K_3 + J_1\,K_4 + \frac{1}{2}\,K_{17} - 2\,J_1\,J_{17} + J_1\,K_6)\,K_1 + (-J_3 + K_{10} - 2\,K_9)\,K_2 + (2\,J_{17} + K_7)\,K_4 + (-J_4 - \frac{1}{2}\,K_9)\,K_5 + K_6^2 + (2\,J_{17} + \frac{1}{2}\,J_2 + \frac{1}{2}\,J_1^2)\,K_6 + J_{22} - 2\,K_{28} + 2\,J_1\,K_{24} - K_{21}\,K_9 - J_1\,K_{13}$,
Q_{23}	$-J_1\,K_1^3 + (-\frac{1}{2}\,J_2 + 2\,K_4 - \frac{1}{2}\,K_7)\,K_1^2 + (2\,K_{12} + 2\,K_{17} - 2\,K_{13} - J_1\,J_{17} + J_{20})\,K_1 + (-J_5 + J_4 + K_{10})\,K_2 + K_3\,K_7 + K_4^2 + (J_{17} - K_7)\,K_4 + (-J_5 + K_{10})\,K_5 + \frac{1}{2}\,K_7^2 + 2\,K_{33} - 2\,K_{28} + 2\,K_{27} + J_1\,K_{12} + J_{23} - J_1\,K_{13} - J_5\,K_{21} - K_6\,K_7 + J_1\,K_{25} + \frac{1}{2}\,J_2\,K_7 + J_1\,K_{17}$,
Q_{24}	$-2\,K_1^4 + (4\,K_6 - \frac{5}{4}\,J_2 - K_3)\,K_1^2 + (4\,K_{17} - J_1\,J_{17} - 3\,J_1\,K_7 - J_1\,K_3 + 2\,K_{12} - \frac{1}{2}\,J_1\,K_4 + 2\,J_{20} + 4\,K_8 + 4\,K_{24})\,K_1 + (-2\,K_9 + 3\,J_3 - \frac{1}{2}\,J_5 - K_{10})\,K_2 + K_3^2 + (\frac{1}{2}\,J_2 + 4\,K_7 - 2\,K_4 - \frac{1}{2}\,J_1^2)\,K_3 + \frac{5}{2}\,K_4^2 + 3\,K_4\,K_7 + (4\,K_{10} - J_5)\,K_5 - 4\,K_6^2 + (\frac{1}{2}\,J_1^2 - 4\,J_{17} - \frac{1}{2}\,J_2)\,K_6 + 2\,K_7^2 + (-\frac{1}{4}\,J_1^2 + \frac{3}{2}\,J_2)\,K_7 + J_1\,K_8 + 4\,K_{21}\,K_9 + \frac{7}{2}\,J_1\,K_{12} - \frac{5}{2}\,J_1\,K_{13} + 3\,J_1\,K_{17} + (-J_4 + J_5)\,K_{21} + J_{24} + 4\,K_{27} + 4\,K_{33} + J_1\,K_{25} - 2\,K_{28}$,
Q_{25}	$(J_3 - \frac{1}{2}\,J_4 + \frac{1}{2}\,J_5 - K_{10} + 2\,K_9)\,K_1 + J_{25} - K_{15} + \frac{1}{2}\,J_1\,K_9 - K_{18}$,
Q_{26}	$(J_3 + K_9)\,K_1 + J_{26} - K_{16}$,
Q_{27}	$J_{27} + (11\,K_3^2 + (\frac{1}{2}\,J_1^2 - \frac{5}{2}\,K_4 - \frac{1}{2}\,J_2)\,K_3 - 2\,K_4^2 + (\frac{7}{2}\,K_7 + \frac{1}{2}\,J_2 - \frac{1}{2}\,J_1^2 - K_6)\,K_4 + (-7\,J_3 + 3\,J_5)\,K_5 + \frac{5}{2}\,K_6^2 + (-\frac{9}{2}\,J_1^2 + 7\,J_2)\,K_6 - \frac{5}{4}\,K_7^2 + (\frac{7}{2}\,J_1^2 - \frac{27}{4}\,J_2)\,K_7 + \frac{9}{2}\,J_1\,K_{13} + 6\,J_1\,K_{17} + (2\,J_5 - 2\,J_4)\,K_{21} - 3\,J_{24} + 2\,K_{28} - 2\,K_{27} - \frac{3}{2}\,J_{23} + \frac{9}{2}\,J_{17}^2 - 4\,J_1\,K_{25})K_2 + ((-\frac{5}{2}\,J_1^2 - \frac{5}{2}\,K_6 + \frac{5}{2}\,K_7)\,K_5 + 2\,J_1\,K_{11} + (\frac{3}{2}\,J_2 - \frac{3}{2}\,J_1^2 - 9\,J_{17})\,K_{21} - 6\,K_{32})\,K_4 - \frac{13}{2}\,K_4^2\,K_5 + (-7\,J_1\,K_{11} + (-2\,J_2 - 6\,J_{17})\,K_{21} + \frac{9}{2}\,K_{32} + 3\,J_1\,K_{22} + 3\,K_{23})\,K_7 + ((J_1^2 + K_6 - \frac{11}{2}\,J_2 - \frac{1}{2}\,K_7)\,K_5 - 4\,J_1\,K_{31} + 4\,K_{32} + \frac{9}{2}\,J_1\,K_{11})\,K_3 + \frac{1}{2}\,K_1^4\,K_5 - 2\,K_{10}\,K_5^2 + (-2\,K_{31} - 12\,K_{11} + 2\,K_{22} + J_1\,K_{21})\,K_8+$

