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Sharper bounds for ψ, θ, π, p_k

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ABSTRACT. In default of a proof of the RIEMANN hypothesis, the best estimates for $\psi(x)$ and $\theta(x)$ and hence of $\pi(x)$, p_k and other functions of the primes, depend on the current state of knowledge of the zeros of $\zeta(s)$. With a better knowledge about the zeros of the RIEMANN ζ function, we can show sharper bounds for $\psi(x), \theta(x), \pi(x)$ and primes p_k.

1. INTRODUCTION

In many applications it is useful to have explicit error bounds in the prime number theorem. ROSSER and SCHOENFELD developed an analytic method which combines a numerical verification of the RIEMANN hypothesis with a zero-free region. The aim of this paper is to find sharper bounds for the CHEBYSHEV's functions $\psi(x)$, the logarithm of the least common multiple of all integers not exceeding x, and $\theta(x)$, the product of all primes not exceeding x:

$$\theta(x) = \sum_{p \leqslant x} \ln p, \qquad \psi(x) = \sum_{\substack{p,\nu \\ p^\nu \leqslant x}} \ln p$$

where sum runs over primes p and respectively over powers of primes p^ν. The Prime Number Theorem could be written as follows:

$$\psi(x) = x + o(x), \quad x \to +\infty.$$

An equivalent formulation of the above theorem should be: for all $\varepsilon > 0$, there exists $x_0 = x_0(\varepsilon)$ such that

$$|\psi(x) - x| < \varepsilon\, x \qquad \text{for } x \geqslant x_0$$

or

$$|\theta(x) - x| < \varepsilon\, x \qquad \text{for } x \geqslant x_0.$$

This article hangs up on some known results: the most important works on effective results have been shown by ROSSER & SCHOENFELD [13, 14, 16], ROBIN [9] & MASSIAS [6] and COSTA PEREIRA [4].

The proofs for estimates of $\psi(x)$ in [14] are based on the verification of RIEMANN hypothesis to a given height and an explicit zero-free region for $\zeta(s)$ whose form is essentially that the classical one of DE LA VALLÉE POUSSIN. ROSSER & SCHOENFELD have shown that the first 3 502 500 zeros of $\zeta(s)$ are on the critical strip. VAN DE LUNE *et al* [17] have shown that the first 1 500 000 000 zeros are

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on the critical strip. We will improve bounds for $\psi(x)$ and $\theta(x)$. We will prove the following results:

$$\begin{array}{rcll}
|\psi(x)-x| & \leqslant & 0.006409\dfrac{x}{\ln x} & \text{for } x \geqslant \exp(22),\\
|\theta(x)-x| & \leqslant & 0.006788\dfrac{x}{\ln x} & \text{for } x \geqslant 10\ 544\ 111,\\
|\theta(x)-x| & \leqslant & 0.2\dfrac{x}{\ln^2 x} & \text{for } x \geqslant 3594641,\\
|\theta(x)-x| & \leqslant & 515\dfrac{x}{\ln^3 x} & \text{for } x > 1,\\
|\theta(x)-x| & \leqslant & 1717433\dfrac{x}{\ln^4 x} & \text{for } x > 1.
\end{array}$$

We apply these results on p_k, the k^{th} prime, and $\theta(p_k)$. Let's denote by $\ln_2 x$ for $\ln\ln x$.

The asymptotic expansion of p_k is well known; Cesaro [2] then Cipolla [3] expressed it in 1902:

$$p_k = k\left\{\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k} - \frac{\ln_2^2 k - 6\ln_2 k + 11}{2\ln^2 k} + O\left(\left(\frac{\ln_2 k}{\ln k}\right)^3\right)\right\}.$$

A more precise work about this can be find in [10, 15]. We group together the results on p_k and on $\theta(p_k)$:

$$\begin{array}{llcll}
(1) & \theta(p_k) & \geqslant & k\left(\ln k + \ln_2 k - 1 + \dfrac{\ln_2 k - 2.0553}{\ln k}\right) & \text{for } k \geqslant \exp(22),\\
(2) & \theta(p_k) & \leqslant & k\left(\ln k + \ln_2 k - 1 + \dfrac{\ln_2 k - 2}{\ln k}\right) & \text{for } k \geqslant 198,\\
(3) & p_k & \geqslant & k\left(\ln k + \ln_2 k - 1\right) & \text{for } k \geqslant 2,\\
(4) & p_k & \leqslant & k\left(\ln k + \ln_2 k - 0.9484\right) & \text{for } k \geqslant 39017,\\
(5) & p_k & \leqslant & k\left(\ln k + \ln_2 k - 1 + \dfrac{\ln_2 k - 1.8}{\ln k}\right) & \text{for } k \geqslant 27076,\\
(6) & p_k & \geqslant & k\left(\ln k + \ln_2 k - 1 + \dfrac{\ln_2 k - 2.25}{\ln k}\right) & \text{for } k \geqslant 2.
\end{array}$$

The formula (2) has been proved by Robin in [9].

We use the above results to prove that, for $x \geqslant 3275$, the interval

$$\left[x, x + x/(2\ln^2 x)\right]$$

contains at least one prime. Let's denote by $\pi(x)$ the number of primes not greater than x. We show that

$$\frac{x}{\ln x}\left(1 + \frac{1}{\ln x}\right) \underset{x\geqslant 599}{\leqslant} \pi(x) \underset{x>1}{\leqslant} \frac{x}{\ln x}\left(1 + \frac{1.2762}{\ln x}\right).$$

More precise results on $\pi(x)$ are also shown:

$$\begin{aligned}
\pi(x) &\geqslant \frac{x}{\ln x - 1} \text{ for } x \geqslant 5393,\\
\pi(x) &\leqslant \frac{x}{\ln x - 1.1} \text{ for } x \geqslant 60184,\\
\pi(x) &\geqslant \frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{1.8}{\ln^2 x}\right) \text{ for } x \geqslant 32299,\\
\pi(x) &\leqslant \frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{2.51}{\ln^2 x}\right) \text{ for } x \geqslant 355991.
\end{aligned}$$

We gave in § 7.1 some others inequalities.

2. Results of Rosser & Schoenfeld

We rewrite here some results of [14] which we use. We want to bound $\psi(x) - x$ for "moderate" values of x. Remember the used notations:
we want to find ε such that we have, for $x \geqslant \exp(b)$

$$|\psi(x) - x| < \varepsilon x.$$

Let

$$R = 9.645908801,$$
$$X = \sqrt{\ln(x)/R},$$
$$K_\nu(z,x) = \frac{1}{2}\int_x^\infty t^{\nu-1}\exp\{(-1/2)z(t+1/t)\}dt \qquad \text{for } z>0, x \geqslant 0, \text{ real } \nu,$$
$$\phi_m(y) = y^{-m-1}\exp(-X^2/\ln(y/17)),$$
$$R_m(\delta) = \left((1+\delta)^{m+1}+1\right)^m,$$
$$D = 158.84998,$$
$$T_1 = \frac{1}{\delta}\left(\frac{2R_m(\delta)}{2+m\delta}\right)^{1/m}.$$

Let $N(T)$ the number of zeros such that $0 < \gamma \leqslant T$ and its approximation

$$F(T) = \frac{T}{2\pi}\ln\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}$$

and let

$$R(T) = 0.137\ln T + 0.443\ln_2 T + 1.588.$$

Let A such all zeros $\rho = \beta + i\gamma$ of the Riemann ζ function in the critical strip are such that the real part $\beta = 1/2$ for $0 < \gamma \leqslant A$. The number A, used by Rosser & Schoenfeld, is defined as the unique solution of the equation $F(A) = 3,502,500$, which yields $A = 1,894,438.51224\cdots$ and $F(A) = N(A)$. We use a new value of A, defined as the unique solution of the equation $F(A) = 1,500,000,001$. According to [17], this yields $A = 545,439,823.215\cdots$ and $F(A) = N(A)$.

Proposition 1 ([14], Theorem 4)**.** *Let $T_1 \geqslant D$, m be a positive integer and*

$$\begin{aligned}
\Omega_1 &= \frac{2+m\delta}{4\pi}\left\{\left(\ln\frac{T_1}{2\pi}+\frac{1}{m}\right)^2 + 0.038207 + \frac{1}{m^2} - \frac{2.82m}{(m+1)T_1}\right\}\\
\Omega_2 &= \frac{(0.159155)R_m(\delta)z}{2m^2 17^m}\left\{zK_2(z,A') + 2m\ln\left(\frac{17}{2\pi}\right)K_1(z,A')\right\}\\
&\quad + R_m(\delta)\{2R(Y)\phi_m(Y) - R(A)\phi_m(A)\},
\end{aligned}$$

where $z = 2X\sqrt{m} = 2\sqrt{mb/R}$, $A' = (2m/z)\ln(A/17)$ *and*

$$Y = \max\{A,\ 17\exp\sqrt{b/((m+1)R)}\}.$$

If $b > 1/2$ $0 < \delta < (1-\exp(-b))/m$ *then, for all* $x \geqslant \exp(b)$,

$$|\psi(x) - x| < \varepsilon x$$

where $\varepsilon = \Omega_1\exp(-b/2) + \Omega_2\delta^{-m} + m\delta/2 + \exp(-b)\ln(2\pi)$.

As we use this theorem, we write explicitly the proof which doesn't appear in Rosser & Schoenfeld's article; They gave only indications.

