



Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation
ESA - CNRS 6090

Sharper bounds for ψ , θ , π , p_k

Pierre Dusart

Rapport de recherche n° 1998-06

Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. 05 55 45 73 23 - Fax. 05 55 45 73 22 - laco@unilim.fr

<http://www.unilim.fr/laco/>

SHARPER BOUNDS FOR ψ , θ , π , p_k .

PIERRE DUSART

ABSTRACT. In default of a proof of the RIEMANN hypothesis, the best estimates for $\psi(x)$ and $\theta(x)$ and hence of $\pi(x)$, p_k and other functions of the primes, depend on the current state of knowledge of the zeros of $\zeta(s)$. With a better knowledge about the zeros of the RIEMANN ζ function, we can show sharper bounds for $\psi(x)$, $\theta(x)$, $\pi(x)$ and primes p_k .

1. INTRODUCTION

In many applications it is useful to have explicit error bounds in the prime number theorem. ROSSER and SCHOENFELD developed an analytic method which combines a numerical verification of the RIEMANN hypothesis with a zero-free region. The aim of this paper is to find sharper bounds for the CHEBYSHEV's functions $\psi(x)$, the logarithm of the least common multiple of all integers not exceeding x , and $\theta(x)$, the product of all primes not exceeding x :

$$\theta(x) = \sum_{p \leq x} \ln p, \quad \psi(x) = \sum_{\substack{p, \nu \\ p^\nu \leq x}} \ln p$$

where sum runs over primes p and respectively over powers of primes p^ν . The Prime Number Theorem could be written as follows:

$$\psi(x) = x + o(x), \quad x \rightarrow +\infty.$$

An equivalent formulation of the above theorem should be: for all $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that

$$|\psi(x) - x| < \varepsilon x \quad \text{for } x \geq x_0$$

or

$$|\theta(x) - x| < \varepsilon x \quad \text{for } x \geq x_0.$$

This article hangs up on some known results: the most important works on effective results have been shown by ROSSER & SCHOENFELD [13, 14, 16], ROBIN [9] & MASSIAS [6] and COSTA PEREIRA [4].

The proofs for estimates of $\psi(x)$ in [14] are based on the verification of RIEMANN hypothesis to a given height and an explicit zero-free region for $\zeta(s)$ whose form is essentially that the classical one of DE LA VALLÉE POUSSIN. ROSSER & SCHOENFELD have shown that the first 3 502 500 zeros of $\zeta(s)$ are on the critical strip. VAN DE LUNE *et al* [17] have shown that the first 1 500 000 000 zeros are

Date: June 11, 1998.

1991 Mathematics Subject Classification. Primary 11A25, 11A40; Secondary 11L20, 11-04.

Key words and phrases. Estimates of elementary functions.

on the critical strip. We will improve bounds for $\psi(x)$ and $\theta(x)$. We will prove the following results:

$$\begin{aligned} |\psi(x) - x| &\leq 0.006409 \frac{x}{\ln x} && \text{for } x \geq \exp(22), \\ |\theta(x) - x| &\leq 0.006788 \frac{x}{\ln x} && \text{for } x \geq 10\,544\,111, \\ |\theta(x) - x| &\leq 0.2 \frac{x}{\ln^2 x} && \text{for } x \geq 3594641, \\ |\theta(x) - x| &\leq 515 \frac{x}{\ln^3 x} && \text{for } x > 1, \\ |\theta(x) - x| &\leq 1717433 \frac{x}{\ln^4 x} && \text{for } x > 1. \end{aligned}$$

We apply these results on p_k , the k^{th} prime, and $\theta(p_k)$. Let's denote by $\ln_2 x$ for $\ln \ln x$.

The asymptotic expansion of p_k is well known; CESARO [2] then CIPOLLA [3] expressed it in 1902:

$$p_k = k \left\{ \ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k} - \frac{\ln_2^2 k - 6 \ln_2 k + 11}{2 \ln^2 k} + O \left(\left(\frac{\ln_2 k}{\ln k} \right)^3 \right) \right\}.$$

A more precise work about this can be find in [10, 15]. We group together the results on p_k and on $\theta(p_k)$:

$$\begin{aligned} (1) \quad \theta(p_k) &\geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.0553}{\ln k} \right) && \text{for } k \geq \exp(22), \\ (2) \quad \theta(p_k) &\leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k} \right) && \text{for } k \geq 198, \\ (3) \quad p_k &\geq k (\ln k + \ln_2 k - 1) && \text{for } k \geq 2, \\ (4) \quad p_k &\leq k (\ln k + \ln_2 k - 0.9484) && \text{for } k \geq 39017, \\ (5) \quad p_k &\leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k} \right) && \text{for } k \geq 27076, \\ (6) \quad p_k &\geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.25}{\ln k} \right) && \text{for } k \geq 2. \end{aligned}$$

The formula (2) has been proved by ROBIN in [9].

We use the above results to prove that, for $x \geq 3275$, the interval

$$[x, x + x/(2 \ln^2 x)]$$

contains at least one prime. Let's denote by $\pi(x)$ the number of primes not greater than x . We show that

$$\frac{x}{\ln x} \left(1 + \frac{1}{\ln x} \right) \underset{x \geq 599}{\leq} \pi(x) \underset{x > 1}{\leq} \frac{x}{\ln x} \left(1 + \frac{1.2762}{\ln x} \right).$$

More precise results on $\pi(x)$ are also shown:

$$\begin{aligned}\pi(x) &\geq \frac{x}{\ln x - 1} \text{ for } x \geq 5393, \\ \pi(x) &\leq \frac{x}{\ln x - 1.1} \text{ for } x \geq 60184, \\ \pi(x) &\geq \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\right) \text{ for } x \geq 32299, \\ \pi(x) &\leq \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x}\right) \text{ for } x \geq 355991.\end{aligned}$$

We gave in § 7.1 some others inequalities.

2. RESULTS OF ROSSER & SCHOENFELD

We rewrite here some results of [14] which we use. We want to bound $\psi(x) - x$ for “moderate” values of x . Remember the used notations: we want to find ε such that we have, for $x \geq \exp(b)$

$$|\psi(x) - x| < \varepsilon x.$$

Let

$$\begin{aligned}R &= 9.645908801, \\ X &= \sqrt{\ln(x)/R}, \\ K_\nu(z, x) &= \frac{1}{2} \int_x^\infty t^{\nu-1} \exp\{(-1/2)z(t+1/t)\} dt \quad \text{for } z > 0, x \geq 0, \text{ real } \nu, \\ \phi_m(y) &= y^{-m-1} \exp(-X^2/\ln(y/17)), \\ R_m(\delta) &= ((1+\delta)^{m+1} + 1)^m, \\ D &= 158.84998, \\ T_1 &= \frac{1}{\delta} \left(\frac{2R_m(\delta)}{2+m\delta}\right)^{1/m}.\end{aligned}$$

Let $N(T)$ the number of zeros such that $0 < \gamma \leq T$ and its approximation

$$F(T) = \frac{T}{2\pi} \ln \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}$$

and let

$$R(T) = 0.137 \ln T + 0.443 \ln_2 T + 1.588.$$

Let A such all zeros $\rho = \beta + i\gamma$ of the RIEMANN ζ function in the critical strip are such that the real part $\beta = 1/2$ for $0 < \gamma \leq A$. The number A , used by ROSSER & SCHOENFELD, is defined as the unique solution of the equation $F(A) = 3,502,500$, which yields $A = 1,894,438.51224 \dots$ and $F(A) = N(A)$. We use a new value of A , defined as the unique solution of the equation $F(A) = 1,500,000,001$. According to [17], this yields $A = 545,439,823.215 \dots$ and $F(A) = N(A)$.

Proposition 1 ([14], Theorem 4). *Let $T_1 \geq D$, m be a positive integer and*

$$\begin{aligned}\Omega_1 &= \frac{2+m\delta}{4\pi} \left\{ \left(\ln \frac{T_1}{2\pi} + \frac{1}{m} \right)^2 + 0.038207 + \frac{1}{m^2} - \frac{2.82m}{(m+1)T_1} \right\} \\ \Omega_2 &= \frac{(0.159155)R_m(\delta)z}{2m^2 17^m} \left\{ zK_2(z, A') + 2m \ln \left(\frac{17}{2\pi} \right) K_1(z, A') \right\} \\ &\quad + R_m(\delta) \{ 2R(Y)\phi_m(Y) - R(A)\phi_m(A) \},\end{aligned}$$

where $z = 2X\sqrt{m} = 2\sqrt{mb/R}$, $A' = (2m/z)\ln(A/17)$ and

$$Y = \max\{A, 17 \exp \sqrt{b/((m+1)R)}\}.$$

If $b > 1/2$ $0 < \delta < (1 - \exp(-b))/m$ then, for all $x \geq \exp(b)$,

$$|\psi(x) - x| < \varepsilon x$$

where $\varepsilon = \Omega_1 \exp(-b/2) + \Omega_2 \delta^{-m} + m\delta/2 + \exp(-b) \ln(2\pi)$.

As we use this theorem, we write explicitly the proof which doesn't appear in ROSSER & SCHOENFELD's article; They gave only indications.