(continued on the next page)

Q_{27} (end)	$(12\,K_{24} + \frac{1}{2}\,J_1^3 - 6\,J_{20} + 4\,K_{25} + 6\,J_1\,J_{17} - 4\,K_{12} - 5\,K_{13} - \frac{1}{2}\,J_1\,J_2 + K_{17})\,K_{11} - 3\,K_{21}^2\,K_{10} + 4\,K_{25}\,K_{31} - 12\,K_{24}\,K_{31} + (-10\,K_{31} + 6\,J_1\,K_{21})\,K_{12} + (3\,K_{31} - \frac{7}{2}\,J_1\,K_2 - 3\,J_1\,K_5)\,K_1^3 + (-2\,J_1\,K_{21} + 20\,K_{31})\,K_{13} + ((-\frac{5}{2}\,K_6 + \frac{5}{4}\,K_7 + \frac{5}{4}\,J_2 + \frac{7}{2}\,K_3)\,K_2 + (3\,K_5 + K_{21})\,K_3 - 5\,J_1\,K_{22} - 6\,K_{21}\,K_6 + \frac{7}{2}\,K_{32} - 2\,J_2\,K_{21} - \frac{3}{2}\,J_1\,K_{11} - 3\,K_6\,K_5 + K_{21}\,K_4)K_1^2 + (-3\,K_{22} - J_1\,K_{21} - K_{31})\,K_{17} - 6\,J_4\,K_{21}^2 + 3\,K_{21}\,K_6^2 + (-K_{25} + 6\,K_{24} - 2\,J_{19})\,K_{22} + (J_1\,K_{25} + 3\,K_{28} + J_{17}^2 + 2\,J_1\,J_{19} - 3\,K_{27} - J_{23})\,K_{21} + (-\frac{3}{2}\,J_1^2 + \frac{3}{2}\,J_2)\,K_{23} + (3\,J_{17} + \frac{3}{2}\,J_2 - \frac{3}{2}\,J_1^2)\,K_{32} + (\frac{3}{2}\,J_1^3 - J_{19} + 6\,J_{20} - \frac{3}{2}\,J_1\,J_2)\,K_{31} + ((-K_{24} - \frac{1}{2}\,J_1^3 + \frac{1}{2}\,J_1\,J_2 - K_{13} + \frac{9}{2}\,K_8 - 9\,J_1\,K_3 + 2\,J_{20})\,K_2 + (6\,K_{11} + 6\,K_{22})\,K_3 - 4\,K_{22}\,K_4 + (-K_8 + \frac{3}{2}\,J_1\,K_6 + 12\,J_{19} + \frac{1}{2}\,J_1\,J_2 + \frac{1}{2}\,J_1\,K_7 - 8\,J_1\,J_{17} + 3\,K_{25} - \frac{5}{2}\,K_{13})\,K_5 + 3\,J_1\,K_{21}\,K_6 - J_1\,K_{21}\,K_7 + 6\,K_{21}\,K_8 - \frac{1}{2}\,J_2\,K_{11} - 21\,K_{21}\,K_{13} + 6\,J_{19}\,K_{21} + (\frac{3}{2}\,J_2 - 6\,J_{17} - \frac{3}{2}\,J_1^2)\,K_{22} + 2\,J_{21} + 3\,J_{17}\,K_{31})K_1 + (\frac{13}{2}\,J_5 + \frac{7}{2}\,K_{10} - \frac{17}{2}\,J_3)\,K_2^2 + (2\,K_6^2 + (5\,J_2 - 5\,J_1^2 + K_7)\,K_6 - \frac{1}{2}\,K_7^2 + (-\frac{9}{4}\,J_2 + \frac{19}{4}\,J_1^2)\,K_7 + 8\,J_1\,K_{13} + \frac{1}{4}\,J_2^2 + 2\,J_1\,K_{24} - 5\,J_1\,K_{12} + 12\,K_{28} - 4\,J_{24} - J_1\,K_{17} - 6\,J_1^2\,J_{17} - 3\,K_{21}\,K_9 - 12\,K_{27} - \frac{1}{4}\,J_1^4 - 3\,K_{33} + \frac{11}{2}\,J_1\,K_8)K_5 + (3\,K_{21}\,K_7 + 6\,J_1\,K_{11} + (15\,J_{17} + 6\,J_2)\,K_{21} + 3\,K_{32} - 12\,K_{23} + 2\,J_1\,K_{31})\,K_6,$