Proof: In all this proof, the numbering refers to [14] and some non-defined notations too. As $T_2 = 0$, $S_3(m,\delta) = 0$.

$$S_4(m,\delta) = 2\sum_{\substack{\beta>1/2\\ \gamma>0}} \frac{R_m(\delta)\exp(-X^2\ln(\gamma/17))}{\delta^m|\rho(\rho+1)\cdots(\rho+m)|} < \frac{R_m(\delta)}{\delta^m}\sum_{\gamma>A}\phi(\gamma)$$

by writing $\phi(y) := \phi_m(y)$. By using corollary of lemma 7, ($\phi(y) := \phi_m(y)$)

$$\sum_{\gamma>A}\phi(\gamma) \leqslant \{\frac{1}{2\pi} + q(y)\}\int_A^{\infty}\phi(y)\ln\left(\frac{y}{2\pi}\right)dy + E_0$$

where $q(y) = \frac{0.137\ln y + 0.443}{y\ln y\ln\left(\frac{y}{2\pi}\right)}$ and $E_0 = 2R(Y)\phi(Y) - R(A)\phi(A)$ because $N(A) = F(A)$ with $Y = \max(A, W)$ where W satisfies $(W-y)\phi'(y) \geqslant 0$ for $y > A$. Let's compute W.

$$\phi'(y) = \phi'_m(y) = y^{-m-2}\exp(-X^2/\ln(y/17))\left(-(m+1) + \frac{X^2}{\ln^2(y/17)}\right).$$

Hence $\phi'_m(y) \geqslant 0$ for $y \geqslant 17\exp\left(\sqrt{\frac{X^2}{m+1}}\right)$. According to (3.16),

$$\int_A^{+\infty}\phi_m(y)\ln\left(\frac{y}{2\pi}\right)dy = \frac{z^2}{2m^2+17^m}K_2(z,A') + \frac{z\ln(17/(2\pi))}{m17^m}K_1(z,A')$$

where $z = 2X\sqrt{m}$ and $A' = (2m/z)\ln(A/17)$. Moreover, $q(Y) \leqslant q(A) \leqslant 7\cdot10^{-9}$. So for $x \geqslant \exp(b)$,

$$S_4(m,\delta) < \frac{R_m(\delta)}{\delta^m}(\frac{1}{2\pi}+7\cdot10^{-9})\left(\frac{z^2}{2m^2+17^m}K_2(z,A') + \frac{z\ln\left(\frac{17}{2\pi}\right)}{m17^m}K_1(z,A')\right)+E_0$$

where $z = 2\sqrt{\frac{mb}{R}}$, $A' = (2m/z)\ln(A/17)$.

We deduce that

$$S_3(m,\delta) + S_4(m,\delta) \leqslant \Omega_2\delta^{-m}.$$

By using the upper bound of $S_1(m,\delta) + S_2(m,\delta)$ given in this previous lemma, we conclude by lemma 8 that

$$\frac{1}{x}|\psi(x) - (x - \ln(2\pi) - 1/2\cdot\ln(1-1/x^2))| \leqslant \Omega_1/\sqrt{x} + \Omega_2\delta^{-m} + m\delta/2$$

thus

$$|\psi(x) - x| \leqslant (\Omega_1\exp(-b/2) + \Omega_2\delta^{-m} + m\delta/2 + \exp(-b)\ln(2\pi))x.$$

□

3. Results on $\psi(x)$ and $\theta(x)$

We group here together some known results which we will use in our proofs.

Proposition 2. *(i) -* $\theta(x) \leqslant \psi(x)$ *for all* x.
(ii) - $\psi(x) - \theta(x) < \sqrt{x} + \frac{6}{5}\sqrt[3]{x}$ *for* $10^8 \leqslant x \leqslant 10^{16}$.
(iii) - $\psi(x) - \theta(x) < 1.43\sqrt{x}$ *for all* $x > 0$.
(iv) - $\theta(x) < x$ *for* $0 < x \leqslant 10^{11}$.
(v) - $\theta(x) < 1.000081x$ *for* $x > 0$.
(vi) - $|\theta(x) - x| < 0.0077629\frac{x}{\ln x}$ *for* $x > 1.04 \cdot 10^7$.

Proof: The first inequality can be easily deduced from

$$\psi(x) = \sum_{k=1}^{\infty} \theta(x^{1/k}).$$

The second is given in [4]. The next one is theorem 13 of [13]. The three last ones are taken from [16] p. 360. □

For "large" values of x, we use the following theorem (theorem 11 p. 342 of [16]):

Theorem 1. *Let* $R = 9.645908801$ *and* $X = \sqrt{\frac{\ln x}{R}}$.

$$|\psi(x) - x| < x\varepsilon_0(x) \qquad \text{for } x \geqslant 17,$$
$$|\theta(x) - x| < x\varepsilon_0(x) \qquad \text{for } x \geqslant 101$$

where $\varepsilon_0(x) = \sqrt{8/(17\pi)}X^{1/2}\exp(-X)$.

Theorem 2. *Let* b *be a positive real. For* $x \geqslant \exp(b)$,

$$|\ \psi(x) - x\ | \leqslant \varepsilon x,$$

where ε *is given in table 1.*

Corollary 1. *For* $x \geqslant \exp(50)$, *we have*

$$|\ \psi(x) - x\ | \leqslant 0.905 \cdot 10^{-7}x.$$

Proof: We make no mathematical change of the Rosser & Schoenfeld's method. We only change the value A by a bigger one. Apply proposition 1 with $A = 545,439,823.215\cdots$ as defined in section 2. Evaluation of the quantities of proposition 1, for example with the computer algebra `Maple` [1], gives new bounds given in table 1.

If we take $A = 1,894,438.51224\cdots$, this implementation permits to recover the results found by Rosser and Schoenfeld.

In particular, we compute the value of ε for $x \geqslant e^{50}$. Choose $\delta = 0.947265625 \cdot 10^{-8}$ and $m = 18$. By computation, we find a value for ε lightly less than $0.905 \cdot 10^{-7}$. This value has been verified with `Pari`. □

We need afterwards to have bounds for $\theta(x)$ to 10^{11}. The first method is to compute $\theta(x)$ until 10^{11} (we need a powerful computer). The second method uses the remark that the table of [1], which gives the upper and the lower bounds of $\mathrm{Li}(x) - \pi(x)$ by intervals up to 10^{11}, permits to find precise bounds for $\theta(x)$ up to 10^{11}. The results are in table 2.

We construct this table in the following manner:
Let $\rho_1(a,b) = \min_{p_k\in[a,b]}(\mathrm{Li}(p_k) - k)$ and $R_1(a,b) = \max_{p_k\in[a,b]}(\mathrm{Li}(p_k) - k)$. Write

[1] We thank the `Médicis` group for the use of their computers.

$r(a,b) = \min_{x\in[a,b]}(\mathrm{Li}(x) - \pi(x))$ and $R(a,b) = \max_{x\in[a,b]}(\mathrm{Li}(x) - \pi(x))$. Note that $r(a,b) \geqslant \min(\rho_1(a,b) - 1/2, \mathrm{Li}(a) - \pi(a))$ and $R(a,b) \leqslant \max(R_1(a+1,b) + 1 + 1/2, \mathrm{Li}(b) - \pi(b))$. The interval $[0, 10^{11}]$ is split into intervals $[a_i, b_i]$.

$$\begin{aligned}
\theta(x) &= \pi(x)\ln(x) - \int_2^x \frac{\pi(y)}{y}dy \\
&= \pi(x)\ln(x) - \int_2^{100} \frac{\pi(y)}{y}dy - \int_{100}^x \frac{\pi(y) - \mathrm{Li}(y)}{y} + \frac{\mathrm{Li}(y)}{y}dy \\
&\geqslant x - (\mathrm{Li}(x) - \pi(x))\ln x + C + \sum_{\substack{[a_i,b_i]\\ 100\leqslant a_i,b_i\leqslant b}} r(a_i,b_i)\ln\left(\frac{b_i}{a_i}\right) + r(b,x)\ln\left(\frac{x}{b}\right) \qquad (7)
\end{aligned}$$

where $C = \mathrm{Li}(100)\ln(100) - 100 - \int_2^{100} \frac{\pi(y)}{y}dy$ and b is the upper bound of the last interval which doesn't contain x. If we don't know the value of $\pi(x)$, we can bound the difference $\mathrm{Li}(x) - \pi(x)$ by $R(a,x)$.

We have an upper bound of $\theta(x)$ by replacing in formula (7) all r by R and conversely R by r.

Theorem 3. 1. $|\psi(x) - x| \leqslant 0.006409\frac{x}{\ln x}$ *for* $x \geqslant \exp(22)$.
2. $|\theta(x) - x| \leqslant 0.006788\frac{x}{\ln x}$ *for* $x \geqslant 10\ 544\ 111$.

Proof: We use table 1 by intervals:
if $x \geqslant \exp(22)$ then $|\psi(x) - x| \leqslant 2.78652 \cdot 10^{-4}x \leqslant 23 * 2.78652 \cdot 10^{-4}\frac{x}{\ln x} \leqslant 6.409 \cdot 10^{-3}\frac{x}{\ln x}$ for $x \leqslant \exp(23)$.
If $x \geqslant \exp(23)$ then $|\psi(x) - x| \leqslant 1.8436 \cdot 10^{-4}x \leqslant 5.5308 \cdot 10^{-3}\frac{x}{\ln x}$ for $x \leqslant \exp(30)$.
If $x \geqslant \exp(30)$ then $|\psi(x) - x| \leqslant 0.978 \cdot 10^{-5}x \leqslant 5.868 \cdot 10^{-3}\frac{x}{\ln x}$ for $x \leqslant \exp(600)$.
If $x \geqslant \exp(600)$ then $|\psi(x) - x| \leqslant 0.75 \cdot 10^{-7}x \leqslant 1.5 \cdot 10^{-4}\frac{x}{\ln x}$ for $x \leqslant \exp(2000)$.
Moreover,
$|\psi(x) - \theta(x)| < \sqrt{x} + \frac{6}{5}\sqrt[3]{x}$ for $10^8 \leqslant x \leqslant 10^{16}$ thus
$|\psi(x) - \theta(x)| < 0.00037871\frac{x}{\ln x}$ for $\exp(22) \leqslant x \leqslant \exp(30)$
and
$|\psi(x) - \theta(x)| < 1.43\sqrt{x}$ for $x > 0$ thus
$|\psi(x) - \theta(x)| < 1.32 \cdot 10^{-5}\frac{x}{\ln x}$ for $x \geqslant \exp(30)$.

