



Laboratoire d'Arithmétique, de Calcul formel et d'Optimisation
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Estimates of $\theta(x;k,l)$ for large values of x

Pierre Dusart

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Université de Limoges, 123 avenue Albert Thomas, 87060 Limoges Cedex
Tél. 05 55 45 73 23 - Fax. 05 55 45 73 22 - laco@unilim.fr

<http://www.unilim.fr/laco/>

ESTIMATES OF $\theta(x; k, l)$ FOR LARGE VALUES OF x .

PIERRE DUSART

ABSTRACT. We extend a result of RAMARÉ & RUMELY, 1996 [3] about CHEBYSHEV function θ in arithmetic progressions. We find a map $\varepsilon(x)$ such that $|\theta(x; k, l) - x/\varphi(k)| < x\varepsilon(x)$ and $\varepsilon(x) = O(\frac{1}{\ln^a x})$ ($\forall a > 0$) whereas $\varepsilon(x)$ is a constant in [3]. Now we are able to show that

$$|\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}$$

and, for $x \geq 151$,

$$\pi(x; 3, l) > \frac{x}{2 \ln x}.$$

1. INTRODUCTION

Let $R = 9.645908801$ and $X = \sqrt{\frac{\ln x}{R}}$. ROSSER [6] and SCHOENFELD [7] (Th. 11 p. 342) showed that, for $x \geq 101$,

$$|\theta(x) - x|, |\psi(x) - x| < x\varepsilon(x)$$

where

$$\varepsilon(x) = \sqrt{\frac{8}{17\pi}} X^{1/2} \exp(-X).$$

We adapt their work to the case of arithmetic progressions. Let us recall the usual notations for nonnegative real x ,

$$\theta(x; k, l) = \sum_{\substack{p \equiv l \pmod{k} \\ p \leq x}} \ln p, \quad \text{where } p \text{ is a prime number}$$

$$\psi(x; k, l) = \sum_{\substack{n \equiv l \pmod{k} \\ n \leq x}} \Lambda(n), \quad \text{where } \Lambda \text{ is Von Mangoldt's function}$$

and φ is Euler's function. We show, for $x \geq x_0(k)$ where $x_0(k)$ can be easily computed, that

$$|\theta(x; k, l) - x/\varphi(k)|, |\psi(x; k, l) - x/\varphi(k)| < x\varepsilon(x)$$

where

$$\varepsilon(x) = 3 \sqrt{\frac{k}{\varphi(k)C}} X^{1/2} \exp(-X).$$

We apply the above results for $k = 3$. For small values, we use RAMARÉ & RUMELY's results. We show that for $x > 1$,

$$(1) \quad |\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}.$$

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If we assume that the Generalized Riemann Hypothesis is true then we can show that, for $x \geq 10^{10}$ and $k \leq 432$,

$$|\psi(x; k, l) - x/\varphi(k)| < \frac{12}{33\pi} \sqrt{x} \ln^2 x.$$

Let define, as usual, $\pi(x)$ the number of primes not greater than x . In 1962, ROSSER & SCHOENFELD ([5] p. 69) found a lower bound for $\pi(x)$:

$$(2) \quad \pi(x) > \frac{x}{\ln x} \quad \text{for } x \geq 17.$$

Letting

$$\pi(x; k, l) = \sum_{p \leq x, p \equiv l \pmod{k}} 1,$$

we show an analogous result in the case of arithmetic progression with $k = 3$ and $l = 1$ or 2 ,

$$\pi(x; 3, l) > \frac{x}{2 \ln x} \quad \text{for } x \geq 151.$$

This result, deduced from (1), implies (2) and can't be shown with RAMARÉ & RUMELY's results.

The method used for $k = 3$ can also be applied for other fixed integer k .

2. INTRODUCING LEMMAS

Notations: we will always denote by ρ a non trivial zero of the Dirichlet's function L , that is to say a zero such that $0 < \Re \rho < 1$. We write $\rho = \beta + i\gamma$. Let $\wp(\chi)$ be the set of the zeros ρ of the function $L(s, \chi)$, with $0 < \beta < 1$.

For a positive real H , following RAMARÉ & RUMELY, we say that GRH(k, H) holds¹ if, for all χ modulo k , all the non trivial zeros of $L(s, \chi)$ with $|\gamma| \leq H$ are such that $\beta = 1/2$.

As in ROSSER & SCHOENFELD (in [6, 7] where the case $k = 1$ is studied), we must know the distribution of $L(s, \chi)$'s zeros; namely find a real H such that GRH(k, H) is satisfied and a zero-free region.

2.1. Zero-free region.

Theorem 1 (Ramaré & Rumely [3]). *If χ is a character with conductor k , $H \geq 1000$ and $\rho = \beta + i\gamma$ is a zero of $L(s, \chi)$ with $|\gamma| \geq H$ then there exists a computable constant $C_1(\chi, H)$ such that*

$$1 - \beta \geq \frac{1}{R \ln(k|\gamma|/C_1(\chi, H))}.$$

Examples:

k	H_k	$C_1(\chi, H_k)$
1	545000000	38.31
3	10000	20.92
420	2500	56.59

Proof: See Definition and (3.6.6) of Ramaré & Rumely [3]. □

Remark: For $k \geq 1$ and $H_k \geq 1000$, $C_1(\chi, H) \geq C_1(\chi_0, 1000) \geq 9.14$.