Q_{28}	$J_{28} + 2\,K_{29} + J_1\,K_{27} + 2\,J_1\,K_{33} - J_2\,K_1^3 - 4\,K_{21}\,K_{14} + 2\,J_5\,K_{31} + (4\,J_1\,K_6 + J_1\,K_4 + J_1\,K_7 - 2\,K_{12} + 6\,K_{25})\,K_3 - J_1\,K_{28} + (2\,J_1\,K_{10} - 2\,J_{26} - 2\,K_{14} - 4\,K_{15} - \frac{1}{2}\,J_1\,J_5 - J_6 + 2\,K_{16} + \frac{1}{2}\,J_1\,J_4)\,K_2 - 2\,J_5\,K_{11} + (-2\,J_1\,K_6 + J_{19} + 5\,K_{17} + \frac{1}{2}\,J_1\,J_{17})\,K_4 + (-J_1\,K_7 - \frac{1}{4}\,J_1^3 - 3\,J_1\,K_3)\,K_1^2 + (-2\,J_6 + K_{14} - 2\,J_1\,J_4)\,K_5 + (-\frac{1}{2}\,J_1\,J_2 + \frac{1}{2}\,J_1^3 - 4\,K_8 + K_{12} + 2\,K_{17} - 2\,J_1\,J_{17})\,K_6 + (2\,J_{20} + K_{17} + K_8 + 2\,K_{25} - 3\,J_{19} + 2\,J_1\,J_{17} + \frac{1}{2}\,J_1\,J_2 - \frac{1}{4}\,J_1^3 + \frac{5}{2}\,K_{12} - -\frac{1}{2}\,J_1^2)\,K_{25} + (2\,J_{17} + \frac{1}{2}\,J_2)\,K_{12} + (J_3 - J_5)\,K_{22} + ((3\,J_4 - 2\,K_9 + J_3)\,K_2 + 4\,K_3^2 + (K_4 - \frac{1}{2}\,K_6 - J_{17})\,K_3 + 2\,K_4^2 + (-K_9 - \frac{1}{2}\,J_3 + \frac{1}{2}\,J_4)\,K_5 - K_6\,K_7 + (-J_2 - 2\,J_{17})\,K_7 - 2\,K_{21}\,K_9 + 2\,J_1\,K_{12} + (-J_3 + J_4)\,K_{21} - \frac{1}{2}\,J_1\,J_{20} - 4\,J_1\,K_{25} - \frac{1}{2}\,J_1^2\,J_{17} + 2\,J_1\,K_{24} + J_{24})K_1 + (4\,J_{17} + 3\,J_2 - J_1^2)\,K_{17} - \frac{1}{2}\,J_1^2\,K_{13} + (\frac{1}{2}\,J_1\,J_5 - J_6)\,K_{21} + J_1\,K_7^2,$
Q_{29}	$-K_1^3\,K_6 + (-\frac{3}{2}\,K_{13} - J_1\,J_{17} - 2\,K_{24} - 7\,K_8)\,K_1^2 + ((-6\,J_5 + 5\,J_3 + J_4)\,K_2 + 3\,K_3^2 + 2\,K_4^2 + (-J_{17} - K_6 - J_1^2)\,K_4 - K_{10}\,K_5 + K_6^2 + (-3\,J_1^2 + K_7)\,K_6 - J_{17}^2 - J_{23} + 3\,J_1^2\,K_7 - 2\,K_{27} - 2\,J_1\,J_{20} + 2\,J_{22} + J_1\,K_{25} + 2\,K_{28})K_1 + (-J_6 + J_1\,J_4 - 17\,K_{15} + J_{25} + 5\,K_{16} + 12\,K_{14} - 4\,J_{26})\,K_2 + (9\,J_1\,K_6 - 12\,K_{12} + K_{17} + J_1\,K_4 + 11\,K_{13} - 6\,J_1\,K_7)\,K_3 + J_1\,K_4^2 + (K_{25} + 2\,J_{20} - 2\,K_{13} + 2\,J_{19} - 5\,J_1\,K_7 + 2\,J_1\,K_6 + 4\,K_{24})\,K_4 + (4\,J_{25} - K_{14} - 8\,J_{26} - K_{18} - 6\,J_1\,J_3 + 3\,J_1\,J_4 - 4\,K_{15} - 2\,J_6)\,K_5 + (5\,K_{13} + 4\,K_{24} - \frac{1}{2}\,J_1\,J_2 + \frac{1}{2}\,J_1^3 - 3\,K_8 + K_{17} + 2\,K_{25})\,K_6 + (-K_{12} + 12\,K_8 + \frac{5}{2}\,K_{13})\,K_7 + (K_{22} + 4\,K_{31})\,K_9 + (-4\,K_{11} - 4\,K_{31})\,K_{10} + 4\,J_{17}\,K_{12} + (\frac{1}{2}\,J_2 - \frac{1}{2}\,J_1^2 - 2\,J_{17})\,K_{13} + 2\,K_{21}\,K_{18} + 2\,J_4\,K_{22} + (-J_2 + J_1^2)\,K_{24} + 2\,J_{17}\,K_{25} + J_{29},$