For $x \geqslant \exp(2000)$, we use theorem 1 to conclude that

$$|\psi(x) - x| \text{ and } |\theta(x) - x| \leqslant 0.00164\frac{x}{\ln x} \quad \text{for } x \geqslant \exp(2000).$$

With Table 2 and a computer verification, we extend the result on θ, showed until now for $x \geqslant \exp(22)$, until 10 544 111. This shows that the result of proposition 2(vi) of Rosser & Schoenfeld is almost best possible. □

The above result gives a formula with order 1 for power of logarithm. We need sometimes the next orders. We obtain better results than Rosser & Schoenfeld ones.

Theorem 4. *Let* $\eta_2 = 3.965$, $\eta_3 = 515$ *and* $\eta_4 = 1717433$.
For $x > 1$,

$$|\theta(x) - x| < \eta_k\frac{x}{\ln^k x}.$$

(We can choose $\eta_2 = 0.2$ *for* $x \geqslant 3594641$.*)*

Proof: In all cases, we use table 1 by intervals. Starting with $k = 2$, we show that

$$|\theta(x) - x| < 0.2\frac{x}{\ln^2 x}.$$

For $x \geqslant \exp(3220)$, according to theorem 1 we have

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2 \frac{x}{\ln^2 x}$$

with $\eta_2 = 0.19923$ for $x \geqslant \exp(3220)$.

For $x \geqslant 1.04 \cdot 10^7$,	$|\theta(x) - x| \leqslant 0.0077629 \frac{x}{\ln x}$	$\leqslant 0.1941 \frac{x}{\ln^2 x}$	for $x \leqslant e^{25}$.
For $x \geqslant e^{25}$,	$|\psi(x) - x| \leqslant 10^{-4} x$	$\leqslant 0.09 \frac{x}{\ln^2 x}$	for $x \leqslant e^{30}$.
For $x \geqslant e^{30}$,	$|\psi(x) - x| \leqslant 10^{-5} x$	$\leqslant 0.1 \frac{x}{\ln^2 x}$	for $x \leqslant e^{100}$.
For $x \geqslant e^{100}$,	$|\psi(x) - x| \leqslant 0.9 \cdot 10^{-7} x$	$\leqslant 0.1521 \frac{x}{\ln^2 x}$	for $x \leqslant e^{1300}$.
For $x \geqslant e^{1300}$,	$|\psi(x) - x| \leqslant 0.6 \cdot 10^{-7} x$	$\leqslant 0.1944 \frac{x}{\ln^2 x}$	for $x \leqslant e^{1800}$.
For $x \geqslant e^{1800}$,	$|\psi(x) - x| \leqslant 0.42 \cdot 10^{-7} x$	$\leqslant 0.168 \frac{x}{\ln^2 x}$	for $x \leqslant e^{2000}$.
For $x \geqslant e^{2000}$,	$|\psi(x) - x| \leqslant 0.37 \cdot 10^{-7} x$	$\leqslant 0.196 \frac{x}{\ln^2 x}$	for $x \leqslant e^{2300}$.
For $x \geqslant e^{2300}$,	$|\psi(x) - x| \leqslant 0.292 \cdot 10^{-7} x$	$\leqslant 0.1825 \frac{x}{\ln^2 x}$	for $x \leqslant e^{2500}$.
For $x \geqslant e^{2500}$,	$|\psi(x) - x| \leqslant 0.244 \cdot 10^{-7} x$	$\leqslant 0.18453 \frac{x}{\ln^2 x}$	for $x \leqslant e^{2750}$.
For $x \geqslant e^{2750}$,	$|\psi(x) - x| \leqslant 0.19 \cdot 10^{-7} x$	$\leqslant 0.197 \frac{x}{\ln^2 x}$	for $x \leqslant e^{3220}$.

Moreover,

for $\exp(25) \leqslant x \leqslant \exp(30)$,	$|\psi(x) - \theta(x)|$	$< \sqrt{x} + \frac{6}{5}\sqrt[3]{x}$	$< 0.0023725 \frac{x}{\ln^2 x}$,
and for $x \geqslant \exp(30)$,	$|\psi(x) - \theta(x)|$	$< 1.43\sqrt{x}$	$< 0.0004 \frac{x}{\ln^2 x}$.

Now we are interested in case $k = 3$. As $|\theta(x) - x| \leqslant 0.0077629 \frac{x}{\ln x}$, we have

$$|\theta(x) - x| \leqslant 310.516 \frac{x}{\ln^3 x} \qquad \text{for } x \leqslant \exp(200).$$

Note that for $x \geqslant \exp(200)$, the difference between θ and ψ is negligible since, for $x \geqslant \exp(200)$,

$$|\psi(x) - \theta(x)| < 1.43\sqrt{x} \leqslant 0.5 \cdot 10^{-36} \frac{x}{\ln^3 x}.$$

For $x \geqslant e^{200}$,	$|\theta(x) - x| \leqslant 0.8561317 \cdot 10^{-7} x$	$\leqslant 500 \frac{x}{\ln^3 x}$	for $x \leqslant e^{1800}$.
For $x \geqslant e^{1800}$,	$|\theta(x) - x| \leqslant 0.419134 \cdot 10^{-7} x$	$\leqslant 510 \frac{x}{\ln^3 x}$	for $x \leqslant e^{2300}$.
For $x \geqslant e^{2300}$,	$|\theta(x) - x| \leqslant 0.2917036 \cdot 10^{-7} x$	$\leqslant 456 \frac{x}{\ln^3 x}$	for $x \leqslant e^{2500}$.
For $x \geqslant e^{2500}$,	$|\theta(x) - x| \leqslant 0.243946 \cdot 10^{-7} x$	$\leqslant 508 \frac{x}{\ln^3 x}$	for $x \leqslant e^{2750}$.
For $x \geqslant e^{2750}$,	$|\theta(x) - x| \leqslant 0.1877 \cdot 10^{-7} x$	$\leqslant 507 \frac{x}{\ln^3 x}$	for $x \leqslant e^{3000}$.
For $x \geqslant e^{3000}$,	$|\theta(x) - x| \leqslant 0.137602 \cdot 10^{-7} x$	$\leqslant 514 \frac{x}{\ln^3 x}$	for $x \leqslant e^{3341}$.

Now apply theorem 1 for $x \geqslant \exp(3341)$; we find

$$|\theta(x) - x| \leqslant 514.826 \frac{x}{\ln^3 x}.$$

We use the same methods for case $k = 4$. As $|\theta(x) - x| \leqslant 0.0077629 \frac{x}{\ln x}$, we have

$$|\theta(x) - x| \leqslant 1676786.4 \frac{x}{\ln^3 x} \qquad \text{for } x \leqslant \exp(600).$$

Note that, for $x \geqslant \exp(600)$, the difference between θ and ψ is negligible seeing that, for $x \geqslant \exp(600)$,

$$|\psi(x) - \theta(x)| < 1.43\sqrt{x} \leqslant 10^{-119} \frac{x}{\ln^4 x}.$$

For $x \geqslant e^{600}$,	$|\theta(x) - x| \leqslant 0.744205 \cdot 10^{-7} x$	$\leqslant \frac{1190728x}{\ln^4 x}$	for $x \leqslant e^{2000}$.
For $x \geqslant e^{2000}$,	$|\theta(x) - x| \leqslant 0.3675 \cdot 10^{-7} x$	$\leqslant \frac{1435547x}{\ln^4 x}$	for $x \leqslant e^{2500}$.
For $x \geqslant e^{2500}$,	$|\theta(x) - x| \leqslant 0.243946 \cdot 10^{-7} x$	$\leqslant \frac{1395162x}{\ln^4 x}$	for $x \leqslant e^{2750}$.
For $x \geqslant e^{2750}$,	$|\theta(x) - x| \leqslant 0.1877 \cdot 10^{-7} x$	$\leqslant \frac{1520370x}{\ln^4 x}$	for $x \leqslant e^{3000}$.
For $x \geqslant e^{3000}$,	$|\theta(x) - x| \leqslant 0.137602 \cdot 10^{-7} x$	$\leqslant \frac{1716527x}{\ln^4 x}$	for $x \leqslant e^{3342}$.