From now on,

$$C_1(k) = \min_{\chi \pmod{k}} C_1(\chi, H_\chi).$$

¹Note that our GRH is a restriction of the usual Generalized Riemann Hypothesis.

2.2. GRH(k, H) and $N(T, \chi)$.

Lemma 1 (McCurley [1]). *Let $C_2 = 0.9185$ and $C_3 = 5.512$. Write $F(y, \chi) = \frac{y}{\pi} \ln \left(\frac{ky}{2\pi e} \right)$ and $R(y, \chi) = C_2 \ln(ky) + C_3$. If χ is a character of Dirichlet with conductor k , if $T \geq 1$ is a real number and if $N(T, \chi)$ denotes the number of zeros $\beta + i\gamma$ of $L(s, \chi)$ in the rectangle $0 < \beta < 1$, $|\gamma| \leq T$, then*

$$|N(T, \chi) - F(T, \chi)| \leq R(T, \chi).$$

Lemma 2 (deduced from Rumely [3], Theorem 2.1.1 p. 399).

- GRH(k, H) is true for $H = 10000$ and $k \leq 13$.
- GRH($k, 2500$) is true for sets
 - $E_1 = \{k \leq 72\}$,
 - $E_2 = \{k \leq 112, k \text{ not prime}\}$,
 - $E_3 = \{116, 117, 120, 121, 124, 125, 128, 132, 140, 143, 144, 156, 163, 169, 180, 216, 243, 256, 360, 420, 432\}$.

2.3. Estimates of $|\psi(x; k, l) - x/\varphi(k)|$ using properties of zeros of $L(s, \chi)$.

Theorem 2 (McCurley [1]). *Let $x \geq 1$ be a real number, m and k two positive integers, δ a real number such that $0 < \delta < \frac{x-2}{mx}$ and H a positive real.*

Let

$$(3) \quad A(m, \delta) = \frac{1}{\delta^m} \sum_{j=0}^m \binom{m}{j} (1 + j\delta)^{m+1}.$$

Assume GRH($k, 1$). Then

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} \left| \psi(y; k, l) - \frac{y}{\varphi(k)} \right| &< A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ &+ \left(1 + \frac{m\delta}{2} \right) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x \end{aligned}$$

where \sum_{χ} denotes the summation over all characters modulo k , $\tilde{R} = \varphi(k)[(f(k) + 0, 5) \ln x + 4 \ln k + 13, 4]$ and $f(k) = \sum_{p|k} \frac{1}{p-1}$.

2.4. One more explicit form of estimates. The next lemma can be found in [3] with the difference that authors assumed GRH(k, H) but they used in fact only GRH($k, 1$). Since we must apply it with $T > H$, we rewrite the proof.

Lemma 3. *Let χ be a character modulo k . Assume GRH($k, 1$). Then, for some $T \geq 1$, we have*

$$\sum_{\substack{|\gamma| \leq T \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \tilde{E}(T)$$

with $\tilde{E}(T) = \frac{1}{2\pi} \ln^2(T) + \frac{\ln(\frac{k}{2\pi e})}{\pi} \ln(T) + C_2 + 2 \left(\frac{1}{\pi} \ln \left(\frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right)$.

Proof: For $|\gamma| \leq 1$, we have GRH($k, 1$) and so

$$\sum_{\substack{|\gamma| \leq 1 \\ \rho \in \varphi(\chi)}} \frac{1}{|\rho|} \leq \sum_{\substack{|\gamma| \leq 1 \\ \rho \in \varphi(\chi)}} \frac{1}{|1/2 + i\gamma|} \leq 2N(1, \chi).$$

For $|\gamma| > 1$,

$$\sum_{\substack{1 < |\gamma| \leq T \\ \rho \in \wp(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{dN(t, \chi)}{t} = \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} - \frac{N(1, \chi)}{1}.$$

Thus

$$\sum_{\substack{|\gamma| \leq T \\ \rho \in \wp(\chi)}} \frac{1}{|\rho|} \leq \int_1^T \frac{N(t, \chi)}{t^2} dt + \frac{N(T, \chi)}{T} + N(1, \chi).$$

We conclude by lemma 1:

$$\begin{aligned} \int_1^T \frac{N(t, \chi)}{t^2} dt &\leq \int_1^T \frac{F(t, \chi) + R(t, \chi)}{t^2} dt \\ &= \frac{1}{\pi} \int_1^T \frac{\ln(kt/(2\pi e))}{t} dt + C_2 \int_1^T \frac{\ln(kt)}{t^2} dt + C_3 \int_1^T \frac{1}{t^2} dt \\ &= \frac{1}{\pi} \left[\frac{1}{2} \ln^2 \left(\frac{kT}{2\pi e} \right) \right]_1^T \\ &\quad + C_2 \left\{ \left[-\frac{\ln(kt)}{t} \right]_1^T + \int_1^T \frac{1}{t^2} dt \right\} + C_3 [-1/t]_1^T \\ &= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln \left(\frac{k}{2\pi e} \right) \ln T + C_2 \left(-\frac{\ln(kT)}{T} + \ln k - \frac{1}{T} + 1 \right) \\ &\quad + C_3(1 - 1/T) \end{aligned}$$