Proof: In all this proof, the numbering refers to [14] and some non-defined notations too. As $T_2 = 0$, $S_3(m, \delta) = 0$.

$$S_4(m, \delta) = 2 \sum_{\substack{\beta > 1/2 \\ \gamma > 0}} \frac{R_m(\delta) \exp(-X^2 \ln(\gamma/17))}{\delta^m |\rho(\rho+1) \cdots (\rho+m)|} < \frac{R_m(\delta)}{\delta^m} \sum_{\gamma > A} \phi(\gamma)$$

by writing $\phi(y) := \phi_m(y)$. By using corollary of lemma 7, $(\phi(y) := \phi_m(y))$

$$\sum_{\gamma > A} \phi(\gamma) \leq \left\{ \frac{1}{2\pi} + q(y) \right\} \int_A^\infty \phi(y) \ln \left(\frac{y}{2\pi} \right) dy + E_0$$

where $q(y) = \frac{0.137 \ln y + 0.443}{y \ln y \ln(\frac{y}{2\pi})}$ and $E_0 = 2R(Y)\phi(Y) - R(A)\phi(A)$ because $N(A) = F(A)$ with $Y = \max(A, W)$ where W satisfies $(W - y)\phi'(y) \geq 0$ for $y > A$. Let's compute W .

$$\phi'(y) = \phi'_m(y) = y^{-m-2} \exp(-X^2/\ln(y/17)) \left(-(m+1) + \frac{X^2}{\ln^2(y/17)} \right).$$

Hence $\phi'_m(y) \geq 0$ for $y \geq 17 \exp \left(\sqrt{\frac{X^2}{m+1}} \right)$. According to (3.16),

$$\int_A^{+\infty} \phi_m(y) \ln \left(\frac{y}{2\pi} \right) dy = \frac{z^2}{2m^2 + 17^m} K_2(z, A') + \frac{z \ln(17/(2\pi))}{m17^m} K_1(z, A')$$

where $z = 2X\sqrt{m}$ and $A' = (2m/z)\ln(A/17)$. Moreover, $q(Y) \leq q(A) \leq 7 \cdot 10^{-9}$. So for $x \geq \exp(b)$,

$$S_4(m, \delta) < \frac{R_m(\delta)}{\delta^m} \left(\frac{1}{2\pi} + 7 \cdot 10^{-9} \right) \left(\frac{z^2}{2m^2 + 17^m} K_2(z, A') + \frac{z \ln(\frac{17}{2\pi})}{m17^m} K_1(z, A') \right) + E_0$$

where $z = 2\sqrt{\frac{mb}{R}}$, $A' = (2m/z)\ln(A/17)$.

We deduce that

$$S_3(m, \delta) + S_4(m, \delta) \leq \Omega_2 \delta^{-m}.$$

By using the upper bound of $S_1(m, \delta) + S_2(m, \delta)$ given in this previous lemma, we conclude by lemma 8 that

$$\frac{1}{x} |\psi(x) - (x - \ln(2\pi) - 1/2 \cdot \ln(1 - 1/x^2))| \leq \Omega_1/\sqrt{x} + \Omega_2 \delta^{-m} + m\delta/2$$

thus

$$|\psi(x) - x| \leq (\Omega_1 \exp(-b/2) + \Omega_2 \delta^{-m} + m\delta/2 + \exp(-b) \ln(2\pi))x.$$

□

3. RESULTS ON $\psi(x)$ AND $\theta(x)$

We group here together some known results which we will use in our proofs.

Proposition 2. (i) - $\theta(x) \leq \psi(x)$ for all x .

(ii) - $\psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5}\sqrt[3]{x}$ for $10^8 \leq x \leq 10^{16}$.

(iii) - $\psi(x) - \theta(x) < 1.43\sqrt{x}$ for all $x > 0$.

(iv) - $\theta(x) < x$ for $0 < x \leq 10^{11}$.

(v) - $\theta(x) < 1.000081x$ for $x > 0$.

(vi) - $|\theta(x) - x| < 0.0077629\frac{x}{\ln x}$ for $x > 1.04 \cdot 10^7$.

Proof: The first inequality can be easily deduced from

$$\psi(x) = \sum_{k=1}^{\infty} \theta(x^{1/k}).$$

The second is given in [4]. The next one is theorem 13 of [13]. The three last ones are taken from [16] p. 360. \square

For “large” values of x , we use the following theorem (theorem 11 p. 342 of [16]):

Theorem 1. Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$.

$$|\psi(x) - x| < x\varepsilon_0(x) \quad \text{for } x \geq 17,$$

$$|\theta(x) - x| < x\varepsilon_0(x) \quad \text{for } x \geq 101$$

where $\varepsilon_0(x) = \sqrt{8/(17\pi)}X^{1/2} \exp(-X)$.

Theorem 2. Let b be a positive real. For $x \geq \exp(b)$,

$$|\psi(x) - x| \leq \varepsilon x,$$

where ε is given in table 1.

Corollary 1. For $x \geq \exp(50)$, we have

$$|\psi(x) - x| \leq 0.905 \cdot 10^{-7}x.$$

Proof: We make no mathematical change of the ROSSER & SCHOENFELD’s method. We only change the value A by a bigger one. Apply proposition 1 with $A = 545,439,823.215 \dots$ as defined in section 2. Evaluation of the quantities of proposition 1, for example with the computer algebra `Maple`¹, gives new bounds given in table 1.

If we take $A = 1,894,438.51224 \dots$, this implementation permits to recover the results found by ROSSER and SCHOENFELD.

In particular, we compute the value of ε for $x \geq e^{50}$. Choose $\delta = 0.947265625 \cdot 10^{-8}$ and $m = 18$. By computation, we find a value for ε lightly less than $0.905 \cdot 10^{-7}$. This value has been verified with `Pari`. \square

We need afterwards to have bounds for $\theta(x)$ to 10^{11} . The first method is to compute $\theta(x)$ until 10^{11} (we need a powerful computer). The second method uses the remark that the table of [1], which gives the upper and the lower bounds of $\text{Li}(x) - \pi(x)$ by intervals up to 10^{11} , permits to find precise bounds for $\theta(x)$ up to 10^{11} . The results are in table 2.

We construct this table in the following manner:

Let $\rho_1(a, b) = \min_{p_k \in [a, b]} (\text{Li}(p_k) - k)$ and $R_1(a, b) = \max_{p_k \in [a, b]} (\text{Li}(p_k) - k)$. Write

¹We thank the `Médicis` group for the use of their computers.

$r(a, b) = \min_{x \in [a, b]} (\text{Li}(x) - \pi(x))$ and $R(a, b) = \max_{x \in [a, b]} (\text{Li}(x) - \pi(x))$. Note that $r(a, b) \geq \min(\rho_1(a, b) - 1/2, \text{Li}(a) - \pi(a))$ and $R(a, b) \leq \max(R_1(a + 1, b) + 1 + 1/2, \text{Li}(b) - \pi(b))$. The interval $[0, 10^{11}]$ is split into intervals $[a_i, b_i]$.

$$\begin{aligned} \theta(x) &= \pi(x) \ln(x) - \int_2^x \frac{\pi(y)}{y} dy \\ &= \pi(x) \ln(x) - \int_2^{100} \frac{\pi(y)}{y} dy - \int_{100}^x \frac{\pi(y) - \text{Li}(y)}{y} + \frac{\text{Li}(y)}{y} dy \\ (7) \quad &\geq x - (\text{Li}(x) - \pi(x)) \ln x + C + \sum_{\substack{[a_i, b_i] \\ 100 \leq a_i, b_i \leq b}} r(a_i, b_i) \ln \left(\frac{b_i}{a_i} \right) + r(b, x) \ln \left(\frac{x}{b} \right) \end{aligned}$$

where $C = \text{Li}(100) \ln(100) - 100 - \int_2^{100} \frac{\pi(y)}{y} dy$ and b is the upper bound of the last interval which doesn't contain x . If we don't know the value of $\pi(x)$, we can bound the difference $\text{Li}(x) - \pi(x)$ by $R(a, x)$.

We have an upper bound of $\theta(x)$ by replacing in formula (7) all r by R and conversely R by r .

Theorem 3. 1. $|\psi(x) - x| \leq 0.006409 \frac{x}{\ln x}$ for $x \geq \exp(22)$.
2. $|\theta(x) - x| \leq 0.006788 \frac{x}{\ln x}$ for $x \geq 10\,544\,111$.

Proof: We use table 1 by intervals:

if $x \geq \exp(22)$ then $|\psi(x) - x| \leq 2.78652 \cdot 10^{-4} x \leq 23 * 2.78652 \cdot 10^{-4} \frac{x}{\ln x} \leq 6.409 \cdot 10^{-3} \frac{x}{\ln x}$ for $x \leq \exp(23)$.

If $x \geq \exp(23)$ then $|\psi(x) - x| \leq 1.8436 \cdot 10^{-4} x \leq 5.5308 \cdot 10^{-3} \frac{x}{\ln x}$ for $x \leq \exp(30)$.

If $x \geq \exp(30)$ then $|\psi(x) - x| \leq 0.978 \cdot 10^{-5} x \leq 5.868 \cdot 10^{-3} \frac{x}{\ln x}$ for $x \leq \exp(600)$.

If $x \geq \exp(600)$ then $|\psi(x) - x| \leq 0.75 \cdot 10^{-7} x \leq 1.5 \cdot 10^{-4} \frac{x}{\ln x}$ for $x \leq \exp(2000)$.

Moreover,

$|\psi(x) - \theta(x)| < \sqrt{x} + \frac{6}{5} \sqrt[3]{x}$ for $10^8 \leq x \leq 10^{16}$ thus

$|\psi(x) - \theta(x)| < 0.00037871 \frac{x}{\ln x}$ for $\exp(22) \leq x \leq \exp(30)$

and

$|\psi(x) - \theta(x)| < 1.43\sqrt{x}$ for $x > 0$ thus

$|\psi(x) - \theta(x)| < 1.32 \cdot 10^{-5} \frac{x}{\ln x}$ for $x \geq \exp(30)$.