Now apply theorem 1 for $x \geqslant \exp(3342)$: we find
$$|\theta(x) - x| \leqslant 1717433 \frac{x}{\ln^4 x}.$$
We verify by computer for $x \leqslant 1,04 \cdot 10^7$. We find that $|\theta(x) - x| \leqslant 0.2 \frac{x}{\ln^2 x}$ for $x \geqslant 3594641$ and $|\theta(x) - x| \leqslant 3.9648085 \cdots \frac{x}{\ln^2 x}$ for $x > 1$ (the value $3.9648085 \cdots$ is choosen for p_{17}). □

4. Results on p_k and $\theta(p_k)$

For some steps of demonstrations, we use the following results:

Lemma 1.
$$\begin{array}{llll} p_k & \geqslant & k(\ln k) & \text{for } k \geqslant 2, \\ p_k & \leqslant & k(\ln k + \ln_2 k) & \text{for } k \geqslant 6, \\ p_k & \leqslant & k \ln p_k & \text{for } k \geqslant 4, \\ p_k & \geqslant & k(\ln p_k - 2) & \text{for } k \geqslant 5. \end{array}$$

Proof: These inequalities have been proved by Rosser. The first inequality can be found in [11]; the next one in [12]. The last ones can be easily deduced from
$$\frac{x}{\ln x} < \pi(x) < \frac{x}{\ln x - 2}$$
(see [12]). □

Rosser showed that $p_k \geqslant k \ln k$ and improved his result with Schoenfeld by showing that $p_k \geqslant k(\ln k + \ln_2 k - 3/2)$ (see [14]). En 1983, Robin [9] succeeded to prove that
$$p_k \geqslant k(\ln k + \ln_2 k - 1.0072629).$$
Massias & Robin [6] are able to show that
$$p_k \geqslant k(\ln k + \ln_2 k - 1)$$
for $k \leqslant \exp(598)$ and $k \geqslant \exp(1800)$.

We have show that for all $k \geqslant 2$ in [5].

Theorem 5. *For $k \geqslant 2$, we have*
$$p_k \geqslant k(\ln k + \ln_2 k - 1)$$
where p_k is k^{th} prime number.

First method to find a lower bound for $\theta(p_k)$.

Proposition 3. *Let*
$$f_\beta(k) := k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k}\right).$$
Assume that for $k \geqslant k_0$
$$|\theta(p_k) - p_k| \leqslant ck.$$
Let β be a real satisfying
$$\begin{aligned} \beta(\ln k_0 - 1) \quad \geqslant \quad & \ln k_0(\ln_2 k_0 + 1) + 1 - \ln_2 k_0 \\ & - \ln^2 k_0 \ln\left(1 + \frac{\ln_2 k_0 - 1}{\ln k_0} + \frac{\ln_2 k_0 - \beta - c\ln k_0}{\ln^2 k_0}\right). \end{aligned}$$
If $\theta(p_{k_0}) \geqslant f_\beta(k_0)$ then
$$\theta(p_k) \geqslant f_\beta(k) \qquad \text{for } k \geqslant k_0.$$

Proof: Assume that

$$p_k \geqslant h_a(k) := k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - a}{\ln k}\right).$$

Hence

$$\theta(p_k) - \theta(p_{k_0}) = \sum_{n=k_0+1}^{k} \ln p_n \geqslant \sum_{n=k_0+1}^{k} \ln h_a(n).$$

Moreover

$$f'_\beta(k) = \ln k + \ln_2 k + \frac{\ln_2 k - \beta + 1}{\ln k} - \frac{\ln_2 k - (\beta+1)}{\ln^2 k}$$

and

$$\ln p_k \geqslant \ln h_a = \ln k + \ln_2 k + \ln\left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k}\right).$$

It's sufficient to show when $f'_\beta \leqslant \ln h_a$ because

$$f_\beta(k) - f_\beta(k_0) = \int_{k_0}^{k} f'_\beta(x)dx \leqslant \sum_{n=k_0+1}^{k} \ln h_a(n) \leqslant \theta(p_k) - \theta(p_{k_0}),$$

equivalently

$$\frac{\ln_2 k + 1 - \beta}{\ln k} - \frac{\ln_2 k - (\beta+1)}{\ln^2 k} \leqslant \ln\left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k}\right)$$

$$\beta(\ln k - 1) \geqslant \ln k(\ln_2 k + 1) - \ln^2 k \ln\left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - a}{\ln^2 k}\right) + 1 - \ln_2 k.$$

Since $p_k \geqslant \theta(p_k) - ck$, we can choose $a = \beta + c\ln k$. Then we find that

$$\beta(\ln k - 1) \geqslant (\ln k)(\ln_2 k + 1) - (\ln^2 k)\ln\left(1 + \frac{\ln_2 k - 1}{\ln k} + \frac{\ln_2 k - \beta - c\ln k}{\ln^2 k}\right) + 1 - \ln_2 k.$$

□

We apply proposition 3. For $c = 0.007$, the computed values of β are:

$\ln k_0$	β
22	2.0553
23	2.0532
25	2.04975
30	2.04397
100	2.02961
1000	2.0156
2000	2.0128

Corollary 2. *For $p_k \geqslant 10^{11}$,*

$$\theta(p_k) \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.0548}{\ln k}\right).$$

Proof: We take $k_0 = 4118054814$ and $c = 0.006788$. We obtain $\beta = 2.0548$. Using table 2, we check $\theta(10^{11}) \geqslant f_{2.0548}(k_0)$. □

The major drawback of the above method is that we must check if $\theta(p_{k_0}) \geqslant f_\beta(k_0)$ which needs a computation of $\theta(x)$ for large x. The next method doesn't need this hypothesis.

Second method to find a lower bound for $\theta(p_k)$.

Proposition 4. *Let* $k_0 \geqslant \exp(\exp(3))$. *Assume that*

$$|\theta(x) - x| \leqslant c\frac{x}{\ln x} \qquad \text{for } x \geqslant p_{k_0}.$$

Then

$$\theta(p_k) \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k}\right) \text{ for } k \geqslant k_0$$

with β *solution of the following equality*

$$\begin{aligned}\beta &= \frac{\ln k_0(1+\alpha) - \ln_2 k_0 + 1}{\ln k_0 - 1}\\ \alpha &= \frac{1}{2}\frac{(\ln_2 k_0 - a)^2}{\ln k_0} + a\\ a &= 1 - \frac{\ln_2 k_0 - \beta}{\ln k_0} + c\end{aligned}$$

Proof: Applying lemma 1 of [9] for $a = 1$ and $\ln k_0 \geqslant \exp(a+2)$ in the same way than theorem 7 of [9], we find a first value of β written β_0. Since $|\theta(p_k) - p_k| \leqslant ck$, we can apply lemma 1 one more time with the new value of a: we choose

$$a = 1 - \frac{\ln_2 k_0 - \beta_0}{\ln k_0} + c.$$

In fact the series $\{\beta_k\}_k$ is a convergent one. We determine the limit by solving the equation of the fixed point for $k = k_0$

$$\beta = \frac{\ln k\left(1 + \frac{1}{2}\frac{(\ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} - c)^2}{\ln k} + 1 - \frac{\ln_2 k - \beta}{\ln k} + c\right) - \ln_2 k + 1}{\ln k - 1}.$$

When k_0 grows to infinity, β decreases to $2 + c$. □

Applying the previous proposition for $c = 0.007$, the computed values of β are:

$\ln k_0$	β
exp(3)	2.0675
25	2.056
30	2.04938
1000	2.01356
2000	2.0128

Corollary 3. *For* $k \geqslant \exp(30)$,

$$\theta(p_k) \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.05}{\ln k}\right).$$

Theorem 6.

$$p_k \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.25}{\ln k}\right) \qquad \text{for } k \geqslant 2.$$

Proof: Since $\theta(x) < x$ for $0 < x < 10^{11}$, we deduce immediately

$$p_k \geqslant \theta(p_k) \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1454}{\ln k}\right).$$

For $p_k \geqslant \exp(4000)$, it follows from theorem 1 that

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2\frac{x}{\ln^2 x}$$

with $\eta_2 = 0.040033$ for $x \geqslant \exp(4000)$. Thus, for $p_k \geqslant \exp(4000)$,

$$p_k \geqslant \theta(p_k) - \eta_2 \frac{p_k}{\ln^2 p_k} \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2.1 - 0.040033}{\ln k}\right).$$

For $10^{11} \leqslant p_k \leqslant \exp(4000)$,

$$\begin{aligned} p_k &\geqslant \theta(p_k)\left(\frac{1}{1+\varepsilon}\right) \geqslant (1-\varepsilon)\theta(p_k) \\ &\geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k} - \varepsilon\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k}\right)\right). \end{aligned}$$

with $\beta = 2.0548$. Let's study the map

$$f(\varepsilon, k) := -\beta - \varepsilon\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \beta}{\ln k}\right)\ln k.$$

We choose at the first time $\varepsilon = \varepsilon(10^{11}) = 0.00008$. We verify that $f(\varepsilon, \exp(30)) > -2.1319$ and so on, by small intervals up to $\exp(4000)$. We deduce that $f(\varepsilon, k) > -2.25$ for p_k between $\exp(22)$ and $\exp(4000)$. □

Theorem 7.

$$p_k \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k}\right) \qquad \textit{for } k \geqslant 27076.$$

Proof: Assume first that (see lemma 1)

$$\frac{p_k}{\ln^2 p_k} \leqslant \frac{k}{\ln k}.$$

For $x \geqslant 3594641$, we have establish that

$$|\theta(x) - x| < x\varepsilon(x) < \eta_2 \frac{x}{\ln^2 x}$$

with $\eta_2 = 0.2$. Thus, for $p_k \geqslant 3594641$,

$$p_k \leqslant \theta(p_k) + \eta_2 \frac{p_k}{\ln^2 p_k} \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2 + 0.2}{\ln k}\right).$$

We terminate the proof by a computer check. □

Theorem 8. *For $k \geqslant 39017$, we have*

$$p_k \leqslant k(\ln k + \ln_2 k - 0.9484).$$

Proof: As (see theorem 7)

$$p_k \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k}\right),$$

we have, for $x \geqslant \exp(33)$,

$$p_k \leqslant k(\ln k + \ln_2 k - 0.9484).$$

¿From

$$|\theta(p_k) - p_k| \leqslant \varepsilon(p_k)p_k,$$

we deduce that

$$p_k \leqslant \theta(p_k) + \varepsilon(p_k)p_k.$$

As

$$\theta(p_k) \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 2}{\ln k}\right),$$

it remains to show that

$$g(k) := -1 + \frac{\ln_2 k - 2}{\ln k} + \varepsilon(p_k)\frac{p_k}{k} < -0.9484$$

for $4118054813 \leqslant k \leqslant \exp(30)$. If we choose (cf. [6])

$$p_k \leqslant k(\ln k + \ln_2 k - 0.9427)$$

and $\varepsilon(p_k) = 0.000079989 + 1/\sqrt{10^{11}}$ for $10^{11} \leqslant p_k \leqslant \exp(30)$ and next $\varepsilon(p_k) = 10^{-5} + 1.43/\sqrt{\exp 30}$ for $p_k \geqslant \exp(30)$, we obtain the result for $p_k \geqslant 10^{11}$.