In the same way, we have an upper bound of

$$\frac{N(T, \chi)}{T} \text{ with } \frac{F(T, \chi) + R(T, \chi)}{T}$$

and

$$N(1, \chi) \text{ with } F(1, \chi) + R(1, \chi).$$

Finally, we obtain

$$\sum_{\substack{|\gamma| \leq T \\ \rho \in \wp(\chi)}} \frac{1}{|\rho|} \leq \frac{1}{2\pi} \ln^2(T) + \frac{\ln \left(\frac{k}{2\pi e} \right)}{\pi} \ln(T) + C_2 + 2 \left(\frac{1}{\pi} \ln \left(\frac{k}{2\pi} \right) + C_2 \ln k + C_3 \right) - \frac{C_2}{T}.$$

□

Using the facts that:

- if ρ is a zero of $L(s, \chi)$ then $\bar{\rho}$ is zero of $L(s, \chi)$ too,
 - these zeros are symmetrical with to the line $\Re(z) = 1/2$,
- we obtain lemma 4 taken directly from [3].

Lemma 4 ([3]). *Let*

$$(4) \quad \phi_m(t) = \frac{1}{t^{m+1}} \exp \left(\frac{-\ln x}{R \ln(kt/C_1(k))} \right)$$

with $R = 9.645908801$.

$$\sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \frac{x^\beta}{|\gamma|^{m+1}} + \sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\bar{\chi})}} \frac{x^\beta}{|\gamma|^{m+1}} \leq x \sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \phi_m(\gamma) + \sqrt{x} \sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\bar{\chi})}} \frac{1}{|\gamma|^{m+1}}.$$

Let us rewrite lemma 7 of [6] to adapt it at the new functions $F(y, \chi)$ and $R(y, \chi)$ which we use.

Lemma 5. Write $N(y) = N(y, \chi)$, $F(y) = F(y, \chi)$ and $R(y) = R(y, \chi)$. Let $1 < U \leq V$ and $\phi(y)$ a positive and differentiable function for $U < y < V$. Let $(W - y)\phi'(y) \geq 0$ for $U < y < V$, where W not necessarily belong to $[U, V]$. Let Y one of the numbers U, V, W which is not the greatest and not the lowest. Choose $j = 0$ or 1 depending on $(-1)^j(V - W) \geq 0$. Then

$$\sum_{U < |\gamma| \leq V} \phi(\gamma) \leq \frac{1}{\pi} \int_U^V \phi(y) \ln \left(\frac{ky}{2\pi} \right) dy + (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy + B_j(Y, U, V)$$

where $B_j(Y, U, V) = \{1 + (-1)^j\}R(Y)\phi(Y) + \{N(V) - F(V) - (-1)^j R(V)\}\phi(V) - \{N(U) - F(U) + R(U)\}\phi(U)$.

Proof: We have

$$\sum_{U < |\gamma| \leq V} \phi(\gamma) = - \int_U^V N(y)\phi'(y)dy + N(V)\phi(V) - N(U)\phi(U).$$

- $j = 1$. We have $W > V$ and so $Y = \min(V, W) = V$. According to theorem 1, $N(y) \geq F(y) - R(y)$.

$$\begin{aligned} \sum_{U < |\gamma| \leq V} \phi(\gamma) &\leq [(N(y) - F(y) + R(y))\phi(y)]_U^V + \frac{1}{\pi} \int_U^V \ln \left(\frac{ky}{2\pi} \right) \phi(y) dy \\ &\quad - \int_U^V R'(y)\phi(y) dy \end{aligned}$$

because $F'(y) = \ln \left(\frac{ky}{2\pi e} \right) + 1 = \ln \left(\frac{ky}{2\pi} \right)$. Moreover,

$$- \int_U^V R'(y)\phi(y) dy = -C_2 \int_U^V \frac{\phi(y)}{y} dy.$$

- $j = 0$. We have $V > W$. Take $Y = \max(U, W)$. Split the integral in Y . This way, $-\phi'(y) \leq 0$ for $y \in [U, Y]$ and $-\phi'(y) \geq 0$ for $y \in [Y, V]$. Replacing $N(y)$ by $F(y) - R(y)$ in the first part and by $F(y) + R(y)$ in the second part, we obtain

$$\begin{aligned} \sum_{U < |\gamma| \leq V} \phi(\gamma) &\leq \frac{1}{\pi} \int_U^V \ln \left(\frac{ky}{2\pi} \right) \phi(y) dy + \int_Y^V R'(y)\phi(y) dy - \int_U^Y R'(y)\phi(y) dy \\ &\quad + B_0(Y, U, V). \end{aligned}$$

Moreover,

$$\int_Y^V R'(y)\phi(y) dy \leq (-1)^j C_2 \int_U^V \frac{\phi(y)}{y} dy$$

and

$$- \int_U^Y R'(y)\phi(y) dy \leq 0.$$

□

Theorem 3. Let $k \geq 1$ an integer, $H \geq 1000$ a real number. Assume GRH(k, H). Let x_0 a real, m a positive integer and $\delta > 0$ a real number such that $0 < \delta <$