For $x \geq \exp(2000)$, we use theorem 1 to conclude that

$$|\psi(x) - x| \text{ and } |\theta(x) - x| \leq 0.00164 \frac{x}{\ln x} \text{ for } x \geq \exp(2000).$$

With Table 2 and a computer verification, we extend the result on θ , showed until now for $x \geq \exp(22)$, until $10\,544\,111$. This shows that the result of proposition 2(vi) of ROSSER & SCHOENFELD is almost best possible. \square

The above result gives a formula with order 1 for power of logarithm. We need sometimes the next orders. We obtain better results than ROSSER & SCHOENFELD ones.

Theorem 4. Let $\eta_2 = 3.965$, $\eta_3 = 515$ and $\eta_4 = 1717433$.

For $x > 1$,

$$|\theta(x) - x| < \eta_k \frac{x}{\ln^k x}.$$

(We can choose $\eta_2 = 0.2$ for $x \geq 3594641$.)

Proof: In all cases, we use table 1 by intervals. Starting with $k = 2$, we show that

$$|\theta(x) - x| < 0.2 \frac{x}{\ln^2 x}.$$

For $x \geq \exp(3220)$, according to theorem 1 we have

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2 \frac{x}{\ln^2 x}$$

with $\eta_2 = 0.19923$ for $x \geq \exp(3220)$.

For $x \geq 1.04 \cdot 10^7$,	$ \theta(x) - x \leq 0.0077629 \frac{x}{\ln x}$	$\leq 0.1941 \frac{x}{\ln^2 x}$	for $x \leq e^{25}$.
For $x \geq e^{25}$,	$ \psi(x) - x \leq 10^{-4}x$	$\leq 0.09 \frac{x}{\ln^2 x}$	for $x \leq e^{30}$.
For $x \geq e^{30}$,	$ \psi(x) - x \leq 10^{-5}x$	$\leq 0.1 \frac{x}{\ln^2 x}$	for $x \leq e^{100}$.
For $x \geq e^{100}$,	$ \psi(x) - x \leq 0.9 \cdot 10^{-7}x$	$\leq 0.1521 \frac{x}{\ln^2 x}$	for $x \leq e^{1300}$.
For $x \geq e^{1300}$,	$ \psi(x) - x \leq 0.6 \cdot 10^{-7}x$	$\leq 0.1944 \frac{x}{\ln^2 x}$	for $x \leq e^{1800}$.
For $x \geq e^{1800}$,	$ \psi(x) - x \leq 0.42 \cdot 10^{-7}x$	$\leq 0.168 \frac{x}{\ln^2 x}$	for $x \leq e^{2000}$.
For $x \geq e^{2000}$,	$ \psi(x) - x \leq 0.37 \cdot 10^{-7}x$	$\leq 0.196 \frac{x}{\ln^2 x}$	for $x \leq e^{2300}$.
For $x \geq e^{2300}$,	$ \psi(x) - x \leq 0.292 \cdot 10^{-7}x$	$\leq 0.1825 \frac{x}{\ln^2 x}$	for $x \leq e^{2500}$.
For $x \geq e^{2500}$,	$ \psi(x) - x \leq 0.244 \cdot 10^{-7}x$	$\leq 0.18453 \frac{x}{\ln^2 x}$	for $x \leq e^{2750}$.
For $x \geq e^{2750}$,	$ \psi(x) - x \leq 0.19 \cdot 10^{-7}x$	$\leq 0.197 \frac{x}{\ln^2 x}$	for $x \leq e^{3220}$.

Moreover,

$$\begin{aligned} \text{for } \exp(25) \leq x \leq \exp(30), \quad & |\psi(x) - \theta(x)| < \sqrt{x} + \frac{6}{5} \sqrt[3]{x} < 0.0023725 \frac{x}{\ln^2 x}, \\ \text{and for } x \geq \exp(30), \quad & |\psi(x) - \theta(x)| < 1.43\sqrt{x} < 0.0004 \frac{x}{\ln^2 x}. \end{aligned}$$

Now we are interested in case $k = 3$. As $|\theta(x) - x| \leq 0.0077629 \frac{x}{\ln x}$, we have

$$|\theta(x) - x| \leq 310.516 \frac{x}{\ln^3 x} \quad \text{for } x \leq \exp(200).$$

Note that for $x \geq \exp(200)$, the difference between θ and ψ is negligible since, for $x \geq \exp(200)$,

$$|\psi(x) - \theta(x)| < 1.43\sqrt{x} \leq 0.5 \cdot 10^{-36} \frac{x}{\ln^3 x}.$$

For $x \geq e^{200}$,	$ \theta(x) - x \leq 0.8561317 \cdot 10^{-7}x$	$\leq 500 \frac{x}{\ln^3 x}$	for $x \leq e^{1800}$.
For $x \geq e^{1800}$,	$ \theta(x) - x \leq 0.419134 \cdot 10^{-7}x$	$\leq 510 \frac{x}{\ln^3 x}$	for $x \leq e^{2300}$.
For $x \geq e^{2300}$,	$ \theta(x) - x \leq 0.2917036 \cdot 10^{-7}x$	$\leq 456 \frac{x}{\ln^3 x}$	for $x \leq e^{2500}$.
For $x \geq e^{2500}$,	$ \theta(x) - x \leq 0.243946 \cdot 10^{-7}x$	$\leq 508 \frac{x}{\ln^3 x}$	for $x \leq e^{2750}$.
For $x \geq e^{2750}$,	$ \theta(x) - x \leq 0.1877 \cdot 10^{-7}x$	$\leq 507 \frac{x}{\ln^3 x}$	for $x \leq e^{3000}$.
For $x \geq e^{3000}$,	$ \theta(x) - x \leq 0.137602 \cdot 10^{-7}x$	$\leq 514 \frac{x}{\ln^3 x}$	for $x \leq e^{3341}$.

Now apply theorem 1 for $x \geq \exp(3341)$; we find

$$|\theta(x) - x| \leq 514.826 \frac{x}{\ln^3 x}.$$

We use the same methods for case $k = 4$. As $|\theta(x) - x| \leq 0.0077629 \frac{x}{\ln x}$, we have

$$|\theta(x) - x| \leq 1676786.4 \frac{x}{\ln^3 x} \quad \text{for } x \leq \exp(600).$$

Note that, for $x \geq \exp(600)$, the difference between θ and ψ is negligible seeing that, for $x \geq \exp(600)$,

$$|\psi(x) - \theta(x)| < 1.43\sqrt{x} \leq 10^{-119} \frac{x}{\ln^4 x}.$$

For $x \geq e^{600}$,	$ \theta(x) - x \leq 0.744205 \cdot 10^{-7}x$	$\leq \frac{1190728x}{\ln^4 x}$	for $x \leq e^{2000}$.
For $x \geq e^{2000}$,	$ \theta(x) - x \leq 0.3675 \cdot 10^{-7}x$	$\leq \frac{1435547x}{\ln^4 x}$	for $x \leq e^{2500}$.
For $x \geq e^{2500}$,	$ \theta(x) - x \leq 0.243946 \cdot 10^{-7}x$	$\leq \frac{1395162x}{\ln^4 x}$	for $x \leq e^{2750}$.
For $x \geq e^{2750}$,	$ \theta(x) - x \leq 0.1877 \cdot 10^{-7}x$	$\leq \frac{1520370x}{\ln^4 x}$	for $x \leq e^{3000}$.
For $x \geq e^{3000}$,	$ \theta(x) - x \leq 0.137602 \cdot 10^{-7}x$	$\leq \frac{1716527x}{\ln^4 x}$	for $x \leq e^{3342}$.

Now apply theorem 1 for $x \geq \exp(3342)$: we find

$$|\theta(x) - x| \leq 1717433 \frac{x}{\ln^4 x}.$$

We verify by computer for $x \leq 1,04 \cdot 10^7$. We find that $|\theta(x) - x| \leq 0.2 \frac{x}{\ln^2 x}$ for $x \geq 3594641$ and $|\theta(x) - x| \leq 3.9648085 \cdots \frac{x}{\ln^2 x}$ for $x > 1$ (the value $3.9648085 \cdots$ is chosen for p_{17}). \square

4. RESULTS ON p_k AND $\theta(p_k)$

For some steps of demonstrations, we use the following results:

Lemma 1.

$$\begin{array}{ll} p_k \geq k(\ln k) & \text{for } k \geq 2, \\ p_k \leq k(\ln k + \ln_2 k) & \text{for } k \geq 6, \\ p_k \leq k \ln p_k & \text{for } k \geq 4, \\ p_k \geq k(\ln p_k - 2) & \text{for } k \geq 5. \end{array}$$

Proof: These inequalities have been proved by ROSSER. The first inequality can be found in [11]; the next one in [12]. The last ones can be easily deduced from

$$\frac{x}{\ln x} < \pi(x) < \frac{x}{\ln x - 2}$$

(see [12]). \square

ROSSER showed that $p_k \geq k \ln k$ and improved his result with SCHOENFELD by showing that $p_k \geq k(\ln k + \ln_2 k - 3/2)$ (see [14]). En 1983, ROBIN [9] succeeded to prove that

$$p_k \geq k(\ln k + \ln_2 k - 1.0072629).$$

MASSIAS & ROBIN [6] are able to show that

$$p_k \geq k(\ln k + \ln_2 k - 1)$$

for $k \leq \exp(598)$ and $k \geq \exp(1800)$.

We have show that for all $k \geq 2$ in [5].

Theorem 5. For $k \geq 2$, we have

$$p_k \geq k(\ln k + \ln_2 k - 1)$$

where p_k is k^{th} prime number.

First method to find a lower bound for $\theta(p_k)$.