Now we study if the result is true for $p_k \leqslant 10^{11}$. Write $\alpha = 0.9484$ and

$$f(k) = k(\ln k + \ln_2 k - \alpha).$$

Thanks to Brent [1], we have the lower and the upper bounds of the difference of $\mathrm{Li}(x) - \pi(x)$ into various intervals up to 10^{11}. Hence if, for $k \in [k_0, k_1]$,

$$\mathrm{Li}(f(k)) - \pi(p_k) \geqslant M_{p_{k_0}, p_{k_1}} := \max_{k\in[k_0,k_1]} (\mathrm{Li}(p_k) - \pi(p_k))$$

we could deduce that

$$p_k \leqslant f(k).$$

We consider $g(k) = \mathrm{Li}(f(k)) - k$.

$$g'(k) = \frac{1 + 1/\ln k - \alpha - \ln(1 + (\ln_2 k - \alpha)/\ln k)}{\ln k + \ln_2 k + \ln(1 + (\ln_2 k - \alpha)/\ln k)}$$

admits a minimum for $\ln k \approx \exp(2+\alpha+2/\exp(2+\alpha))$. For $\alpha = 0.9484$, this lower bound is positive. We show, thanks to the following table, that

$$\mathrm{Li}(f(k)) - \pi(p_k) \geqslant M_{p_{k_0}, p_{k_1}}$$

for $170368 \leqslant k \leqslant 4118054813$ ($2312573 \leqslant p_k \leqslant 10^{11}$). The values of $\pi(x)$ are extracted from table 3 of Riesel [8] (Ex.: $\pi(10^7) = 664579$)

k	$p_k \leqslant \cdot$	$\mathrm{Li}(f(k)) - k$	$R_1 = M_{p_{k_0}, p_{k_1}}$
170368	$2.315 \cdot 10^6$	261.0004	$M_{2\cdot 10^6, 5\cdot 10^6}$=261
348512	$5 \cdot 10^6$	414.1091	$M_{5\cdot 10^6, 10^7}$=346
664578	10^7	634.5851	$M_{10^7, 2\cdot 10^7}$=435
1270606	$2 \cdot 10^7$	983.965	$M_{2\cdot 10^7, 5\cdot 10^7}$=692
3001133	$5 \cdot 10^7$	1788.864	$M_{5\cdot 10^7, 5\cdot 10^8}$=1724
26355866	$5 \cdot 10^8$	8890.45	$M_{5\cdot 10^7, 10^{10}}$=7048
455052510	10^{10}	93238.1	$M_{10^{10}, 10^{11}}$=17065

For $k = 39017..170367$, a check has been made thanks to `Pari` system. □

5. Interval which contains at least one prime

We already know the result of Schoenfeld [16] showing that, for $x \geqslant 2010759.9$, the interval $]x, x + x/16597[$ contains at least one prime. We improve this result with the following proposition. You can see also [7].

Proposition 5. *For $k \geqslant 463$,*

$$p_{k+1} \leqslant p_k \left(1 + \frac{1}{2\ln^2 p_k}\right).$$

Proof: Assume that, for $k \geqslant k_0$,

$$p_k \geqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_0}{\ln k}\right)$$

and

$$p_k \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_1}{\ln k}\right).$$

$$\begin{aligned}
p_k\left(1+\frac{\gamma}{\ln^2 p_k}\right) - p_{k+1} &= p_k - p_{k+1} + \gamma\frac{p_k}{\ln^2 p_k}\\
\geqslant\ & k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - \alpha_0}{\ln k}\right) + \gamma\frac{p_k}{\ln^2 p_k}\\
& -(k+1)\left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)}\right)\\
=\ & k\left(\ln k - \ln(k+1) + \ln_2 k - \ln_2(k+1) + \frac{\ln_2 k - \alpha_0}{\ln k} - \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)}\right)\\
& -\left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)}\right) + \gamma\frac{p_k}{\ln^2 p_k}
\end{aligned}$$

But,

$$\ln(k) - \ln(k+1) = -\ln(1+1/k),$$

$$\ln_2(k) - \ln_2(k+1) = -\ln\left(1+\frac{\ln(1+1/k)}{\ln k}\right),$$

$$\frac{\ln_2 k - \alpha_0}{\ln k} - \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)} \geqslant (\alpha_1 - \alpha_0)/\ln(k+1)$$

because $\frac{\ln x}{x}$ is decreasing for $x \geqslant 1/e$ hence $f(k) \geqslant f(k+1)$ where $f(x) := \frac{\ln_2 x}{\ln x}$. We choose γ such that

$$\begin{aligned}
\gamma\frac{p_k}{\ln^2 p_k} \quad \geqslant\ & k\left(\ln(1+1/k) + \ln\left(1+\frac{\ln(1+1/k)}{\ln k}\right) + (\alpha_0-\alpha_1)/\ln(k+1)\right)\\
& + \left(\ln(k+1) + \ln_2(k+1) - 1 + \frac{\ln_2(k+1) - \alpha_1}{\ln(k+1)}\right); \qquad (8)
\end{aligned}$$

We must choose $\gamma > \alpha_0 - \alpha_1$ to make the inequality true for large k. As

$$\frac{p_k}{\ln^2 p_k} \geqslant \frac{k\ln k}{(\ln k + \ln_2 k)^2},$$

the inequality (8) is satisfied with $\gamma = 1/2$ if

$$\ln(k+1)/2 \geqslant (\alpha_0 - \alpha_1)(\ln k + \ln_2 k).$$

For $\alpha_0 = 2.25$ and $\alpha_1 = 1.8$, this holds for $k \geqslant \exp(82)$. Theorem 12 of [16] showed that the interval $]x, x + x/16597[$ contains at least one prime for $x \geqslant 2010759.9$. Apply it for $x = p_k$, this yields the result for $2010759.9 \leqslant p_k \leqslant \exp(91)$. According to [16] p. 355,

$$p_{n+1} - p_n \leqslant 652 \text{ for } p_n \leqslant 2.686 \cdot 10^{12},$$

and so

$$p_{k+1} \leqslant p_k\left(1 + \frac{1}{2\ln^2 p_k}\right)$$

when

$$\frac{p_k}{2\ln^2 p_k} \geqslant \frac{k}{2(\ln k + \ln_2 k)} \geqslant 652$$

that is to say for $k \geqslant 2 \cdot 10^4$. For $k = 463..20000$, a direct check with computer conclude the proof. □

Theorem 9. *For all $x \geqslant 3275$, there exists a prime p such that*

$$x < p \leqslant x\left(1 + \frac{1}{2\ln^2 x}\right).$$

This result is better that ROSSER & SCHOENFELD's one for $x \geqslant 3 \cdot 10^{39}$.
Proof: Let $x \geqslant 2$. There exists $k \in \mathbb{N}^*$ such that

$$p_k \leqslant x < p_{k+1}.$$

As the map $x \mapsto x(1 + 1/(2\ln^2 x))$ is increasing,

$$p_k \leqslant x \Rightarrow p_k\left(1 + \frac{1}{2\ln^2 p_k}\right) \leqslant x\left(1 + \frac{1}{2\ln^2 x}\right).$$

According to proposition 5, for $k \geqslant 463$,

$$p_{k+1} \leqslant p_k\left(1 + \frac{1}{2\ln^2 p_k}\right).$$

We deduce that, for $x \geqslant p_{463} = 3299$,

$$x < p_{k+1} \leqslant x\left(1 + \frac{1}{2\ln^2 x}\right).$$

Moreover, for $x \geqslant 3274.0111$,

$$3299 < x\left(1 + \frac{1}{2\ln^2 x}\right).$$

□

6. Results on $\pi(x)$

Remember that

$$\pi(x) = \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{2}{\ln^2 x} + O(\frac{1}{\ln^3 x})\right).$$

Theorem 10.
1. - $\frac{x}{\ln x}\left(1 + \frac{1}{\ln x}\right) \leqslant \pi(x)$ *for* $x \geqslant 599$.
2. - $\pi(x) \leqslant \frac{x}{\ln x}\left(1 + \frac{1.2762}{\ln x}\right)$ *for* $x > 1$ *(the value 1.2762 is chosen for* $x = p_{258} = 1627$*).*
3. - $\pi(x) \leqslant \frac{x}{\ln x}\left(1 + \frac{1.0992}{\ln x}\right)$ *for* $x \geqslant 1.332 \cdot 10^{10}$.
4. - $\pi(x) \leqslant \frac{x}{\ln x - 1.1}$ *for* $x \geqslant 60184$.
5. - $\pi(x) \geqslant \frac{x}{\ln x - 1}$ *for* $x \geqslant 5393$.
6. - $\frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\right) \leqslant \pi(x)$ *for* $x \geqslant 32299$.
7. - $\pi(x) \leqslant \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{2.51}{\ln^2 x}\right)$ *for* $x \geqslant 355991$.