$(x_0 - 2)/(mx_0)$ and Y defined as in lemma 5. We write

$$(5) \quad \tilde{A} = \frac{1}{\pi} \int_H^\infty \phi_m(y) \ln\left(\frac{ky}{2\pi}\right) dy + C_2 \int_H^\infty \frac{\phi_m(y)}{y} dy$$

$$(6) \quad \tilde{B}_H = B_0(Y, H, \infty)$$

$$(7) \quad \tilde{C}_H = \frac{1}{m\pi H^m} \left(\ln\left(\frac{kH}{2\pi}\right) + 1/m \right)$$

$$(8) \quad \tilde{D}_H = \left(2C_2 \ln(kH) + 2C_3 + \frac{C_2}{m+1} \right) / H^{m+1}$$

Then for all $x \geq x_0$, we have

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(x; k, l) - \frac{y}{\varphi(k)}| &\leq A(m, \delta) \frac{\varphi(k)}{2} \left(\tilde{A} + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H) / \sqrt{x} \right) \\ &\quad + \left(1 + \frac{m\delta}{2} \right) \varphi(k) \tilde{E}(H) / \sqrt{x} + \frac{m\delta}{2} + \tilde{R}/x. \end{aligned}$$

Proof: According to theorem 2:

$$\begin{aligned} \frac{\varphi(k)}{x} \max_{1 \leq y \leq x} |\psi(y; k, l) - \frac{y}{\varphi(k)}| &< A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \wp(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ &\quad + \left(1 + \frac{m\delta}{2} \right) \sum_{\chi} \sum_{\substack{\rho \in \wp(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \frac{m\delta}{2} + \tilde{R}/x \end{aligned}$$

We treat separately the different parts:

- By lemma 4,

$$\sum_{\substack{\rho \in \wp(\chi) \\ |\gamma| > H}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \leq \frac{x}{2} \sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \phi_m(\gamma) + \frac{\sqrt{x}}{2} \sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \frac{1}{|\gamma|^{m+1}}.$$

Let, for $m \geq 0$,

$$(9) \quad W_m = \frac{C_1(k)}{k} \exp(X/\sqrt{m+1}).$$

Using lemma 5 with $U = H$, $V = \infty$ and $W = W_m$,

$$\sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \phi_m(\gamma) \leq \tilde{A} + \tilde{B}_H.$$

Integration by parts gives,

$$\sum_{\substack{|\gamma| \geq H \\ \rho \in \wp(\chi)}} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_H + \tilde{D}_H.$$

- By lemma 2,

$$\sum_{\substack{\rho \in \wp(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} < \tilde{E}(H) / \sqrt{x}.$$

□

2.5. **Leading term: (\tilde{A}).** To have an upper bound of the leading term, we proceed as ROSSER & SCHOENFELD with upper bounds on the integrals. The three next lemmas are taken directly from [6].

Lemma 6 (Functions like Bessel's ones). *Let*

$$K_\nu(z, u) = \frac{1}{2} \int_u^\infty t^{\nu-1} H^z(t) dt$$

where $z > 0, u \geq 0$ and

$$H^z(t) = \{H(t)\}^z = \exp\left\{-\frac{z}{2}(t + 1/t)\right\}.$$

Moreover we have that $K_\nu(z, 0) = K_\nu(z)$. Remember that

$$(10) \quad K_1(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{3}{8z}\right)$$

$$(11) \quad K_2(z) \leq \sqrt{\frac{\pi}{2z}} \exp(-z) \left(1 + \frac{15}{8z} + \frac{105}{128z^2}\right)$$

Lemma 7.

$$K_\nu(z, x) + K_{-\nu}(z, x) = K_\nu(z).$$

Hence $K_\nu(z, x) \leq K_\nu(z)$ ($\nu \geq 0$).

Lemma 8. *Let*

$$Q_\nu(z, x) = x^{\nu+1} \exp\{-z(t + 1/t)/2\}.$$

If $z > 0$ and $x > 1$ then

$$K_1(z, x) < Q_1(z, x)$$

and

$$K_2(z, x) < (x + 2/z)Q_1(z, x).$$

The leading term can be expressed with Bessel's functions. Let $X = \sqrt{\frac{\ln x}{R}}$, $z_m = 2X\sqrt{m} = 2\sqrt{\frac{m \ln x}{R}}$ and $U_m = \frac{2m}{z_m} \ln\left(\frac{kH}{C_1(k)}\right) = \sqrt{\frac{Rm}{\ln x}} \ln\left(\frac{kH}{C_1(k)}\right)$.