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Q_{30}	$(3\,K_9-2\,K_{10}-\frac{1}{2}\,J_4)\,K_1^2+(-J_1\,K_{10}+\frac{1}{2}\,J_1\,K_9-J_{25}+J_{26}-2\,K_{18})\,K_1+$ $(-\frac{3}{2}\,J_8-\frac{1}{2}\,J_7)\,K_2+(-2\,K_9+2\,K_{10})\,K_3+K_4\,K_9+(-2\,J_7+J_9-J_8)\,K_5+$ $(J_4+J_5-J_3)\,K_6+(-K_9+\frac{1}{2}\,J_5)\,K_7+J_1\,K_{18}+J_{17}\,K_9+K_{19}+2\,K_{30}-$ $J_8\,K_{21}-J_{17}\,K_{10},$
Q_{31}	$(-2\,K_{10}-\frac{1}{2}\,J_3+3\,K_9)\,K_1^2+(\frac{1}{4}\,J_1\,J_4-\frac{1}{2}\,J_1\,J_3-2\,K_{18}-\frac{1}{4}\,J_1\,J_5+K_{14}+$ $\frac{1}{2}\,J_6-2\,J_{25})\,K_1-J_7\,K_2-\frac{1}{2}\,J_4\,K_3+(\frac{1}{2}\,J_5+\frac{1}{2}\,J_3)\,K_4+(-3\,J_7+J_9)\,K_5+$ $(J_3+J_5)\,K_6+(-K_9-\frac{3}{2}\,J_3)\,K_7+(2\,J_{17}-\frac{1}{2}\,J_2+\frac{1}{4}\,J_1^2)\,K_9+J_1\,K_{18}+$ $2\,K_{30}-2\,J_9\,K_{21}+\frac{1}{2}\,J_1\,K_{15},$
Q_{32}	$(\frac{1}{2}\,J_4-2\,K_9-2\,J_3)\,K_1^2+(-J_1\,J_3+J_6-J_1\,K_9-J_{25}+2\,K_{15})\,K_1-$ $J_4\,K_3+(-2\,K_{10}+J_4+K_9)\,K_4+(-J_7-J_9)\,K_5+J_4\,K_6-\frac{1}{2}\,J_4\,K_7+$ $J_{17}\,K_9+(\frac{1}{2}\,J_1^2-2\,J_{17}-\frac{1}{2}\,J_2)\,K_{10}+J_1\,K_{16},$
Q_{33}	$2\,J_{17}\,J_5\,K_{21}+K_{17}^2+J_{33}-\frac{1}{4}\,K_7^3-K_{12}^2+9\,K_{25}^2+4\,K_3^3+9\,K_{13}^2-3\,K_{21}\,K_{19}+$ $(6\,J_1\,K_8-5\,K_{21}\,K_9+\frac{3}{4}\,J_1\,K_{17}+(\frac{3}{2}\,J_5+\frac{3}{2}\,J_4)\,K_{21}-10\,K_{27}+\frac{3}{2}\,J_1\,K_{25}+$ $\frac{3}{2}\,J_1\,J_{19}-6\,J_{24}+4\,K_{28}-\frac{3}{2}\,K_{33}-\frac{15}{2}\,J_{23}-\frac{7}{2}\,J_1^2\,J_{17}+\frac{11}{2}\,J_1\,J_{20})K_7+$ $(-\frac{1}{2}\,J_{20}-J_{19})\,K_{24}+(-6\,J_{19}+\frac{3}{2}\,K_{17}-8\,K_{13}+10\,K_{24}+4\,J_1\,J_{17})\,K_{12}+$ $(\frac{3}{2}\,J_1\,J_5+J_{25}+2\,J_{26}-2\,J_1\,J_3)\,K_{31}+2\,K_{16}\,K_{31}-28\,K_{15}\,K_{31}+(10\,K_{13}+$ $6\,K_{25}+3\,K_{12}+K_{17})\,K_8+(-\frac{3}{2}\,J_{20}+3\,J_{19})\,K_{25}-J_7\,K_5^2-3\,J_8\,K_{21}^2+$ $((\frac{3}{2}\,J_{17}+2\,J_1^2)\,K_{21}-\frac{11}{2}\,J_{18}-10\,J_1\,K_{31}+12\,K_{32})\,K_9+(\frac{3}{2}\,J_2+3\,J_{17})\,K_{33}+$ $(-3\,J_1\,K_{22}+13\,J_1\,K_{11}+2\,J_{18}-\frac{1}{2}\,J_1^2\,K_{21}-6\,K_{32}+3\,K_{23})\,K_{10}+(4\,K_{18}+$ $11\,J_6-2\,K_{14}+\frac{1}{2}\,J_1\,J_5+16\,K_{16}-8\,K_{15}-3\,J_1\,J_3+\frac{3}{2}\,J_1\,J_4)\,K_{11}+$ $(\frac{3}{2}\,K_{17}-8\,K_{25}+6\,J_{19}-6\,K_{24})\,K_{13}-6\,K_3^2\,K_4+((2\,J_1\,J_3-6\,J_1\,K_{10}-$ $12\,J_{26}+9\,J_1\,K_9-\frac{29}{4}\,K_{18}+6\,K_{14})\,K_2+(2\,J_{19}+6\,K_{12}-J_1\,K_4)\,K_3+$ $3\,K_4\,K_{24}+(-2\,K_{16}+2\,J_{26}-6\,K_{15}-3\,K_{14}-J_1\,J_5)\,K_5+(-J_1^3+3\,K_{17}+$ $2\,K_8)\,K_6-K_7\,K_{17}-7\,J_{17}\,K_8+13\,K_{22}\,K_9+2\,J_1\,K_{21}\,K_{10}+\frac{1}{2}\,J_2\,K_{12}+$ $(J_2+2\,J_1^2)\,K_{13}+\frac{7}{2}\,J_{17}\,K_{17}-6\,J_{26}\,K_{21}+(8\,J_4-7\,J_3-J_5)\,K_{22}+2\,J_1^2\,K_{24}-$ $\frac{3}{2}\,J_{17}\,K_{25}-6\,J_1\,K_{27}+(-11\,J_4+20\,J_3)\,K_{31}+5\,J_{29}-4\,J_{28}-\frac{3}{2}\,J_1\,J_{22})K_1+$ $(13\,K_{31}-3\,K_{22})\,K_{14}-6\,J_7\,K_2^2+6\,K_4^2\,K_6+\frac{3}{4}\,J_2\,K_7^2+((9\,J_5-6\,J_4+$ $12\,K_{10}-4\,J_3)\,K_3+(-15\,K_9+4\,J_4-J_3-10\,J_5)\,K_4+(-\frac{59}{4}\,J_7+\frac{3}{4}\,J_8)\,K_5+$ $(17\,K_{10}-\frac{51}{4}\,K_9)\,K_6+\frac{3}{2}\,K_9\,K_7+(7\,J_2+3\,J_{17}-6\,J_1^2)\,K_9+(-\frac{15}{4}\,J_2+$ $2\,J_{17})\,K_{10}+6\,K_{19}-10\,K_{30}+\frac{11}{2}\,J_1\,J_{26}-3\,J_1\,K_{16}-5\,J_1\,J_{25}-2\,J_1\,K_{15}+$ $2\,J_1\,K_{14})K_2-5\,K_{31}\,K_{18}+(\frac{1}{4}\,J_1\,J_2+6\,J_{19}-7\,K_{24}+3\,K_{25})\,K_{17}+(-K_4^2+$ $9\,J_{17}\,K_4-6\,J_4\,K_5+4\,J_1^2\,K_6-\frac{21}{4}\,J_2\,K_7-7\,J_1\,K_{12}+J_1\,K_{17}+(3\,J_4+$ $2\,J_3+3\,J_5)\,K_{21}-8\,K_{28}-\frac{3}{2}\,J_{23}+7\,K_{33}-\frac{3}{2}\,J_{17}^2+6\,K_{27})K_3+(-\frac{1}{2}\,K_{17}+$ $\frac{3}{4}\,J_1\,J_2-\frac{3}{2}\,J_1\,J_{17})\,K_1^3+(2\,K_{10}\,K_6+(\frac{1}{2}\,J_1^2-\frac{3}{2}\,J_2)\,K_9-3\,K_{30}-4\,J_{10}-$ $\frac{1}{4}\,J_2\,J_5+J_1\,K_{14}-3\,K_{19}-J_1\,K_{18}-\frac{1}{2}\,J_1^2\,J_5+3\,J_2\,J_3+6\,J_1\,K_{15}-4\,J_2\,J_4+$ $2\,J_1\,J_{25}-\frac{9}{2}\,J_2\,K_{10}-4\,J_1\,J_6)K_5+$