Let's start with the first three inequalities.

Lower bound of $\pi(x)$. • We began with $x \geqslant 10^8$.

$$\pi(x) = \pi(p) - \frac{\theta(p)}{\ln(p)} + \frac{\theta(x)}{\ln(x)} + \int_p^x \frac{\theta(y)dy}{y\ln^2(y)}.$$

But $\theta(x) \geqslant x - 0.024\frac{x}{\ln(x)}$ for $x \geqslant 758711$ ([16], Th.7 p.357). Hence, for $p \geqslant 758711$, this yields

$$\pi(x) \geqslant \frac{\theta(x)}{\ln(x)} + \frac{x}{\ln^2(x)} + \pi(p) - \frac{\theta(p)}{\ln(p)} - \frac{p}{\ln^2(p)} + (2-0.024)\int_p^x \frac{dy}{\ln^3(y)}.$$

Since $\theta(x) \geqslant x - 0.0077629\frac{x}{\ln(x)}$ for $x \geqslant 1.04 \cdot 10^7$,

$$\pi(x) \geqslant \frac{x}{\ln x}\left(1 + \frac{1-0.008}{\ln(x)}\right) \text{ for } x \geqslant 10^8$$

because

$$\pi(p) - \frac{\theta(p)}{\ln(p)} - \frac{p}{\ln^2(p)} + (2-0.024)\int_p^x \frac{dy}{\ln^3(y)} \geqslant 0 \text{ for } x \geqslant 10^8,$$

by choosing

$$p = p_{61000} = 760267 \text{ and } \theta(p) < p.$$

• According to [13](Th 16, p. 72), we have for $11 < x \leqslant 10^8$,

$$\mathrm{Li}(x) - \mathrm{Li}(\sqrt{x}) < \pi(x).$$

For $1859 \leqslant x \leqslant 10^8$,

$$\mathrm{Li}(x) - \mathrm{Li}(\sqrt{x}) \geqslant \frac{x}{\ln x}\left(1 + \frac{0.992}{\ln x}\right).$$

For $k = 1..284$ ($p_{284} = 1861$), we check by computer that

$$\pi(p_k - 1/2) = k - 1 \geqslant \frac{p_k}{\ln p_k}\left(1 + \frac{0.992}{\ln p_k}\right),$$

inequality true for $k \geqslant 110$. Hence $\pi(x) \geqslant \frac{x}{\ln x}\left(1 + \frac{0.992}{\ln x}\right)$ for $x \geqslant p_{109} = 599$.

We will obtain a better result with an upper bound for θ in order 2 in $\ln x$.

Upper bound for $\pi(x)$. According to [16] (p. 360), we know that :

$$\theta(x) < x \qquad \text{for } 0 < x \leqslant 10^{11},$$

$$\theta(x) < 1.000081x \qquad \text{for } x > 0,$$

and that

$$|\theta(x) - x| \leqslant 0.0077926\frac{x}{\ln x} \qquad \text{for } x \geqslant 1.04 \cdot 10^7.$$

Let $b = 0.0077926$, $c = 1.000081$ and $K = 10^{11}$.

$$\begin{aligned}
\pi(x) &= \frac{\theta(x)}{\ln x} + \int_2^x \frac{\theta(y)dy}{y\ln^2 y} \\
&= \frac{\theta(x)}{\ln x} + \int_2^K \frac{\theta(y)dy}{y\ln^2 y} + \int_K^x \frac{\theta(y)dy}{y\ln^2 y} \\
&< \frac{x}{\ln x}\left(1+\frac{b}{\ln x}\right) + \int_2^K \frac{dy}{\ln^2 y} + \int_K^x \frac{dy}{\ln^2 y} + b\int_K^x \frac{dy}{\ln^3 y} \text{ for } x \geqslant K \\
&= \frac{x}{\ln x}\left(1+\frac{b}{\ln x}\right) + \left[\frac{y}{\ln^2 y}\right]_2^x + 2\int_2^x \frac{dy}{\ln^3 y} + b\int_K^x \frac{dy}{\ln^3 y} \\
&< \frac{x}{\ln x}\left(1+\frac{b+1}{\ln x}\right) + 2\frac{x}{\ln^3 x} + 6\int_2^x \frac{dy}{\ln^4 y} + b\left[\frac{y}{\ln^3 y}\right]_K^x + 3b\int_K^x \frac{dy}{\ln^4 y}
\end{aligned}$$

But

$$\int_K^x \frac{dy}{\ln^4 y} = \int_K^{\sqrt{x}} \frac{dy}{\ln^4 y} + \int_{\sqrt{x}}^x \frac{dy}{\ln^4 y} < (\sqrt{x}-K)/\ln^4(K) + \frac{x-\sqrt{x}}{\ln^4\sqrt{x}} \text{ if } \sqrt{x} \geqslant K.$$

To have

$$\pi(x) \leqslant \frac{x}{\ln x}\left(1+\frac{\beta}{\ln x}\right)$$

we choose

$$\beta > b+1+\frac{2+b}{\ln x}+\frac{\ln^2 x}{x}\left(6\int_2^K \frac{dy}{\ln^4 x} - b\frac{K}{\ln^3 K} + (6+3b)\left(\frac{\sqrt{x}-K}{\ln^4 K} + \frac{x-\sqrt{x}}{\ln^4\sqrt{x}}\right)\right).$$

We obtain $\beta \geqslant 1.03$ when $x \geqslant \exp(100)$.

$$\begin{aligned}
\pi(x) &= \pi(K) - \frac{\theta(K)}{\ln K} + \frac{\theta(x)}{\ln x} + \int_K^x \frac{\theta(y)dy}{y\ln^2 y} \\
&< \pi(K) - \frac{\theta(K)}{\ln K} + c\frac{x}{\ln x} + c\int_K^x \frac{dy}{\ln^2 y} \\
&= \pi(K) - \frac{\theta(K)}{\ln K} + c\frac{K}{\ln K} + c\,(\mathrm{Li}(x) - \mathrm{Li}(K)) \\
&= M + c\cdot \mathrm{Li}(x) \qquad \text{where } M \text{ is a constant.}
\end{aligned}$$

Write

$$\Delta(x) = \frac{x}{\ln x}\left(1+\frac{\beta}{\ln x}\right) - (M + c\cdot\mathrm{Li}(x)).$$

$$\Delta'(x) = \big((1-c)\ln^2(x) + (\beta-1)\ln(x) - 2\beta\big)/\ln^3 x.$$

Δ' equals zero when

$$\ln x = \frac{-(\beta-1) \pm \sqrt{(\beta-1)^2 + 8\beta(1-c)}}{2(1-c)}.$$

We can take $\pi(K) = 4118054813$ and $\frac{\theta(K)}{\ln K} > \frac{K}{\ln K}\left(1 - \frac{0.007}{\ln K}\right)$ hence, for $\beta = 1.0992$, the local minimum $x_0 \approx \exp(22.5775)$ ($x_0 < K$) is negative but $\Delta(K)$ is positive. Hence, for $x \in [K, \exp(100)]$,

$$\pi(x) < \frac{x}{\ln x}\left(1+\frac{1.0992}{\ln x}\right).$$

By consulting table given in [1], we show that the result remains true for $x \geqslant 1.332 \cdot 10^{10}$. According to [13] (Th. 16 p. 72) and [1], we have for $x \leqslant K$,

$$\pi(x) < \mathrm{Li}(x).$$

Let's study the difference

$$\Delta(x,\beta) = \frac{x}{\ln x}\left(1+\frac{\beta}{\ln x}\right) - \mathrm{Li}(x).$$

This function admits a minimum at point $x = \exp(2\beta/(\beta-1))$. For $\beta_0 := 1.276103273$, the value of this minimum equals to -9.972985. We consult the table of Brent [1]. This yields the lower difference between $\pi(x)$ and $\mathrm{Li}(x)$ in different intervals $[10^n, 10^{n+1}]$ with $n = 0..10$. We deduce that

$$\frac{x}{\ln x}\left(1+\frac{\beta_0}{\ln x}\right) - \pi(x) = \frac{x}{\ln x}\left(1+\frac{\beta_0}{\ln x}\right) - \mathrm{Li}(x) + (\mathrm{Li}(x) - \pi(x)) > 0$$

for $x \geqslant 10^4$. For $k = 1..1230$, we verify that

$$\pi(p_k) = k \leqslant \frac{p_k}{\ln p_k}\left(1+\frac{\beta_0}{\ln p_k}\right).$$

We choose the value for β_0 such that it holds for all integers k, including $k = 258$.