Lemma 9.

$$\begin{aligned} \tilde{A} &= \frac{2 \ln x}{\pi Rm} \left(\frac{k}{C_1(k)}\right)^m K_2(z_m, U_m) \\ &+ \frac{2}{\pi} \ln\left(\frac{C_1(k)}{2\pi}\right) \sqrt{\frac{\ln x}{Rm}} \left(\frac{k}{C_1(k)}\right)^m K_1(z_m, U_m) \\ &+ 2C_2 \sqrt{\frac{\ln x}{R(m+1)}} \left(\frac{k}{C_1(k)}\right)^{m+1} K_1(z_{m+1}, U_{m+1}). \end{aligned}$$

Proof: This is only straightforward algebraic manipulations; for example, we write

$$I = \int_H^\infty \frac{C_2}{y^{m+1}} \exp\left(\frac{-\ln x}{R \ln(ky/C_1(k))}\right) \frac{dy}{y}.$$

Changing variables:

$$\begin{aligned} t &= \sqrt{\frac{R(m+1)}{\ln x}} \ln\left(\frac{ky}{C_1(k)}\right) \\ dt &= \sqrt{\frac{R(m+1)}{\ln x}} \frac{dy}{y} \end{aligned}$$

Now

$$\begin{aligned} \exp\left(\frac{-\ln x}{R \ln(ky/C_1(k))}\right) &= \exp\left(\frac{-\ln x}{Rt/\sqrt{\frac{R(m+1)}{\ln x}}}\right) \\ &= \exp\left(\sqrt{\frac{(m+1)\ln x}{R}} \frac{1}{t}\right) = \exp\left(\frac{-z_{m+1}}{2} \frac{1}{t}\right) \end{aligned}$$

and

$$\frac{1}{y^{m+1}} = \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(-\frac{(m+1)t}{\sqrt{\frac{R(m+1)}{\ln x}}}\right) = \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(-t \frac{z_{m+1}}{2}\right).$$

Hence

$$I = \int_{U_{m+1}}^{\infty} C_2 \sqrt{\frac{\ln x}{R(m+1)}} \left(\frac{k}{C_1(k)}\right)^{m+1} \exp\left(\frac{-z_{m+1}}{2}(t+1/t)\right).$$

□

2.6. Study of $f(k)$ which appears in the expression of \tilde{R} . Remember that $f(k) = \sum_{p|k} \frac{1}{p-1}$.

Lemma 10. *For an integer $k \geq 1$,*

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

Proof: We prove by recursion that

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

For $k = 1$, it is obvious. For $k = 2$, $f(k) = 1 \leq \frac{\ln 2}{\ln 2}$. Assume $f(k) \leq \frac{\ln k}{\ln 2}$ holds for $k \leq n$.

Find an upper bound for $f(n+1)$. If $(n+1)$ is prime, then $f(n+1) = 1/n \leq \ln n / \ln 2$. If $(n+1)$ is not prime, then there exists $p \leq n$, which divides n .

If $p = 2$ and $2^\alpha \parallel n+1$,

$$\begin{aligned} f(n+1) &= f\left(\frac{n+1}{2^\alpha} \cdot 2^\alpha\right) = f\left(\frac{n+1}{2^\alpha}\right) + f(2) \\ &= 1 + f\left(\frac{n+1}{2^\alpha}\right) \leq \frac{\ln(n+1)}{\ln 2} + 1 - \frac{\ln 2}{\ln 2} \\ &\leq \frac{\ln(n+1)}{\ln 2}. \end{aligned}$$

If $p > 2$ and $p^\alpha \parallel n+1$,

$$\begin{aligned} f(n+1) &= f\left(\frac{n+1}{p^\alpha} \cdot p^\alpha\right) = f\left(\frac{n+1}{p^\alpha}\right) + f(p) \\ &= \frac{1}{p-1} + f\left(\frac{n+1}{p^\alpha}\right) \leq \ln(n+1) + \frac{1}{p-1} - \ln p \\ &\leq \frac{\ln(n+1)}{\ln 2} \quad \text{because } \frac{1}{p-1} - \ln p < 0 \text{ for } p > 2. \end{aligned}$$

□

3. METHOD WITH $m = 1$.

Theorem 4. Let $X = \sqrt{\frac{\ln x}{R}}$, $H \geq 1000$, an integer $k \leq H$ and

$$\varepsilon = 2\sqrt{\frac{k\varphi(k)}{C_1(k)\sqrt{\pi}}} \left(1 + \frac{1}{2X}(15/16 + \ln(C_1(k)/(2\pi)))\right) X^{3/4} \exp(-X).$$

Assume $GRH(k, H)$. If $\varepsilon \leq 0.588$ and $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$ then

$$|\psi(x; k, l) - x/\varphi(k)| \leq \varepsilon x/\varphi(k).$$

Proof: Choose $m = 1$. If $W_1 \leq H$ then $Y = H$ and $\tilde{B}_H = (F(H) + R(H) - N(H))\phi_1(H)$. If $W_1 > H$ then $Y = W_1$. For $y > e^e$, $R(y)/\ln y$ is decreasing.

$$\begin{aligned} \tilde{B}_H &< 2R(Y)\phi_1(Y) < 2\frac{R(H)}{\ln H}\phi_1(W_1)\ln W_1 \\ &= 2\frac{R(H)}{\ln H} \left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right) \phi_1(W_1) \\ &= 2\frac{R(H)}{\ln H} \left(\frac{X}{\sqrt{2}} + \ln\left(\frac{C_1(k)}{k}\right)\right) (k/C_1(k))^2 \exp(-2\sqrt{2}X) \end{aligned}$$

From the two values of \tilde{B}_H , we need only to consider the above one assuming that $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$.