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Q_{33} (end)	$(-5\,K_9\,K_5-\frac{3}{4}\,J_2\,K_7-4\,K_{21}\,K_{10}-2\,J_1\,K_{13}+(-J_3+J_4)\,K_{21}+\frac{3}{2}\,J_{17}^2+7\,K_{33})\,K_4+(-\frac{29}{4}\,K_{10}\,K_2+\frac{1}{2}\,J_1^2\,K_3+(-\frac{1}{2}\,K_{10}+8\,J_3-3\,J_4-\frac{1}{2}\,K_9+3\,J_5)\,K_5+(-3\,J_2-\frac{5}{2}\,J_1^2)\,K_6+J_1\,K_{13}+2\,J_3\,K_{21}+3\,K_{27}+\frac{1}{4}\,K_7^2-2\,J_1\,J_{20}-\frac{3}{4}\,J_1\,K_{17}-\frac{3}{4}\,J_2\,K_7)K_1^2+(\frac{17}{2}\,J_1^2-3\,J_2-\frac{5}{2}\,K_7)\,K_6^2+(J_3-\frac{3}{2}\,J_5-3\,J_4)\,K_{23}+(-20\,J_1\,K_{13}-7\,K_{21}\,K_{10}+10\,K_{33}+\frac{27}{2}\,K_{21}\,K_9+K_7^2-10\,K_{27}-2\,J_{24}-5\,J_{17}\,K_7-J_3\,K_{21}+6\,J_1\,K_{12}-7\,J_1\,K_8)K_6-\frac{7}{2}\,J_{25}\,K_{22}+8\,J_{17}\,K_{27}-\frac{9}{2}\,J_{17}\,K_{28}+\frac{3}{2}\,J_1\,K_{29},$
Q_{34}	$J_1\,K_{30}+2\,K_{21}\,K_{20}+J_{34}+(-7\,J_3-4\,J_4)\,K_{12}+\frac{1}{2}\,J_2\,K_{18}+(J_5+J_4+5\,J_3)\,K_{24}+(2\,J_8\,K_2+(-2\,K_{10}-2\,J_3+4\,J_5)\,K_3+(2\,J_3-2\,K_{10}-6\,J_5)\,K_4+(-\frac{1}{2}\,J_8-\frac{1}{2}\,J_7)\,K_5+(3\,J_5-J_4)\,K_6+\frac{3}{2}\,K_9\,K_7+(J_{17}+J_1^2)\,K_9-4\,J_1\,K_{14}-J_1^2\,K_{10}-J_{17}\,J_3+2\,K_{19}-\frac{3}{2}\,J_1\,K_{16}+4\,J_1\,K_{15})K_1-\frac{1}{2}\,K_1^3\,K_9+4\,K_8\,K_9+J_{17}\,K_{14}-5\,J_{17}\,K_{15}-2\,J_3\,K_{25}-J_9\,K_{22}-2\,J_{13}\,K_{21}-J_7\,K_{11}+(2\,K_{15}-5\,J_{26}+10\,J_1\,K_{10}+10\,K_{16}+J_{25})\,K_3-3\,K_{20}\,K_2+(7\,J_{26}-4\,J_{25}+10\,K_{14}-8\,K_{16}-\frac{17}{2}\,J_1\,K_{10}-4\,J_1\,K_9-5\,K_{18})\,K_4+(\frac{1}{2}\,K_{18}-3\,K_{14}+K_{16}+\frac{5}{2}\,J_1\,K_9+J_1\,J_5)\,K_1^2-J_9\,K_{31}+(4\,J_{13}+K_{20})\,K_5+(-J_1\,K_{10}+\frac{1}{2}\,J_1\,J_5)\,K_6+(5\,J_4-3\,J_3)\,K_{13}+(-2\,K_{24}-J_1\,J_{17}+6\,K_{13})\,K_{10}+(3\,J_{20}+2\,K_{24}-K_{17}-2\,K_{13}-K_{12})\,K_9+(-\frac{1}{2}\,K_{18}-5\,J_6-\frac{1}{2}\,J_1\,K_9-J_{25})\,K_7,$