Proposition 3. Let

$$f_\beta(k) := k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} \right).$$

Assume that for $k \geq k_0$

$$|\theta(p_k) - p_k| \leq ck.$$

Let β be a real satisfying

$$\begin{aligned} \beta(\ln k_0 - 1) &\geq \ln k_0(\ln_2 k_0 + 1) + 1 - \ln_2 k_0 \\ &\quad - \ln^2 k_0 \ln \left(1 + \frac{\ln_2 k_0 - 1}{\ln k_0} + \frac{\ln_2 k_0 - \beta - c \ln k_0}{\ln^2 k_0} \right). \end{aligned}$$

If $\theta(p_{k_0}) \geq f_\beta(k_0)$ then

$$\theta(p_k) \geq f_\beta(k) \quad \text{for } k \geq k_0.$$

Proof: Assume that

$$p_k \geq h_a(k) := k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - a}{\ln k} \right).$$

Hence

$$\theta(p_k) - \theta(p_{k_0}) = \sum_{n=k_0+1}^k \ln p_n \geq \sum_{n=k_0+1}^k \ln h_a(n).$$

Moreover

$$f'_\beta(k) = \ln k + \ln_2 k + \frac{\ln_2 k - \beta + 1}{\ln k} - \frac{\ln_2 k - (\beta + 1)}{\ln^2 k}$$

and

$$\ln p_k \geq \ln h_a = \ln k + \ln_2 k + \ln \left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k} \right).$$

It's sufficient to show when $f'_\beta \leq \ln h_a$ because

$$f_\beta(k) - f_\beta(k_0) = \int_{k_0}^k f'_\beta(x) dx \leq \sum_{n=k_0+1}^k \ln h_a(n) \leq \theta(p_k) - \theta(p_{k_0}),$$

equivalently

$$\frac{\ln_2 k + 1 - \beta}{\ln k} - \frac{\ln_2 k - (\beta + 1)}{\ln^2 k} \leq \ln \left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k} \right)$$

$$\beta(\ln k - 1) \geq \ln k(\ln_2 k + 1) - \ln^2 k \ln \left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k} \right) + 1 - \ln_2 k.$$

Since $p_k \geq \theta(p_k) - ck$, we can choose $a = \beta + c \ln k$. Then we find that

$$\begin{aligned} \beta(\ln k - 1) &\geq (\ln k)(\ln_2 k + 1) - (\ln^2 k) \ln \left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - \beta - c \ln k}{\ln^2 k} \right) \\ &\quad + 1 - \ln_2 k. \end{aligned}$$

□

We apply proposition 3. For $c = 0.007$, the computed values of β are:

$\ln k_0$	β
22	2.0553
23	2.0532
25	2.04975
30	2.04397
100	2.02961
1000	2.0156
2000	2.0128

Corollary 2. For $p_k \geq 10^{11}$,

$$\theta(p_k) \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.0548}{\ln k} \right).$$

Proof: We take $k_0 = 4118054814$ and $c = 0.006788$. We obtain $\beta = 2.0548$. Using table 2, we check $\theta(10^{11}) \geq f_{2.0548}(k_0)$. □

The major drawback of the above method is that we must check if $\theta(p_{k_0}) \geq f_\beta(k_0)$ which needs a computation of $\theta(x)$ for large x . The next method doesn't need this hypothesis.

Second method to find a lower bound for $\theta(p_k)$.

Proposition 4. *Let $k_0 \geq \exp(\exp(3))$. Assume that*

$$|\theta(x) - x| \leq c \frac{x}{\ln x} \quad \text{for } x \geq p_{k_0}.$$

Then

$$\theta(p_k) \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} \right) \quad \text{for } k \geq k_0$$

with β solution of the following equality

$$\begin{aligned} \beta &= \frac{\ln k_0(1 + \alpha) - \ln_2 k_0 + 1}{\ln k_0 - 1} \\ \alpha &= \frac{1}{2} \frac{(\ln_2 k_0 - a)^2}{\ln k_0} + a \\ a &= 1 - \frac{\ln_2 k_0 - \beta}{\ln k_0} + c \end{aligned}$$

Proof: Applying lemma 1 of [9] for $a = 1$ and $\ln k_0 \geq \exp(a + 2)$ in the same way than theorem 7 of [9], we find a first value of β written β_0 . Since $|\theta(p_k) - p_k| \leq ck$, we can apply lemma 1 one more time with the new value of a : we choose

$$a = 1 - \frac{\ln_2 k_0 - \beta_0}{\ln k_0} + c.$$

In fact the series $\{\beta_k\}_k$ is a convergent one. We determine the limit by solving the equation of the fixed point for $k = k_0$

$$\beta = \frac{\ln k \left(1 + \frac{1}{2} \frac{(\ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} - c)^2}{\ln k} + 1 - \frac{\ln_2 k - \beta}{\ln k} + c \right) - \ln_2 k + 1}{\ln k - 1}.$$

When k_0 grows to infinity, β decreases to $2 + c$. \square

Applying the previous proposition for $c = 0.007$, the computed values of β are:

$\ln k_0$	β
$\exp(3)$	2.0675
25	2.056
30	2.04938
1000	2.01356
2000	2.0128

Corollary 3. *For $k \geq \exp(30)$,*

$$\theta(p_k) \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.05}{\ln k} \right).$$

Theorem 6.

$$p_k \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.25}{\ln k} \right) \quad \text{for } k \geq 2.$$

Proof: Since $\theta(x) < x$ for $0 < x < 10^{11}$, we deduce immediately

$$p_k \geq \theta(p_k) \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k} \right).$$

For $p_k \geq \exp(4000)$, it follows from theorem 1 that

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2 \frac{x}{\ln^2 x}$$

with $\eta_2 = 0.040033$ for $x \geq \exp(4000)$. Thus, for $p_k \geq \exp(4000)$,

$$p_k \geq \theta(p_k) - \eta_2 \frac{p_k}{\ln^2 p_k} \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1 - 0.040033}{\ln k} \right).$$

For $10^{11} \leq p_k \leq \exp(4000)$,

$$\begin{aligned} p_k &\geq \theta(p_k) \left(\frac{1}{1 + \varepsilon} \right) \geq (1 - \varepsilon)\theta(p_k) \\ &\geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} - \varepsilon \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} \right) \right). \end{aligned}$$

with $\beta = 2.0548$. Let's study the map

$$f(\varepsilon, k) := -\beta - \varepsilon \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} \right) \ln k.$$

We choose at the first time $\varepsilon = \varepsilon(10^{11}) = 0.00008$. We verify that $f(\varepsilon, \exp(30)) > -2.1319$ and so on, by small intervals up to $\exp(4000)$. We deduce that $f(\varepsilon, k) > -2.25$ for p_k between $\exp(22)$ and $\exp(4000)$. \square

Theorem 7.

$$p_k \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k} \right) \quad \text{for } k \geq 27076.$$

Proof: Assume first that (see lemma 1)

$$\frac{p_k}{\ln^2 p_k} \leq \frac{k}{\ln k}.$$

For $x \geq 3594641$, we have establish that

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2 \frac{x}{\ln^2 x}$$

with $\eta_2 = 0.2$. Thus, for $p_k \geq 3594641$,

$$p_k \leq \theta(p_k) + \eta_2 \frac{p_k}{\ln^2 p_k} \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2 + 0.2}{\ln k} \right).$$

We terminate the proof by a computer check. \square

Theorem 8. For $k \geq 39017$, we have

$$p_k \leq k(\ln k + \ln_2 k - 0.9484).$$

Proof: As (see theorem 7)

$$p_k \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k} \right),$$

we have, for $x \geq \exp(33)$,

$$p_k \leq k(\ln k + \ln_2 k - 0.9484).$$

From

$$|\theta(p_k) - p_k| \leq \varepsilon(p_k)p_k,$$

we deduce that

$$p_k \leq \theta(p_k) + \varepsilon(p_k)p_k.$$

As

$$\theta(p_k) \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k} \right),$$

it remains to show that

$$g(k) := -1 + \frac{\ln_2 k - 2}{\ln k} + \varepsilon(p_k) \frac{p_k}{k} < -0.9484$$

for $4118054813 \leq k \leq \exp(30)$. If we choose (cf. [6])

$$p_k \leq k(\ln k + \ln_2 k - 0.9427)$$

and $\varepsilon(p_k) = 0.000079989 + 1/\sqrt{10^{11}}$ for $10^{11} \leq p_k \leq \exp(30)$ and next $\varepsilon(p_k) = 10^{-5} + 1.43/\sqrt{\exp 30}$ for $p_k \geq \exp(30)$, we obtain the result for $p_k \geq 10^{11}$.