Using the upper bounds (10) and (11) in lemma 9

$$\begin{aligned} \tilde{A} &< 2\left(\frac{k}{C_1(k)}\right) \left[\sqrt{\frac{\pi}{4X}} \exp(-2X) \left(1 + \frac{15}{16X} + \frac{105}{512X^2}\right) X^2/\pi \right. \\ &\quad \left. + \frac{1}{\pi} \ln \frac{C_1(k)}{2\pi} X \sqrt{\frac{\pi}{4X}} \exp(-2X) \left(1 + \frac{3}{16X}\right) \right. \\ &\quad \left. + C_2 \frac{kX}{C_1(k)\sqrt{2}} \sqrt{\frac{\pi}{4\sqrt{2}X}} \exp(-2\sqrt{2}X) \left(1 + \frac{3}{16\sqrt{2}X}\right) \right] \\ &< \frac{1}{\sqrt{\pi}} \frac{k}{C_1(k)} X^{3/2} \exp(-2X) \left[1 + \left(\frac{15}{16} + \ln \frac{C_1(k)}{2\pi}\right) \frac{1}{2X}\right]^2 = F_1 \end{aligned}$$

Observe that (see next lemma)

$$\tilde{A} + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R} \frac{2}{x\varphi(k)} < F_1.$$

We must choose δ to minimize

$$\frac{A(1, \delta)}{2} \varphi(k) F_1 + \delta/2.$$

Write $f = \varphi(k)F_1$. We have a second degree equation which has one positive solution $\delta = \frac{f + \sqrt{2f}}{1-f}$; This yields $\varepsilon(x) < \delta$. It is a simple matter to prove that for $0 \leq f < 0.588$,

$$\frac{f + \sqrt{2f}}{1-f} < 2\sqrt{f}.$$

It is obvious that δ satisfies hypothesis since

$$0 < \delta < 0.54 < x/(x-2).$$

□

Lemma 11.

$$\tilde{A} + \tilde{B}_H + (\tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H))/\sqrt{x} + \tilde{R} \frac{2}{x\varphi(k)} < F_1.$$

Proof: First prove that $\tilde{A} + \tilde{B}_H < F_1$.

$$\begin{aligned} F_1 &= \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \left(1 + (15/16 + \ln(C_1(k)/2\pi))/X \right. \\ &\quad \left. + (225/1024 + \frac{15}{32} \ln(C_1(k)/2\pi) + \frac{1}{4} \ln^2(C_1(k)/2\pi))/X^2 \right) \\ \tilde{A} &< \frac{k}{C_1(k)\sqrt{\pi}} X^{3/2} e^{-2X} \left(1 + \frac{15}{16X} + \frac{105}{512X^2} + \ln\left(\frac{C_1(k)}{2\pi}\right) \left(\frac{1}{X} + \frac{3}{16X^2}\right) \right. \\ &\quad \left. + C_2 \frac{k\pi}{C_1(k)\sqrt{2\sqrt{2}}} \exp(-2(\sqrt{2}-1)X) (1/X + 3/(16\sqrt{2}X^2)) \right) \\ \tilde{B}_H &< \frac{k}{\sqrt{\pi}C_1(k)} X^{3/2} \exp(-2X) \exp(-2(\sqrt{2}+1)X) \\ &\quad \times \left[\frac{2k\sqrt{\pi}}{C_1(k)\ln H} (C_2 \ln(kH) + C_3) \left(\frac{1}{\sqrt{2X}} + \frac{1}{X\sqrt{X}} \ln(C_1(k)/k) \right) \right] \end{aligned}$$

This yields $F_1 - \tilde{A} - \tilde{B}_H > 0$ if

$$\begin{aligned} &\frac{1}{X^2} \left(\frac{15}{1024} + \frac{9}{32} \ln\left(\frac{C_1(k)}{2\pi}\right) + \frac{1}{4} \ln^2\left(\frac{C_1(k)}{2\pi}\right) \right) \\ &> \frac{C_2\sqrt{\pi}k}{C_1(k)} \exp(-2(\sqrt{2}-1)X) \frac{1}{\sqrt{2X}} \\ &\quad \times \left[\sqrt{\frac{\pi}{2\sqrt{2}}} \left(\sqrt{\frac{2}{X}} + \frac{3}{16X^{3/2}} \right) + 2 \left(1 + \frac{\ln k + C_2/C_3}{\ln H} \right) \left(1 + \frac{\sqrt{2}}{X} \ln \frac{C_1(k)}{k} \right) \right] \\ &> \frac{C_2k\sqrt{\pi}}{C_1(k)} \exp(-2(\sqrt{2}-1)X) \frac{1}{\sqrt{2X}} \cdot 18.2 \end{aligned}$$

since $C_1(k) \leq 32\pi$, $X \geq \sqrt{2} \ln(1000/32\pi)$ and $k \leq H$.