Q_{35}	$,(-11\,K_{12}-2\,K_{25}-2\,K_{24}+17\,K_{13}+4\,K_{17})\,K_9+2\,J_8\,K_{31}+(6\,K_{25}-7\,K_{12}-J_{20}-2\,K_{24}-20\,K_{17})\,K_{10}+(-3\,J_8+12\,J_9-6\,J_7)\,K_{11}+(\frac{1}{2}\,J_5+4\,J_3)\,K_{12}+(\frac{3}{2}\,J_5+J_4-4\,J_3)\,K_{13}+(-\frac{1}{2}\,J_1^2+\frac{1}{2}\,J_2+J_{17})\,K_{14}+J_1\,K_{30}+J_5\,K_{24}+(15\,J_4+4\,J_5+J_3)\,K_{17}+J_1\,K_{19}+(-J_1^2+3\,J_2)\,K_{18}+(-\frac{1}{2}\,J_1\,J_7+\frac{1}{2}\,J_1\,J_8)\,K_{21}+(-\frac{3}{2}\,J_5+2\,J_3)\,K_1^3+(J_9-2\,J_8)\,K_{22}-6\,K_{21}\,K_{20}+(\frac{5}{4}\,J_1\,J_5-4\,J_6-3\,J_1\,J_3+J_1\,J_4)\,K_1^2+(-J_4+J_5)\,K_{25}+(J_7\,K_2+15\,K_3\,K_9+(10\,J_3-6\,J_4-11\,K_9)\,K_4+8\,J_9\,K_5+(26\,J_3-12\,K_{10}-3\,J_5-15\,J_4+22\,K_9)\,K_6+(6\,K_{10}+\frac{3}{2}\,J_4)\,K_7+6\,K_{30}+J_1\,J_{25}-J_2\,K_{10}-2\,J_7\,K_{21})K_1+(-\frac{7}{4}\,J_1\,J_8-5\,K_{20}-\frac{5}{4}\,J_1\,J_7+\frac{1}{2}\,J_{12}-\frac{3}{2}\,J_{13}+5\,J_{11})\,K_2+(-3\,K_{14}+J_6+6\,K_{18}-\frac{3}{2}\,J_1\,K_9+K_{15}+\frac{3}{4}\,J_1\,J_5)\,K_7+(-8\,K_9+7\,K_{10})\,K_8+(3\,J_1\,K_9+6\,J_1\,K_{10}+7\,K_{16}+J_{25}-J_{26}-K_{14}-8\,K_{15})\,K_4+(3\,J_{11}-4\,J_1\,J_9-9\,J_{12})\,K_5+(6\,J_6-26\,K_{15}-20\,K_{18})\,K_6+(2\,J_{26}-3\,J_{25}+K_{14}-5\,J_1\,K_{10}-6\,K_{15}+J_1\,K_9)\,K_3+J_{35},$
Q_{36}	$-\frac{1}{2}\,J_8\,K_1^2+(2\,K_{20}-\frac{1}{4}\,J_1\,J_8-\frac{1}{4}\,J_1\,J_9)\,K_1+(\frac{1}{2}\,J_9+\frac{1}{2}\,J_7)\,K_3+(\frac{1}{2}\,J_8-+(\frac{1}{2}\,J_8-\frac{1}{2}\,J_7)\,K_6-\frac{1}{2}\,J_7\,K_7+(J_3-K_{10})\,K_9+K_{10}^2+(-\frac{1}{2}\,J_5+\frac{1}{2}\,J_4-J_3)\,K_{10}+J_{36}+\frac{1}{2}\,J_1\,K_{20}.$