Next we consider the others formulas. Let

$$x_0 = 1.04 \cdot 10^7, \quad K = \pi(x_0) - \frac{\theta(x_0)}{\ln x_0}.$$

Numerically $\pi(x_0) = 689382$ and $\theta(x_0) = 10395445.63690637\cdots$ Write

$$J(x;a) = K + \frac{x}{\ln x} + a\frac{x}{\ln^3 x} + \int_{x_0}^{x}\left(\frac{1}{\ln^2 y} + a\frac{1}{\ln^4 y}\right)dy$$

Since

$$\pi(x) = \pi(x_0) - \frac{\theta(x_0)}{\ln x_0} + \frac{\theta(x)}{\ln x} + \int_{x_0}^{x}\frac{\theta(y)dy}{y\ln^2 y}$$

and $|\theta(x) - x| \leqslant 0.2\frac{x}{\ln^2 x}$ for $x > x_0$, we have, for $x \geqslant x_0$,

$$J(x;-0.2) \leqslant \pi(x) \leqslant J(x;0.2).$$

Write $M(x;c) = \frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{c}{\ln^2 x}\right)$ for upper bound's function for $\pi(x)$. Let's write the derivatives of $J(x;a)$ and of $M(x;c)$ with respect to x:

$$J'(x;a) = \frac{1}{\ln x} + \frac{a}{\ln^3 x} - 2\frac{a}{\ln^4 x},$$

$$M'(x;c) = \frac{1}{\ln x} + \frac{c-2}{\ln^3 x} - \frac{3c}{\ln^4 x}.$$

We must choose $c > 2 + a$ to have $J' < M'$ when $\ln x > (3c-2a)/(c-2-a)$. Choose $c = 2.51$. We verify by computer that $J(10^{11};0.2) < M(10^{11};2.51)$. Since $\mathrm{Li}(x) < M(x;2.51)$ for $x \geqslant 10^7$, we verify by direct computation for small values of x to obtain

$$\pi(x) < \frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{2.51}{\ln^2 x}\right) \qquad \text{for } x \geqslant 355991.$$

Now write

$$m(x;d) = \frac{x}{\ln x}\left(1+\frac{1}{\ln x}+\frac{d}{\ln^2 x}\right).$$

We study the derivatives: we must choose $d \leqslant 2 - a$ to have $J' > m'$.

Choose $d = 1.8$. As $m(x_0; 1.8) < J(x_0; -0.2)$ and by direct computation for small values, we obtain

$$\pi(x) > \frac{x}{\ln x}\left(1 + \frac{1}{\ln x} + \frac{1.8}{\ln^2 x}\right) \qquad \text{for } x \geqslant 32299.$$

To show that

$$\frac{x}{\ln x - b} \leqslant \pi(x),$$

we proceed in the same way. By comparing with the derivative of $J(x; a)$, we show that we must choose $b \leqslant 1$. For $b = 1$ and $a = -0.2$, $J'(x; -0.2)$ is greater that the derivative of $x/(\ln x - 1)$ for $x > 1$. Moreover, for $x = x_0$,

$$\frac{x_0}{\ln x_0 - 1} < J(x_0; -0.2).$$

We verify by computer that, for $5393 \leqslant x \leqslant 1.04 \cdot 10^7$,

$$\frac{x}{\ln x - 1} \leqslant \pi(x).$$

Now we show that we have, for $x \geqslant 60184$,

$$\pi(x) \leqslant \frac{x}{\ln x - 1.1}.$$

Since

$$\frac{x}{\ln x}\left(1 + \frac{a}{\ln x}\right) < \frac{x}{\ln x - a}$$

for $x > \exp(a)$, an above result shows the result for $x \geqslant 1.332 \cdot 10^{10}$. Now we study if $\pi(x) \leqslant \frac{x}{\ln x - 1.1}$ for $x \leqslant 10^{11}$. Write $k(x) = \frac{x}{\ln x - 1.1}$ and compare $k(x)$ with $\mathrm{Li}(x)$. The derivative of $k'(x) - 1/\ln x$ is

$$-\frac{200 \ln^2 x - 3630 \ln x + 1331}{(10 \ln x - 11)^3 x \ln^2 x}.$$

So we deduce that the difference $k'(x) - 1/\ln x$ is positive for x belonging to $[10^6, 10^{11}]$. Since $\mathrm{Li}(10^6) < k(10^6)$, we deduce that, for $10^6 \leqslant x \leqslant 10^{11}$,

$$\mathrm{Li}(x) < k(x).$$

A computer check concludes the demonstration up to 10^6.

We show that

$$\frac{x}{\ln x}\left(1 + \frac{1}{\ln x}\right) \leqslant \pi(x) \qquad \text{for } x \geqslant 599$$

by using inequality 6 of theorem 10. Next we verify by computer the lower bound is true from $x = 599$.

7. Applications

7.1. **Other inequalities.** Let γ be Euler's constant ($\gamma \approx 0.5772156649$).

Theorem 11. *Let* $B = \gamma + \sum_p (\ln(1 - 1/p) + 1/p) \approx 0.26149\ 72128\ 47643$. *For* $x > 1$,

$$\sum_{p \leqslant x} \frac{1}{p} - \ln_2 x - B \geqslant -\left(\frac{1}{10 \ln^2 x} + \frac{4}{15 \ln^3 x}\right).$$

For $x \geqslant 10372$,

$$\sum_{p \leqslant x} \frac{1}{p} - \ln_2 x - B \leqslant \frac{1}{10 \ln^2 x} + \frac{4}{15 \ln^3 x}.$$

Proof: By (4.20) of [13],

$$\sum_{p\leqslant x}\frac{1}{p}=\ln_2 x+B+\frac{\theta(x)-x}{x\ln x}-\int_x^{\infty}\frac{(\theta(y)-y)(1+\ln y)}{y^2\ln^2 y}dy.$$

Hence

$$|\sum_{p\leqslant x}\frac{1}{p}-\ln_2 x-B|\leqslant\frac{|\theta(x)-x|}{x\ln x}+\int_x^{\infty}\frac{|\theta(y)-y|(1+\ln y)}{y^2\ln^2 y}dy.$$

As $|\theta(x)-x|\leqslant 0.2x/\ln^2 x$ (Theorem 4) and

$$\int_x^{\infty}\frac{1+\ln y}{y\ln^4 y}dy=\frac{1}{2\ln^2 x}+\frac{1}{3\ln^3 x},$$

we have the result for $x\geqslant 3594641$. We conclude by computer's check. □

Theorem 12. *Let* $E=-\gamma-\sum_{n=2}^{\infty}\sum_p(\ln p)/p^n\approx -1.33258\ 22757\ 33221$. *For* $x>1$,

$$\sum_{p\leqslant x}\frac{\ln p}{p}-\ln x-E\geqslant-\left(\frac{0.2}{\ln x}+\frac{0.2}{\ln^2 x}\right).$$

For $x\geqslant 2974$,

$$\sum_{p\leqslant x}\frac{\ln p}{p}-\ln x-E\leqslant\frac{0.2}{\ln x}+\frac{0.2}{\ln^2 x}.$$

Proof: By (4.21) of [13],

$$\sum_{p\leqslant x}\frac{\ln p}{p}=\ln x+E+\frac{\theta(x)-x}{x}-\int_x^{\infty}\frac{\theta(y)-y}{y^2}dy.$$

Hence

$$|\sum_{p\leqslant x}\frac{\ln p}{p}-\ln x-E|\leqslant\frac{|\theta(x)-x|}{x}-\int_x^{\infty}\frac{|\theta(y)-y|}{y^2}dy.$$

As

$$\int_x^{\infty}\frac{dy}{y\ln^2 y}=\frac{1}{\ln x},$$

theorem 4 yields the result for $x\geqslant 3594641$. We conclude by computer's check. □

Theorem 13. *For* $x>1$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)<\frac{e^{-\gamma}}{\ln x}\left(1+\frac{0.2}{\ln^2 x}\right)$$

and

$$\prod_{p\leqslant x}\frac{\ln p}{p}>e^{\gamma}\ln x\left(1-\frac{0.2}{\ln^2 x}\right).$$

For $x\geqslant 2973$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right)>\frac{e^{-\gamma}}{\ln x}\left(1-\frac{0.2}{\ln^2 x}\right)$$

and

$$\prod_{p\leqslant x}\frac{\ln p}{p}<e^{\gamma}\ln x\left(1+\frac{0.2}{\ln^2 x}\right).$$

Proof: By theorem 11 and by definition of B, we have

$$-(\frac{0.1}{\ln^2 x}+\frac{4}{15\ln^3 x}) \leqslant -\gamma - \ln_2 x - \sum_{p>x}\frac{1}{p} - \sum_p \ln(1-1/p) \leqslant \frac{0.1}{\ln^2 x}+\frac{4}{15\ln^3 x}.$$

Let $S=\sum_{p>x}(\ln(1-1/p)+1/p) = -\sum_{n=2}^{\infty}\sum_{p>x}\frac{1}{p^n}$. We have

$$-\gamma-\ln_2 x-\sum_{p\leqslant x}\ln(1-1/p)-S \geqslant -\frac{0.1}{\ln^2 x}-\frac{4}{15\ln^3 x}.$$

Take the exponential of both sides to obtain

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) \leqslant \frac{e^{-\gamma}}{\ln x}\exp(-S+\frac{0.1}{\ln^2 x}+\frac{4}{15\ln^3 x}).$$

We use lower bound for S given in [13] p. 87:

$$S > \frac{-1.02}{(x-1)\ln x}.$$

Hence, for $x \geqslant 100$,

$$\prod_{p\leqslant x}\left(1-\frac{1}{p}\right) \leqslant \frac{e^{-\gamma}}{\ln x}\exp(0.11/\ln^2 x).$$

We have also

$$\prod_{p\leqslant x}\frac{p-1}{p} \geqslant e^{\gamma}\ln x\exp(-0.11/\ln^2 x).$$

In the same way, as

$$-\gamma-\ln_2 x-\sum_{p\leqslant x}\ln(1-1/p)-S \leqslant \frac{0.1}{\ln^2 x}+\frac{4}{15\ln^3 x},$$

we obtain the others inequalities since $S \leqslant 0$. □

Theorem 14. *For $x > 1$,*

$$\pi(2x)-\pi(x) \leqslant \frac{x}{\ln x}$$

For $x \geqslant 1328.5$,

$$\pi(2x)-\pi(x) \geqslant \frac{x}{\ln x}-\frac{0.7x}{\ln^2 x}.$$

Remark: As $\pi(x) > x/\ln x$ for $x \geqslant 17$ (corollary 1 of [13]), we have $\pi(2x) < 2\pi(x)$ for real $x \geqslant 3$.