It remains to prove that

$$\frac{C_1(k)}{kC_2\sqrt{\pi} \cdot 18.2} (15/1024 + \dots) > X^{3/2} \exp(-2(\sqrt{2}-1)X).$$

But for $X \geq \sqrt{2} \ln\left(\frac{kH}{C_1(k)}\right)$,

$$\begin{aligned} X^{3/2} \exp(-2(\sqrt{2}-1)X) &< X^{3/2} \left(\frac{kH}{C_1(k)} \right)^{-3/2} \\ &= \frac{1}{k} \cdot \frac{1}{\sqrt{k}} \left(\frac{\sqrt{2}C_1(k) \ln(kH/C_1(k))}{H} \right)^{3/2} \end{aligned}$$

The map $k \mapsto \frac{1}{\sqrt{k}} \left(\frac{\sqrt{2}C_1(k) \ln(kH/C_1(k))}{H} \right)^{3/2}$ reaches its maximum for $k = \frac{e^3 C_1(k)}{H}$.
Hence

$$X^{3/2} \exp(-2(\sqrt{2}-1)X) < \frac{C_1(k)}{kH} (3\sqrt{2})^{3/2} / e^{3/2}.$$

We must compare

$$\frac{1}{C_2\sqrt{\pi} \cdot 18.2} (15/1024 + \dots) \text{ with } \frac{(3\sqrt{2})^{3/2}}{He^{3/2}}.$$

Since $C_1(k) \geq 9.14$ and $C_2 = 0.9185$, it remains to prove that

$$0.1843 > \frac{(3\sqrt{2})^{3/2}}{He^{3/2}} (\approx 0.002)$$

which is true since $H \geq 1000$.

We show below that the rest $(\tilde{C}_H + \tilde{D}_H)/\sqrt{x} + 3\tilde{E}(H)/\sqrt{x} + \tilde{R} \frac{2}{x\varphi(k)}$ is negligible.

• We will find an upper bound for $A(1, \delta) \frac{\varphi(k)}{2} (\tilde{C}_H + \tilde{D}_H) + \frac{3}{2} \varphi(k) \frac{\tilde{E}(H)}{\sqrt{x}} + \tilde{R}/x$. We assume that $X \geq \sqrt{2} \ln \left(\frac{kH}{C_1(k)} \right)$ and $C_1(k) \leq 32\pi$ hence $X \geq \sqrt{2} \ln \left(\frac{1000}{32\pi} \right) \approx 3.24885$. We have

$$\text{Rest} := \tilde{C}_H + \tilde{D}_H + 3\tilde{E}(H) + \tilde{R}/\sqrt{x} \leq \begin{cases} (\ln H \ln k)^2 & \text{if } k \neq 1 \\ (\ln H)^2 & \text{if } k = 1 \end{cases}$$

Let us treat the case $k \neq 1$. As $X \geq \sqrt{2} \ln \left(\frac{kH}{C_1(k)} \right)$,

$$\exp \left(\frac{X}{\sqrt{2}} \right) \geq \frac{kH}{C_1(k)}.$$

This yields

$$\begin{aligned} \text{Rest} \leq (\ln H \ln k)^2 &\leq \frac{(\ln H \ln k)^2}{\left(\frac{kH}{C_1(k)} \right)^2} \exp(X\sqrt{2}) \\ &\leq C_1^2(k) \frac{1}{e^2} \left(\frac{\ln 1000}{1000} \right)^2 \exp(X\sqrt{2}) \\ &\leq 0.0653 \exp(X\sqrt{2}) \quad \text{because } C_1(k) \leq 32\pi. \end{aligned}$$

Fix $K = 0.0653$. Now compare

$$\varphi(k) \frac{K \exp(X\sqrt{2})}{\sqrt{x}} = \varphi(k) K \exp(X\sqrt{2} - RX^2/2)$$

with

$$\sqrt{\frac{k\varphi(k)}{C_1(k)\sqrt{\pi}}} X^{3/4} \exp(-X).$$

First remark that $\sqrt{\frac{k}{\varphi(k)C_1(k)\sqrt{\pi}}} \geq \sqrt{\frac{1}{32\pi\sqrt{\pi}}}$. We may compute c such that

$$\begin{aligned} 0.0653 \exp(X\sqrt{2} - RX^2/2) &\leq c \sqrt{\frac{1}{32\pi\sqrt{\pi}}} X^{3/4} \exp(-X) \\ \Leftrightarrow c &\geq K \sqrt{32\pi\sqrt{\pi}} \exp(X\sqrt{2} - RX^2/2 + X) X^{-3/4} \\ \Leftrightarrow c &\geq 7.1521 \cdot 10^{-20} \end{aligned}$$

The rest is negligible and absorbed by rounds. \square

4. METHOD WITH $m = 2$.

Corollary 1 (Corollary from lemma 5). *Under the hypothesis of lemma 5, if moreover $\frac{2\pi}{k} < U$ then*

$$\sum_{U < \gamma \leq V} \phi(\gamma) \leq \{1/\pi + (-1)^j q(Y)\} \int_U^V \phi(y) \ln(y/2\pi) dy + B_j(Y, U, V)$$

where $q(y) = (-1)^j \frac{C_2}{y \ln(\frac{ky}{2\pi})}$.