Now we study if the result is true for $p_k \leq 10^{11}$. Write $\alpha = 0.9484$ and

$$f(k) = k(\ln k + \ln_2 k - \alpha).$$

Thanks to BRENT [1], we have the lower and the upper bounds of the difference of $\text{Li}(x) - \pi(x)$ into various intervals up to 10^{11} . Hence if, for $k \in [k_0, k_1]$,

$$\text{Li}(f(k)) - \pi(p_k) \geq M_{p_{k_0}, p_{k_1}} := \max_{k \in [k_0, k_1]} (\text{Li}(p_k) - \pi(p_k))$$

we could deduce that

$$p_k \leq f(k).$$

We consider $g(k) = \text{Li}(f(k)) - k$.

$$g'(k) = \frac{1 + 1/\ln k - \alpha - \ln(1 + (\ln_2 k - \alpha)/\ln k)}{\ln k + \ln_2 k + \ln(1 + (\ln_2 k - \alpha)/\ln k)}$$

admits a minimum for $\ln k \approx \exp(2 + \alpha + 2/\exp(2 + \alpha))$. For $\alpha = 0.9484$, this lower bound is positive. We show, thanks to the following table, that

$$\text{Li}(f(k)) - \pi(p_k) \geq M_{p_{k_0}, p_{k_1}}$$

for $170368 \leq k \leq 4118054813$ ($2312573 \leq p_k \leq 10^{11}$). The values of $\pi(x)$ are extracted from table 3 of RIESEL [8] (Ex.: $\pi(10^7) = 664579$)

k	$p_k \leq \cdot$	$\text{Li}(f(k)) - k$	$R_1 = M_{p_{k_0}, p_{k_1}}$
170368	$2.315 \cdot 10^6$	261.0004	$M_{2 \cdot 10^6, 5 \cdot 10^6} = 261$
348512	$5 \cdot 10^6$	414.1091	$M_{5 \cdot 10^6, 10^7} = 346$
664578	10^7	634.5851	$M_{10^7, 2 \cdot 10^7} = 435$
1270606	$2 \cdot 10^7$	983.965	$M_{2 \cdot 10^7, 5 \cdot 10^7} = 692$
3001133	$5 \cdot 10^7$	1788.864	$M_{5 \cdot 10^7, 5 \cdot 10^8} = 1724$
26355866	$5 \cdot 10^8$	8890.45	$M_{5 \cdot 10^7, 10^{10}} = 7048$
455052510	10^{10}	93238.1	$M_{10^{10}, 10^{11}} = 17065$

For $k = 39017..170367$, a check has been made thanks to **Pari** system. □

5. INTERVAL WHICH CONTAINS AT LEAST ONE PRIME

We already know the result of SCHOENFELD [16] showing that, for $x \geq 2010759.9$, the interval $]x, x + x/16597[$ contains at least one prime. We improve this result with the following proposition. You can see also [7].

Proposition 5. For $k \geq 463$,

$$p_{k+1} \leq p_k \left(1 + \frac{1}{2 \ln^2 p_k} \right).$$

Proof: Assume that, for $k \geq k_0$,

$$p_k \geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_0}{\ln k} \right)$$

and

$$p_k \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_1}{\ln k} \right).$$

$$\begin{aligned} p_k \left(1 + \frac{\gamma}{\ln^2 p_k} \right) - p_{k+1} &= p_k - p_{k+1} + \gamma \frac{p_k}{\ln^2 p_k} \\ &\geq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_0}{\ln k} \right) + \gamma \frac{p_k}{\ln^2 p_k} \\ &\quad - (k+1) \left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \right) \\ &= k \left(\ln k - \ln(k+1) + \ln_2 k - \ln_2(k+1) + \frac{\ln_2 k - \alpha_0}{\ln k} - \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \right) \\ &\quad - \left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \right) + \gamma \frac{p_k}{\ln^2 p_k} \end{aligned}$$

But,

$$\ln(k) - \ln(k+1) = -\ln(1 + 1/k),$$

$$\ln_2(k) - \ln_2(k+1) = -\ln \left(1 + \frac{\ln(1 + 1/k)}{\ln k} \right),$$

$$\frac{\ln_2 k - \alpha_0}{\ln k} - \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \geq (\alpha_1 - \alpha_0)/\ln(k+1)$$

because $\frac{\ln x}{x}$ is decreasing for $x \geq 1/e$ hence $f(k) \geq f(k+1)$ where $f(x) := \frac{\ln_2 x}{\ln x}$. We choose γ such that

$$\begin{aligned} \gamma \frac{p_k}{\ln^2 p_k} &\geq k \left(\ln(1 + 1/k) + \ln \left(1 + \frac{\ln(1 + 1/k)}{\ln k} \right) + (\alpha_0 - \alpha_1)/\ln(k+1) \right) \\ (8) \quad &+ \left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \right); \end{aligned}$$

We must choose $\gamma > \alpha_0 - \alpha_1$ to make the inequality true for large k . As

$$\frac{p_k}{\ln^2 p_k} \geq \frac{k \ln k}{(\ln k + \ln_2 k)^2},$$

the inequality (8) is satisfied with $\gamma = 1/2$ if

$$\ln(k+1)/2 \geq (\alpha_0 - \alpha_1)(\ln k + \ln_2 k).$$

For $\alpha_0 = 2.25$ and $\alpha_1 = 1.8$, this holds for $k \geq \exp(82)$. Theorem 12 of [16] showed that the interval $]x, x + x/16597[$ contains at least one prime for $x \geq 2010759.9$. Apply it for $x = p_k$, this yields the result for $2010759.9 \leq p_k \leq \exp(91)$. According to [16] p. 355,

$$p_{n+1} - p_n \leq 652 \text{ for } p_n \leq 2.686 \cdot 10^{12},$$

and so

$$p_{k+1} \leq p_k \left(1 + \frac{1}{2 \ln^2 p_k} \right)$$

when

$$\frac{p_k}{2 \ln^2 p_k} \geq \frac{k}{2(\ln k + \ln_2 k)} \geq 652$$

that is to say for $k \geq 2 \cdot 10^4$. For $k = 463..20000$, a direct check with computer conclude the proof. \square

Theorem 9. *For all $x \geq 3275$, there exists a prime p such that*

$$x < p \leq x \left(1 + \frac{1}{2 \ln^2 x}\right).$$

This result is better than ROSSER & SCHOENFELD's one for $x \geq 3 \cdot 10^{39}$.

Proof: Let $x \geq 2$. There exists $k \in \mathbb{N}^*$ such that

$$p_k \leq x < p_{k+1}.$$

As the map $x \mapsto x(1 + 1/(2 \ln^2 x))$ is increasing,

$$p_k \leq x \Rightarrow p_k \left(1 + \frac{1}{2 \ln^2 p_k}\right) \leq x \left(1 + \frac{1}{2 \ln^2 x}\right).$$

According to proposition 5, for $k \geq 463$,

$$p_{k+1} \leq p_k \left(1 + \frac{1}{2 \ln^2 p_k}\right).$$

We deduce that, for $x \geq p_{463} = 3299$,

$$x < p_{k+1} \leq x \left(1 + \frac{1}{2 \ln^2 x}\right).$$

Moreover, for $x \geq 3274.0111$,

$$3299 < x \left(1 + \frac{1}{2 \ln^2 x}\right).$$

\square

6. RESULTS ON $\pi(x)$

Remember that

$$\pi(x) = \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{2}{\ln^2 x} + O\left(\frac{1}{\ln^3 x}\right)\right).$$

- Theorem 10.**
1. - $\frac{x}{\ln x} \left(1 + \frac{1}{\ln x}\right) \leq \pi(x)$ for $x \geq 599$.
 2. - $\pi(x) \leq \frac{x}{\ln x} \left(1 + \frac{1.2762}{\ln x}\right)$ for $x > 1$ (the value 1.2762 is chosen for $x = p_{258} = 1627$).
 3. - $\pi(x) \leq \frac{x}{\ln x} \left(1 + \frac{1.0992}{\ln x}\right)$ for $x \geq 1.332 \cdot 10^{10}$.
 4. - $\pi(x) \leq \frac{x}{\ln x - 1.1}$ for $x \geq 60184$.
 5. - $\pi(x) \geq \frac{x}{\ln x - 1}$ for $x \geq 5393$.
 6. - $\frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\right) \leq \pi(x)$ for $x \geq 32299$.
 7. - $\pi(x) \leq \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x}\right)$ for $x \geq 355991$.

Let's start with the first three inequalities.

Lower bound of $\pi(x)$. • We began with $x \geq 10^8$.

$$\pi(x) = \pi(p) - \frac{\theta(p)}{\ln(p)} + \frac{\theta(x)}{\ln(x)} + \int_p^x \frac{\theta(y)dy}{y \ln^2(y)}.$$

But $\theta(x) \geq x - 0.024 \frac{x}{\ln(x)}$ for $x \geq 758711$ ([16], Th.7 p.357). Hence, for $p \geq 758711$, this yields

$$\pi(x) \geq \frac{\theta(x)}{\ln(x)} + \frac{x}{\ln^2(x)} + \pi(p) - \frac{\theta(p)}{\ln(p)} - \frac{p}{\ln^2(p)} + (2 - 0.024) \int_p^x \frac{dy}{\ln^3(y)}.$$

Since $\theta(x) \geq x - 0.0077629 \frac{x}{\ln(x)}$ for $x \geq 1.04 \cdot 10^7$,

$$\pi(x) \geq \frac{x}{\ln x} \left(1 + \frac{1 - 0.008}{\ln(x)} \right) \text{ for } x \geq 10^8$$

because

$$\pi(p) - \frac{\theta(p)}{\ln(p)} - \frac{p}{\ln^2(p)} + (2 - 0.024) \int_p^x \frac{dy}{\ln^3(y)} \geq 0 \text{ for } x \geq 10^8,$$

by choosing

$$p = p_{61000} = 760267 \text{ and } \theta(p) < p.$$

- According to [13](Th 16, p. 72), we have for $11 < x \leq 10^8$,

$$\text{Li}(x) - \text{Li}(\sqrt{x}) < \pi(x).$$

For $1859 \leq x \leq 10^8$,

$$\text{Li}(x) - \text{Li}(\sqrt{x}) \geq \frac{x}{\ln x} \left(1 + \frac{0.992}{\ln x} \right).$$

For $k = 1..284$ ($p_{284} = 1861$), we check by computer that

$$\pi(p_k - 1/2) = k - 1 \geq \frac{p_k}{\ln p_k} \left(1 + \frac{0.992}{\ln p_k} \right),$$

inequality true for $k \geq 110$. Hence $\pi(x) \geq \frac{x}{\ln x} \left(1 + \frac{0.992}{\ln x} \right)$ for $x \geq p_{109} = 599$.