Proof: By theorem 10, we have for $x \geqslant 60184$,

$$\frac{2x}{\ln 2x-1}-\frac{x}{\ln x-1.1} < \pi(2x)-\pi(x) < \frac{2x}{\ln 2x-1.1}-\frac{x}{\ln x-1}.$$

Let $f(x)=\frac{2x}{\ln 2x-1}-\frac{x}{\ln x-1.1}-\frac{x}{\ln x}$. As $1+u < \frac{1}{1-u} < 1+u+2u^2$ for $|u| \leqslant 1/2$,

$$f(x) \geqslant \frac{x}{\ln^2 x}\left(2(1-\ln 2)-1.1-2\frac{1.1^2}{\ln x}\right).$$

In a same way, if we write

$$g(x)=\frac{2x}{\ln 2x-1.1}-\frac{x}{\ln x-1}-\frac{x}{\ln x},$$

we have

$$g(x) \leqslant \frac{x}{\ln^2 x}\left(2(1.1-\ln 2) - 1 - 4\frac{(1.1-\ln 2)^2}{\ln x}\right).$$

□

7.2. **Mandl's conjecture.** We want to show the conjecture of Robert Mandl written in the article of Rosser & Schoenfeld [14] p. 243.

Theorem 15. *Let*

$$S_n = \sum_{k=1}^{n} p_k.$$

Then,

$$S_n \geqslant \frac{n^2}{2}(\ln n + \ln_2 n - 3/2) \quad \textit{for } n \geqslant 305494, \tag{9}$$

$$S_n \leqslant \frac{n^2}{2}(\ln n + \ln_2 n - 1.463) \quad \textit{for } n \geqslant 779. \tag{10}$$

Proof: Write

$$\begin{aligned} s(n) &= \frac{n^2}{2}(\ln n + \ln_2 n - 3/2) \\ f(n) &= s(n) - n\ln n \end{aligned}$$

Hence

$$\begin{aligned} f'(n) &= n\left(\ln n + \ln_2 n - 1 + \frac{1}{2\ln n}\right) - \ln n - 1 \\ f''(n) &= \ln n + \ln_2 n + \frac{3}{2\ln n} - \frac{1}{\ln^2 n} - \frac{1}{n} \end{aligned}$$

We take the Taylor series expansion of f between n and $n+1$:

$$f(n+1) - f(n) = f'(n) + \frac{f''(n_1)}{2} \quad \text{with } n < n_1 < n+1.$$

Since f'' increases,we obtain

$$f(n+1) - f(n) \leqslant f'(n) + \frac{f''(n+1)}{2} \leqslant f'(n) + \ln n.$$

By theorem 6,

$$p_n \geqslant n\left(\ln n + \ln_2 n - 1 + \frac{\ln\ln n - 2.25}{\ln n}\right),$$

this yields $n \geqslant \exp\exp(2.75)$,

$$p_n \geqslant f(n+1) - f(n).$$

For $n_0 = 10^6$,

$$S_{n_0-1} \geqslant f(n_0).$$

Suppose that $S_{n-1} \geqslant f(n)$ up to n. Then

$$S_n = S_{n-1} + p_n \geqslant f(n) + p_n \geqslant f(n+1).$$

Thus, for all $n \geqslant n_0$,

$$S_{n-1} \geqslant f(n)$$

which implies $S_n \geqslant s(n)$ since $p_n \geqslant n\ln n$. We verify that

$$S_n \geqslant s(n) \quad \text{for } n \geqslant 305494$$

by a computer check.

Inequality (10) can be find in [6]. □

We have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} p_k &= \frac{n^2}{2}\left(\ln n + \ln_2 n - \frac{3}{2} + o(1)\right)\\ \frac{1}{2}p_n &= \frac{n^2}{2}\left(\ln n + \ln_2 n - 1 + o(1)\right)\end{aligned}$$

Theorem 16 (Mandl)**.** *For $n \geqslant 9$, we have*

$$(p_1 + p_2 + \cdots + p_n)/n < \frac{1}{2}p_n.$$

Proof: By (10), we have for $n \geqslant 779$,

$$\frac{S_n}{n} \leqslant \frac{n}{2}\left(\ln n + \ln_2 n - 1.463\right).$$

Now, theorem 5 yields the result. □

G. Robin gave me a conjecture concerning a lower bound of the above quantity:

Proposition 6. *For $n \geqslant 2$, we have*

$$p_{\left[\frac{n}{2}\right]} \leqslant \frac{1}{n}\sum_{i=1}^{n} p_i.$$

We have

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n} p_k &= \frac{n^2}{2}\left(\ln n + \ln_2 n - \frac{3}{2} + o(1)\right)\\ np_{[n/2]} &= \frac{n^2}{2}\left(\ln n + \ln_2 n - 1 - \ln 2 + o(1)\right)\end{aligned}$$

Proof: According to theorem 7, we have for $k \geqslant 27076$,

$$p_k \leqslant k\left(\ln k + \ln_2 k - 1 + \frac{\ln_2 k - 1.8}{\ln k}\right).$$

So, for $n \geqslant 54152$,

$$np_{n/2} \leqslant \frac{n^2}{2}\left(\ln n + \ln_2 n - 1 - \ln 2 + \frac{\ln_2 n - 1.8}{\ln n - \ln 2}\right) \leqslant S_n.$$

We make a direct computer check to show that

$$np_{n/2} \leqslant S_n \quad \text{pour } k \leqslant 305494.$$

□

Proposition 7. *For $a \geqslant 2$ and $b \geqslant 2$, or for $b = 1$ and $a \geqslant 5$, we have*

$$p_{ab} > ap_b.$$

For a and b positive integers, we have

$$\frac{ap_{b+1} + bp_{a+1}}{2} < p_{ab+2}.$$

If a and b are integers greater than 91, we have

$$p_{ab} \leqslant ap_b + bp_a.$$

b	m	δ	ε
20	2	0.000128125	0.0006302
21	2	0.0000775	0.000419768506
22	2	0.000049375	0.000278652
23	2	13/400000	0.0001843645
24	2	17/800000	0.000121611962
25	2	0.0000128125	0.00007998895869
30	3	$0.94375 * 10^{-6}$	$0.9778040657 * 10^{-5}$
50	18	$0.947265625 * 10^{-8}$	$0.9049928595 * 10^{-7}$
100	18	$0.9305664063 * 10^{-8}$	$0.8842626429 * 10^{-7}$
200	17	$0.95078125 * 10^{-8}$	$0.8561316979 * 10^{-7}$
400	16	$0.9411132813 * 10^{-8}$	$0.8000089705 * 10^{-7}$
600	15	$0.9296875 * 10^{-8}$	$0.7442047763 * 10^{-7}$
1000	13	$0.905078125 * 10^{-8}$	$0.6337118668 * 10^{-7}$
1300	11	$0.919140625 * 10^{-8}$	$0.5518819789 * 10^{-7}$
1500	10	$0.905078125 * 10^{-8}$	$0.4980115883 * 10^{-7}$
1800	9	$0.83828125 * 10^{-8}$	$0.41913371 * 10^{-7}$
2000	8	$0.8171875 * 10^{-8}$	$0.3674711889 * 10^{-7}$
2300	6	$0.83125 * 10^{-8}$	$0.2917036 * 10^{-7}$
2500	5	$0.8171875 * 10^{-8}$	$0.243946 * 10^{-7}$
2750	4	$0.746875 * 10^{-8}$	$0.18769435073 * 10^{-7}$
3000	3	$0.690625 * 10^{-8}$	$0.137602 * 10^{-7}$
3500	2	$0.409375 * 10^{-8}$	$0.61653 * 10^{-8}$
4000	2	$0.1562500000 * 10^{-8}$	$0.2405714403 * 10^{-8}$

TABLE 1. $|\psi(x) - x| \leqslant \varepsilon x$ for $x \geqslant \exp(b)$

For a and b positive integers, we have

$$p_{ab+2} \leqslant a p_{b+1} + b p_{a+1}.$$

Proof: Use the inequalities (3) and (4). □

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x	$\theta(x) \geqslant \cdot$	$\theta(x) \leqslant \cdot$
10^5	99624.9	99779.3
$2 \cdot 10^5$	199445.0	199629.8
$5 \cdot 10^5$	499216.3	499468.0
10^6	998357.3	998668.6
$2 \cdot 10^6$	1998421.0	1998812.1
$5 \cdot 10^6$	4998331.1	4998889.9
10^7	9994872.2	9995589.7
$2 \cdot 10^7$	19995425.3	19996352.2
$5 \cdot 10^7$	49993085.2	49994491.2
10^8	99986933.8	99988722.5
$2 \cdot 10^8$	199981158.2	199983578.4
$5 \cdot 10^8$	499982199.5	499985664.2
10^9	999966797.9	999971596.9
$2 \cdot 10^9$	1999938007.6	1999944598.4
$5 \cdot 10^9$	4999901997.2	4999911609.0
10^{10}	9999933799.8	9999946821.9
$2 \cdot 10^10$	19999813136.0	19999831825.3
$5 \cdot 10^10$	49999714959.0	49999744514.5
$8 \cdot 10^10$	79999702170.2	79999737244.0
10^{11}	99999720459.8	99999757299.3

TABLE 2. Bounds for $\theta(x)$.

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