References

[1] H.Derksen, H.Kraft, Constructive Theory, Collection Algèbres non commutatives, groupes quatiques et invariants, *Sémin. Congr., 2 (1995),*

SMF, 221 - 224.

[2] J.Dieudonné, J.Carrell, Invariant theory, old and new, *Advances in Math. 4 (1970), 1-80.*

[3] G.B.Gurevitch, Foundations of the theory of algebraic invariants, *Noordhoff, Ed; Groningen (1964).*

[4] D. Hilbert, Invariant Theory, *1893* .

[5] W.J. Haboush, Reductive groups are geometrically reductive *Ann. of Math., 102, 1 (1975, 67 - 83.*

[6] M.Hochster, J.Roberts, Rings of invariants of reductive groups acting on regular rings are Cohen-Macauley, *Advances in Math. 13 (1974), 115-175.*

[7] V. L. Popov, Invariant Theory, *Amer. Math. Soc. Trans. (2), Volume 148 (1991), 99 - 112.*

[8] V. L. Popov, Constructive Invariant Theory, *Astérisque 87-88 (1981), 303 - 334.*

[9] C.S. Sibirskii, Algebraic invariants of differential equations and matrices, *Schtiintsa, Kishinev, Moldova (1976)* (en russe).

[10] C.S. Sibirskii, Introduction to the algebraic theory of invariants of differential equations, *Nonlinear Science, Theory and Applications, Manchester University Press, 1988.*

[11] N.I. Vulpe, Polynomial bases of comitants of differential systems ans their applications in qualitative theory (russian, eng. and french summ.), “Shtintsa”, Kishinev, 1986, 172 pp.1.60r.

[12] N.I. Vulpe, Number of linearly independant comitants of a system of differential equations, *Differential equations, 15, 6, 963-973.*

[13] H. Weyl, The classical groups, their invariants and representations. *Princeton 1939.*

[14] P. A.Worfolk, Zeros of equivariant vector fields : Algorithms for an invariant approach, *J. Symbolic Computation, 17 (1994), 487 - 511.*