We will obtain a better result with an upper bound for θ in order 2 in $\ln x$. Upper bound for $\pi(x)$. According to [16] (p. 360), we know that :

$$\theta(x) < x \quad \text{for } 0 < x \leq 10^{11},$$

$$\theta(x) < 1.000081x \quad \text{for } x > 0,$$

and that

$$|\theta(x) - x| \leq 0.0077926 \frac{x}{\ln x} \quad \text{for } x \geq 1.04 \cdot 10^7.$$

Let $b = 0.0077926$, $c = 1.000081$ and $K = 10^{11}$.

$$\begin{aligned}
\pi(x) &= \frac{\theta(x)}{\ln x} + \int_2^x \frac{\theta(y)dy}{y \ln^2 y} \\
&= \frac{\theta(x)}{\ln x} + \int_2^K \frac{\theta(y)dy}{y \ln^2 y} + \int_K^x \frac{\theta(y)dy}{y \ln^2 y} \\
&< \frac{x}{\ln x} \left(1 + \frac{b}{\ln x}\right) + \int_2^K \frac{dy}{\ln^2 y} + \int_K^x \frac{dy}{\ln^2 y} + b \int_K^x \frac{dy}{\ln^3 y} \text{ for } x \geq K \\
&= \frac{x}{\ln x} \left(1 + \frac{b}{\ln x}\right) + \left[\frac{y}{\ln^2 y}\right]_2^x + 2 \int_2^x \frac{dy}{\ln^3 y} + b \int_K^x \frac{dy}{\ln^3 y} \\
&< \frac{x}{\ln x} \left(1 + \frac{b+1}{\ln x}\right) + 2 \frac{x}{\ln^3 x} + 6 \int_2^x \frac{dy}{\ln^4 y} + b \left[\frac{y}{\ln^3 y}\right]_K^x + 3b \int_K^x \frac{dy}{\ln^4 y}
\end{aligned}$$

But

$$\int_K^x \frac{dy}{\ln^4 y} = \int_K^{\sqrt{x}} \frac{dy}{\ln^4 y} + \int_{\sqrt{x}}^x \frac{dy}{\ln^4 y} < (\sqrt{x} - K)/\ln^4(K) + \frac{x - \sqrt{x}}{\ln^4 \sqrt{x}} \text{ if } \sqrt{x} \geq K.$$

To have

$$\pi(x) \leq \frac{x}{\ln x} \left(1 + \frac{\beta}{\ln x}\right)$$

we choose

$$\beta > b+1 + \frac{2+b}{\ln x} + \frac{\ln^2 x}{x} \left(6 \int_2^K \frac{dy}{\ln^4 x} - b \frac{K}{\ln^3 K} + (6+3b) \left(\frac{\sqrt{x}-K}{\ln^4 K} + \frac{x-\sqrt{x}}{\ln^4 \sqrt{x}}\right)\right).$$

We obtain $\beta \geq 1.03$ when $x \geq \exp(100)$.

$$\begin{aligned}
\pi(x) &= \pi(K) - \frac{\theta(K)}{\ln K} + \frac{\theta(x)}{\ln x} + \int_K^x \frac{\theta(y)dy}{y \ln^2 y} \\
&< \pi(K) - \frac{\theta(K)}{\ln K} + c \frac{x}{\ln x} + c \int_K^x \frac{dy}{\ln^2 y} \\
&= \pi(K) - \frac{\theta(K)}{\ln K} + c \frac{K}{\ln K} + c(\text{Li}(x) - \text{Li}(K)) \\
&= M + c \cdot \text{Li}(x) \quad \text{where } M \text{ is a constant.}
\end{aligned}$$

Write

$$\Delta(x) = \frac{x}{\ln x} \left(1 + \frac{\beta}{\ln x}\right) - (M + c \cdot \text{Li}(x)).$$

$$\Delta'(x) = ((1-c)\ln^2(x) + (\beta-1)\ln(x) - 2\beta) / \ln^3 x.$$

Δ' equals zero when

$$\ln x = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 + 8\beta(1-c)}}{2(1-c)}.$$

We can take $\pi(K) = 4118054813$ and $\frac{\theta(K)}{\ln K} > \frac{K}{\ln K} \left(1 - \frac{0.007}{\ln K}\right)$ hence, for $\beta = 1.0992$, the local minimum $x_0 \approx \exp(22.5775)$ ($x_0 < K$) is negative but $\Delta(K)$ is positive. Hence, for $x \in [K, \exp(100)]$,

$$\pi(x) < \frac{x}{\ln x} \left(1 + \frac{1.0992}{\ln x}\right).$$

By consulting table given in [1], we show that the result remains true for $x \geq 1.332 \cdot 10^{10}$. According to [13] (Th. 16 p. 72) and [1], we have for $x \leq K$,

$$\pi(x) < \text{Li}(x).$$

Let's study the difference

$$\Delta(x, \beta) = \frac{x}{\ln x} \left(1 + \frac{\beta}{\ln x} \right) - \text{Li}(x).$$

This function admits a minimum at point $x = \exp(2\beta/(\beta - 1))$. For $\beta_0 := 1.276103273$, the value of this minimum equals to -9.972985 . We consult the table of BRENT [1]. This yields the lower difference between $\pi(x)$ and $\text{Li}(x)$ in different intervals $[10^n, 10^{n+1}]$ with $n = 0..10$. We deduce that

$$\frac{x}{\ln x} \left(1 + \frac{\beta_0}{\ln x} \right) - \pi(x) = \frac{x}{\ln x} \left(1 + \frac{\beta_0}{\ln x} \right) - \text{Li}(x) + (\text{Li}(x) - \pi(x)) > 0$$

for $x \geq 10^4$. For $k = 1..1230$, we verify that

$$\pi(p_k) = k \leq \frac{p_k}{\ln p_k} \left(1 + \frac{\beta_0}{\ln p_k} \right).$$

We choose the value for β_0 such that it holds for all integers k , including $k = 258$.

Next we consider the others formulas. Let

$$x_0 = 1.04 \cdot 10^7, \quad K = \pi(x_0) - \frac{\theta(x_0)}{\ln x_0}.$$

Numerically $\pi(x_0) = 689382$ and $\theta(x_0) = 10395445.63690637\dots$. Write

$$J(x; a) = K + \frac{x}{\ln x} + a \frac{x}{\ln^3 x} + \int_{x_0}^x \left(\frac{1}{\ln^2 y} + a \frac{1}{\ln^4 y} \right) dy$$

Since

$$\pi(x) = \pi(x_0) - \frac{\theta(x_0)}{\ln x_0} + \frac{\theta(x)}{\ln x} + \int_{x_0}^x \frac{\theta(y) dy}{y \ln^2 y}$$

and $|\theta(x) - x| \leq 0.2 \frac{x}{\ln^2 x}$ for $x > x_0$, we have, for $x \geq x_0$,

$$J(x; -0.2) \leq \pi(x) \leq J(x; 0.2).$$

Write $M(x; c) = \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{c}{\ln^2 x} \right)$ for upper bound's function for $\pi(x)$. Let's write the derivatives of $J(x; a)$ and of $M(x; c)$ with respect to x :

$$J'(x; a) = \frac{1}{\ln x} + \frac{a}{\ln^3 x} - 2 \frac{a}{\ln^4 x},$$

$$M'(x; c) = \frac{1}{\ln x} + \frac{c-2}{\ln^3 x} - \frac{3c}{\ln^4 x}.$$

We must choose $c > 2 + a$ to have $J' < M'$ when $\ln x > (3c - 2a)/(c - 2 - a)$. Choose $c = 2.51$. We verify by computer that $J(10^{11}; 0.2) < M(10^{11}; 2.51)$. Since $\text{Li}(x) < M(x; 2.51)$ for $x \geq 10^7$, we verify by direct computation for small values of x to obtain

$$\pi(x) < \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x} \right) \quad \text{for } x \geq 355991.$$

Now write

$$m(x; d) = \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{d}{\ln^2 x} \right).$$

We study the derivatives: we must choose $d \leq 2 - a$ to have $J' > m'$.

Choose $d = 1.8$. As $m(x_0; 1.8) < J(x_0; -0.2)$ and by direct computation for small values, we obtain

$$\pi(x) > \frac{x}{\ln x} \left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x} \right) \quad \text{for } x \geq 32299.$$

To show that

$$\frac{x}{\ln x - b} \leq \pi(x),$$

we proceed in the same way. By comparing with the derivative of $J(x; a)$, we show that we must choose $b \leq 1$. For $b = 1$ and $a = -0.2$, $J'(x; -0.2)$ is greater than the derivative of $x/(\ln x - 1)$ for $x > 1$. Moreover, for $x = x_0$,

$$\frac{x_0}{\ln x_0 - 1} < J(x_0; -0.2).$$

We verify by computer that, for $5393 \leq x \leq 1.04 \cdot 10^7$,

$$\frac{x}{\ln x - 1} \leq \pi(x).$$

Now we show that we have, for $x \geq 60184$,

$$\pi(x) \leq \frac{x}{\ln x - 1.1}.$$

Since

$$\frac{x}{\ln x} \left(1 + \frac{a}{\ln x} \right) < \frac{x}{\ln x - a}$$

for $x > \exp(a)$, an above result shows the result for $x \geq 1.332 \cdot 10^{10}$. Now we study if $\pi(x) \leq \frac{x}{\ln x - 1.1}$ for $x \leq 10^{11}$. Write $k(x) = \frac{x}{\ln x - 1.1}$ and compare $k(x)$ with $\text{Li}(x)$. The derivative of $k'(x) - 1/\ln x$ is

$$-\frac{200 \ln^2 x - 3630 \ln x + 1331}{(10 \ln x - 11)^3 x \ln^2 x}.$$

So we deduce that the difference $k'(x) - 1/\ln x$ is positive for x belonging to $[10^6, 10^{11}]$. Since $\text{Li}(10^6) < k(10^6)$, we deduce that, for $10^6 \leq x \leq 10^{11}$,

$$\text{Li}(x) < k(x).$$

A computer check concludes the demonstration up to 10^6 .

We show that

$$\frac{x}{\ln x} \left(1 + \frac{1}{\ln x} \right) \leq \pi(x) \quad \text{for } x \geq 599$$

by using inequality 6 of theorem 10. Next we verify by computer the lower bound is true from $x = 599$.

7. APPLICATIONS

7.1. Other inequalities. Let γ be Euler's constant ($\gamma \approx 0.5772156649$).

Theorem 11. Let $B = \gamma + \sum_p (\ln(1 - 1/p) + 1/p) \approx 0.26149\ 72128\ 47643$. For $x > 1$,

$$\sum_{p \leq x} \frac{1}{p} - \ln_2 x - B \geq - \left(\frac{1}{10 \ln^2 x} + \frac{4}{15 \ln^3 x} \right).$$

For $x \geq 10372$,

$$\sum_{p \leq x} \frac{1}{p} - \ln_2 x - B \leq \frac{1}{10 \ln^2 x} + \frac{4}{15 \ln^3 x}.$$

Proof: By (4.20) of [13],

$$\sum_{p \leq x} \frac{1}{p} = \ln_2 x + B + \frac{\theta(x) - x}{x \ln x} - \int_x^\infty \frac{(\theta(y) - y)(1 + \ln y)}{y^2 \ln^2 y} dy.$$

Hence

$$\left| \sum_{p \leq x} \frac{1}{p} - \ln_2 x - B \right| \leq \frac{|\theta(x) - x|}{x \ln x} + \int_x^\infty \frac{|\theta(y) - y|(1 + \ln y)}{y^2 \ln^2 y} dy.$$

As $|\theta(x) - x| \leq 0.2x/\ln^2 x$ (Theorem 4) and

$$\int_x^\infty \frac{1 + \ln y}{y \ln^4 y} dy = \frac{1}{2 \ln^2 x} + \frac{1}{3 \ln^3 x},$$

we have the result for $x \geq 3594641$. We conclude by computer's check. \square

Theorem 12. Let $E = -\gamma - \sum_{n=2}^\infty \sum_p (\ln p)/p^n \approx -1.33258\ 22757\ 33221$. For $x > 1$,

$$\sum_{p \leq x} \frac{\ln p}{p} - \ln x - E \geq -\left(\frac{0.2}{\ln x} + \frac{0.2}{\ln^2 x} \right).$$

For $x \geq 2974$,

$$\sum_{p \leq x} \frac{\ln p}{p} - \ln x - E \leq \frac{0.2}{\ln x} + \frac{0.2}{\ln^2 x}.$$

Proof: By (4.21) of [13],

$$\sum_{p \leq x} \frac{\ln p}{p} = \ln x + E + \frac{\theta(x) - x}{x} - \int_x^\infty \frac{\theta(y) - y}{y^2} dy.$$

Hence

$$\left| \sum_{p \leq x} \frac{\ln p}{p} - \ln x - E \right| \leq \frac{|\theta(x) - x|}{x} - \int_x^\infty \frac{|\theta(y) - y|}{y^2} dy.$$

As

$$\int_x^\infty \frac{dy}{y \ln^2 y} = \frac{1}{\ln x},$$

theorem 4 yields the result for $x \geq 3594641$. We conclude by computer's check. \square

Theorem 13. For $x > 1$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p} \right) < \frac{e^{-\gamma}}{\ln x} \left(1 + \frac{0.2}{\ln^2 x} \right)$$

and

$$\prod_{p \leq x} \frac{\ln p}{p} > e^{\gamma \ln x} \left(1 - \frac{0.2}{\ln^2 x} \right).$$

For $x \geq 2973$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p} \right) > \frac{e^{-\gamma}}{\ln x} \left(1 - \frac{0.2}{\ln^2 x} \right)$$

and

$$\prod_{p \leq x} \frac{\ln p}{p} < e^{\gamma \ln x} \left(1 + \frac{0.2}{\ln^2 x} \right).$$

Proof: By theorem 11 and by definition of B , we have

$$-\left(\frac{0.1}{\ln^2 x} + \frac{4}{15 \ln^3 x}\right) \leq -\gamma - \ln_2 x - \sum_{p>x} \frac{1}{p} - \sum_p \ln(1-1/p) \leq \frac{0.1}{\ln^2 x} + \frac{4}{15 \ln^3 x}.$$

Let $S = \sum_{p>x} (\ln(1-1/p) + 1/p) = -\sum_{n=2}^{\infty} \sum_{p>x} \frac{1}{p^n}$. We have

$$-\gamma - \ln_2 x - \sum_{p \leq x} \ln(1-1/p) - S \geq -\frac{0.1}{\ln^2 x} - \frac{4}{15 \ln^3 x}.$$

Take the exponential of both sides to obtain

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \leq \frac{e^{-\gamma}}{\ln x} \exp\left(-S + \frac{0.1}{\ln^2 x} + \frac{4}{15 \ln^3 x}\right).$$

We use lower bound for S given in [13] p. 87:

$$S > \frac{-1.02}{(x-1) \ln x}.$$

Hence, for $x \geq 100$,

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) \leq \frac{e^{-\gamma}}{\ln x} \exp(0.11/\ln^2 x).$$

We have also

$$\prod_{p \leq x} \frac{p-1}{p} \geq e^{\gamma} \ln x \exp(-0.11/\ln^2 x).$$

In the same way, as

$$-\gamma - \ln_2 x - \sum_{p \leq x} \ln(1-1/p) - S \leq \frac{0.1}{\ln^2 x} + \frac{4}{15 \ln^3 x},$$

we obtain the others inequalities since $S \leq 0$. □

Theorem 14. For $x > 1$,

$$\pi(2x) - \pi(x) \leq \frac{x}{\ln x}$$

For $x \geq 1328.5$,

$$\pi(2x) - \pi(x) \geq \frac{x}{\ln x} - \frac{0.7x}{\ln^2 x}.$$

Remark: As $\pi(x) > x/\ln x$ for $x \geq 17$ (corollary 1 of [13]), we have $\pi(2x) < 2\pi(x)$ for real $x \geq 3$.

Proof: By theorem 10, we have for $x \geq 60184$,

$$\frac{2x}{\ln 2x - 1} - \frac{x}{\ln x - 1.1} < \pi(2x) - \pi(x) < \frac{2x}{\ln 2x - 1.1} - \frac{x}{\ln x - 1}.$$

Let $f(x) = \frac{2x}{\ln 2x - 1} - \frac{x}{\ln x - 1.1} - \frac{x}{\ln x}$. As $1 + u < \frac{1}{1-u} < 1 + u + 2u^2$ for $|u| \leq 1/2$,

$$f(x) \geq \frac{x}{\ln^2 x} \left(2(1 - \ln 2) - 1.1 - 2\frac{1.1^2}{\ln x}\right).$$

In a same way, if we write

$$g(x) = \frac{2x}{\ln 2x - 1.1} - \frac{x}{\ln x - 1} - \frac{x}{\ln x},$$

we have

$$g(x) \leq \frac{x}{\ln^2 x} \left(2(1.1 - \ln 2) - 1 - 4 \frac{(1.1 - \ln 2)^2}{\ln x} \right).$$

□

7.2. Mandl's conjecture. We want to show the conjecture of ROBERT MANDL written in the article of ROSSER & SCHOENFELD [14] p. 243.

Theorem 15. *Let*

$$S_n = \sum_{k=1}^n p_k.$$

Then,

$$(9) \quad S_n \geq \frac{n^2}{2} (\ln n + \ln_2 n - 3/2) \quad \text{for } n \geq 305494,$$

$$(10) \quad S_n \leq \frac{n^2}{2} (\ln n + \ln_2 n - 1.463) \quad \text{for } n \geq 779.$$

Proof: Write

$$\begin{aligned} s(n) &= \frac{n^2}{2} (\ln n + \ln_2 n - 3/2) \\ f(n) &= s(n) - n \ln n \end{aligned}$$

Hence

$$\begin{aligned} f'(n) &= n \left(\ln n + \ln_2 n - 1 + \frac{1}{2 \ln n} \right) - \ln n - 1 \\ f''(n) &= \ln n + \ln_2 n + \frac{3}{2 \ln n} - \frac{1}{\ln^2 n} - \frac{1}{n} \end{aligned}$$

We take the TAYLOR series expansion of f between n and $n+1$:

$$f(n+1) - f(n) = f'(n) + \frac{f''(n_1)}{2} \quad \text{with } n < n_1 < n+1.$$

Since f'' increases, we obtain

$$f(n+1) - f(n) \leq f'(n) + \frac{f''(n+1)}{2} \leq f'(n) + \ln n.$$

By theorem 6,

$$p_n \geq n \left(\ln n + \ln_2 n - 1 + \frac{\ln \ln n - 2.25}{\ln n} \right),$$

this yields $n \geq \exp \exp(2.75)$,

$$p_n \geq f(n+1) - f(n).$$

For $n_0 = 10^6$,

$$S_{n_0-1} \geq f(n_0).$$

Suppose that $S_{n-1} \geq f(n)$ up to n . Then

$$S_n = S_{n-1} + p_n \geq f(n) + p_n \geq f(n+1).$$

Thus, for all $n \geq n_0$,

$$S_{n-1} \geq f(n)$$

which implies $S_n \geq s(n)$ since $p_n \geq n \ln n$. We verify that

$$S_n \geq s(n) \quad \text{for } n \geq 305494$$

by a computer check.

Inequality (10) can be find in [6]. □

We have

$$\begin{aligned}\frac{1}{n} \sum_{k=1}^n p_k &= \frac{n^2}{2} \left(\ln n + \ln_2 n - \frac{3}{2} + o(1) \right) \\ \frac{1}{2} p_n &= \frac{n^2}{2} (\ln n + \ln_2 n - 1 + o(1))\end{aligned}$$

Theorem 16 (Mandl). *For $n \geq 9$, we have*

$$(p_1 + p_2 + \cdots + p_n)/n < \frac{1}{2} p_n.$$

Proof: By (10), we have for $n \geq 779$,

$$\frac{S_n}{n} \leq \frac{n}{2} (\ln n + \ln_2 n - 1.463).$$

Now, theorem 5 yields the result. □

G. ROBIN gave me a conjecture concerning a lower bound of the above quantity:

Proposition 6. *For $n \geq 2$, we have*

$$p_{\lceil \frac{n}{2} \rceil} \leq \frac{1}{n} \sum_{i=1}^n p_i.$$

We have

$$\begin{aligned}\frac{1}{n} \sum_{k=1}^n p_k &= \frac{n^2}{2} \left(\ln n + \ln_2 n - \frac{3}{2} + o(1) \right) \\ np_{\lceil n/2 \rceil} &= \frac{n^2}{2} (\ln n + \ln_2 n - 1 - \ln 2 + o(1))\end{aligned}$$

Proof: According to theorem 7, we have for $k \geq 27076$,

$$p_k \leq k \left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k} \right).$$

So, for $n \geq 54152$,

$$np_{n/2} \leq \frac{n^2}{2} \left(\ln n + \ln_2 n - 1 - \ln 2 + \frac{\ln_2 n - 1.8}{\ln n - \ln 2} \right) \leq S_n.$$

We make a direct computer check to show that

$$np_{n/2} \leq S_n \quad \text{pour } k \leq 305494.$$

□

Proposition 7. *For $a \geq 2$ and $b \geq 2$, or for $b = 1$ and $a \geq 5$, we have*

$$p_{ab} > ap_b.$$

For a and b positive integers, we have

$$\frac{ap_{b+1} + bp_{a+1}}{2} < p_{ab+2}.$$

If a and b are integers greater than 91, we have

$$p_{ab} \leq ap_b + bp_a.$$

b	m	δ	ε
20	2	0.000128125	0.0006302
21	2	0.0000775	0.000419768506
22	2	0.000049375	0.000278652
23	2	13/400000	0.0001843645
24	2	17/800000	0.000121611962
25	2	0.0000128125	0.00007998895869
30	3	$0.94375 * 10^{-6}$	$0.9778040657 * 10^{-5}$
50	18	$0.947265625 * 10^{-8}$	$0.9049928595 * 10^{-7}$
100	18	$0.9305664063 * 10^{-8}$	$0.8842626429 * 10^{-7}$
200	17	$0.95078125 * 10^{-8}$	$0.8561316979 * 10^{-7}$
400	16	$0.9411132813 * 10^{-8}$	$0.8000089705 * 10^{-7}$
600	15	$0.9296875 * 10^{-8}$	$0.7442047763 * 10^{-7}$
1000	13	$0.905078125 * 10^{-8}$	$0.6337118668 * 10^{-7}$
1300	11	$0.919140625 * 10^{-8}$	$0.5518819789 * 10^{-7}$
1500	10	$0.905078125 * 10^{-8}$	$0.4980115883 * 10^{-7}$
1800	9	$0.83828125 * 10^{-8}$	$0.41913371 * 10^{-7}$
2000	8	$0.8171875 * 10^{-8}$	$0.3674711889 * 10^{-7}$
2300	6	$0.83125 * 10^{-8}$	$0.2917036 * 10^{-7}$
2500	5	$0.8171875 * 10^{-8}$	$0.243946 * 10^{-7}$
2750	4	$0.746875 * 10^{-8}$	$0.18769435073 * 10^{-7}$
3000	3	$0.690625 * 10^{-8}$	$0.137602 * 10^{-7}$
3500	2	$0.409375 * 10^{-8}$	$0.61653 * 10^{-8}$
4000	2	$0.1562500000 * 10^{-8}$	$0.2405714403 * 10^{-8}$

TABLE 1. $|\psi(x) - x| \leq \varepsilon x$ for $x \geq \exp(b)$

For a and b positive integers, we have

$$p_{ab+2} \leq ap_{b+1} + bp_{a+1}.$$

Proof: Use the inequalities (3) and (4). □

REFERENCES

- [1] R. P. BRENT, "Irregularities in the Distribution of Primes and Twin Primes", *Math. Of Computation*, Vol. **29**, Number 129 (January 1975) pp. 43-56.
UMT, *Math. Of Computation*, Vol. **30**, Number 134 (April 1976) p. 379.
- [2] E. CESARO, "Sur une formule empirique de M. Pervouchine", *Comptes rendus hebdo. des séances de l'académie des sciences*, Vol. **CXIX**, (1895) pp. 848-849.
- [3] M. CIPOLLA, "La determinazione assintotica dell' n^{imo} numero primo", *Matematiche Napoli*, Vol. **3**, (1902) pp. 132-166.
- [4] N. COSTA PEREIRA, "Estimates for the Chebyshev Function $\psi(x) - \theta(x)$ ", *Math. Of Computation*, Vol. **44**, Number 169 (January 1985) pp. 211-221.
- [5] P. DUSART, "The k^{th} prime is greater than $k(\ln k + \ln \ln k - 1)$ for $k \geq 2$ ", *Math. Of Computation*, to appear.
- [6] JEAN-PIERRE MASSIAS & GUY ROBIN, "Bornes effectives pour certaines fonctions concernant les nombres premiers", *Publication Département Math. de Limoges and Journal Th. Nombres de Bordeaux*, Vol. **8** (1996) pp. 213-238.
- [7] O. RAMARÉ, "Short effective intervals containing primes", *Journal Number Th.*, to appear.
- [8] HANS RIESEL, "Prime Numbers and Computer Methods for Factorization", Birkhäuser (1985)
- [9] GUY ROBIN, "Estimation de la fonction de Tchebychef θ sur le k^{ieme} nombre premier et grandes valeurs de la fonctions $\omega(n)$, nombre de diviseurs premiers de n ", *Acta Arithmetica*, vol. **42**, numéro 4 (1983) pp. 367-389.

x	$\theta(x) \geq \cdot$	$\theta(x) \leq \cdot$
10^5	99624.9	99779.3
$2 \cdot 10^5$	199445.0	199629.8
$5 \cdot 10^5$	499216.3	499468.0
10^6	998357.3	998668.6
$2 \cdot 10^6$	1998421.0	1998812.1
$5 \cdot 10^6$	4998331.1	4998889.9
10^7	9994872.2	9995589.7
$2 \cdot 10^7$	19995425.3	19996352.2
$5 \cdot 10^7$	49993085.2	49994491.2
10^8	99986933.8	99988722.5
$2 \cdot 10^8$	199981158.2	199983578.4
$5 \cdot 10^8$	499982199.5	499985664.2
10^9	999966797.9	999971596.9
$2 \cdot 10^9$	1999938007.6	1999944598.4
$5 \cdot 10^9$	4999901997.2	4999911609.0
10^{10}	9999933799.8	9999946821.9
$2 \cdot 10^{10}$	19999813136.0	19999831825.3
$5 \cdot 10^{10}$	49999714959.0	49999744514.5
$8 \cdot 10^{10}$	79999702170.2	79999737244.0
10^{11}	99999720459.8	99999757299.3

TABLE 2. Bounds for $\theta(x)$.

- [10] GUY ROBIN, "Permanence de relations de récurrence dans certains développements asymptotiques", *Publications de l'institut mathématique*, tome 43 (57), (1988) pp. 17-25.
- [11] J. BARKLEY ROSSER, "the n -th prime is greater than $n \log n$ ", *Proc. London Math. Soc. (2)*, Vol. **45**, (1939) pp. 21-44.
- [12] J. BARKLEY ROSSER, "Explicit Bounds for some functions of prime numbers", *Amer. J. Math.*, Vol. **63**, (1941) pp. 211-232.
- [13] J. BARKLEY ROSSER & L. SCHOENFELD, "Approximate Formulas for Some Functions of Prime Numbers", *Illinois Journal Math.* **6** (1962) pp. 64-94.
- [14] J. BARKLEY ROSSER & L. SCHOENFELD, "Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$ ", *Math. Of Computation*, Vol. **29**, Number 129 (January 1975) pp. 243-269.
- [15] BRUNO SALVY, "Fast computation of some asymptotic functional inverses", *J. Symbolic Computation*, Vol. **17**, (1994) pp. 227-236.
- [16] LOWELL SCHOENFELD, "Sharper Bounds for the Chebyshev Functions $\theta(x)$ and $\psi(x)$.II", *Math. Of Computation*, Vol. **30**, Number 134 (April 1976) pp. 337-360.
- [17] J. VAN DE LUNE, H. J. J. TE RIELE & D.T. WINTER, "On the Zeros of the Riemann Zeta Function in the Critical Strip.IV" *Math. Of Computation*, Vol. **46**, Number 174 (April 1986) pp. 667-681.

DÉPARTEMENT DE MATH., LACO, UPRES A 6090, 123 AVENUE ALBERT THOMAS, 87060
LIMOGES CEDEX
E-mail address: dusart@unilim.fr