Proof: $(-1)^j \frac{R'(y)}{\ln(\frac{ky}{2\pi})} \leq (-1)^j \frac{C_2}{y \ln(\frac{ky}{2\pi})}$ if $\frac{ky}{2\pi} > 1$. □

Lemma 12. *Let $A(m, \delta)$ defined with formula (3). Write*

$$R_m(\delta) = (1 + (1 + \delta)^{m+1})^m.$$

$$\text{Then } A(m, \delta) \leq \frac{R_m(\delta)}{\delta^m}.$$

Proof: The proof appears in [4] p. 222. □

Theorem 5. *Let an integer $k \geq 1$. Remember that $R = 9.645908801$. Let $H \geq 1000$. Assume $GRH(k, H)$. Let $C_1(k)$ defined as in theorem 1. Let X_0, X_1, X_2 et X_3 such that*

$$\frac{e^{X_0}}{\sqrt{X_0}} = H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}, \quad \frac{e^{X_1}}{X_1} = 10\varphi(k),$$

$$X_2 = kC_1(k)/(2\pi\varphi(k)), \quad X_3 = \frac{2k\pi e}{C_1(k)\varphi(k)}.$$

Let $C = \min(C_1(k), 32\pi)$ and $X_4 = \max(10, X_0, X_1, X_2, X_3)$.

Write

$$\varepsilon(X) = 3\sqrt{\frac{k}{\varphi(k)C}} X^{1/2} \exp(-X).$$

then for all real x such that $X = \sqrt{\frac{\ln x}{R}} \geq X_4$ we have

$$|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < \varepsilon \left(\sqrt{\frac{\ln x}{R}} \right) x.$$

Corollary 2. *With the notations of theorem 5, let $X_5 \geq X_4$ and $c := \varepsilon(X_5)$. For $x \geq \exp(RX_5^2)$, we have*

$$|\psi(x; k, l) - x/\varphi(k)|, \quad |\theta(x; k, l) - x/\varphi(k)| < cx.$$

Proof: The idea: fix $m = 2$ and split judiciously the integral in 2 parts with optimal bounds into each part.

Write T in the same form as W_m (formula (9)). Let $T = \frac{C_1(k)}{k} \exp(\nu X)$. Assume that $T \geq H$ and $1/\sqrt{m+1} \leq \nu \leq 1$. Hence $W_m \leq T \leq W_0$. We use the theorem 2

and we split the sums in T :

$$\begin{aligned}
& A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} + \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} \\
&= \left(1 + \frac{m\delta}{2}\right) \sum_{\chi} \left(\sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \sum_{\substack{\rho \in \varphi(\chi) \\ H < |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} \right) \\
&\quad + A(m, \delta) \sum_{\chi} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|} \\
&\leq \left(1 + \frac{m\delta}{2}\right) \varphi(k) \tilde{A}_1 + A(m, \delta) \varphi(k) \tilde{A}_2.
\end{aligned}$$

first part

$$\begin{aligned}
\tilde{A}_1 &= \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} = \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{x^{\beta-1}}{|\rho|} + \sum_{\substack{\rho \in \varphi(\chi) \\ H < |\gamma| \leq T}} \frac{x^{\beta-1}}{|\rho|} \\
&\leq \frac{1}{\sqrt{x}} \sum_{\substack{\rho \in \varphi(\chi) \\ |\gamma| \leq H}} \frac{1}{|\rho|} + \frac{1}{2x} \left(x \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma) + \sqrt{x} \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \frac{1}{|\gamma|} \right) \\
&\leq \tilde{E}(H)/\sqrt{x} + \frac{1}{2} \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \frac{1}{|\gamma|/\sqrt{x}} + \frac{1}{2} \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma) \\
&\leq \tilde{E}(T)/\sqrt{x} + \frac{1}{2} \sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma).
\end{aligned}$$

Apply corollary 1 ($j = 1, m = 0$)

$$\sum_{\substack{\rho \in \varphi(\chi) \\ H \leq |\gamma| \leq T}} \phi_0(\gamma) = \{1/\pi - q(Y)\} \int_H^T \phi_0(y) \ln(y/2\pi) dy + B_1(T, H, T).$$

Moreover, $B_1(T, H, T) < 2R(T)\phi_0(T)$.

We want to find an upper bound of

$$I_1 = \left(\frac{1}{\pi} - q(Y)\right) \int_H^T \phi_0(y) \ln\left(\frac{ky}{2\pi}\right) dy.$$

Write $V'' = X^2 / \ln\left(\frac{kT}{C_1(k)}\right) = X/\nu = Y - 2X - \nu X$ where $Y = X(1 - \nu)^2/\nu$.

Write $U'' = X^2 / \ln\left(\frac{kH}{C_1(k)}\right)$ and $\Gamma(\nu, x) = \int_x^\infty e^{-u} u^{\nu-1} du$. Now

$$\begin{aligned}
\int_U^V \ln\left(\frac{ky}{2\pi}\right) \phi_0(y) dy &= \int_U^V \ln\left(\frac{ky}{2\pi}\right) \exp\left(-X^2 / \ln\left(\frac{ky}{C_1(k)}\right)\right) \frac{dy}{y} \\
&= X^4 \{\Gamma(-2, V'') - \Gamma(-2, U'')\} \\
&\quad + X^2 \ln\left(\frac{C_1(k)}{2\pi}\right) \{\Gamma(-1, V'') - \Gamma(-1, U'')\}
\end{aligned}$$

by writing $y = \frac{C_1(k)}{k} \exp(X^2/y)$. Now if $\nu \leq 1$ and $x > 0$ then $\Gamma(\nu, x) \leq x^{\nu-1} \int_x^\infty e^{-t} dt = x^{\nu-1} e^{-x}$. Hence

$$\int_U^V \ln\left(\frac{ky}{2\pi}\right) \phi_0(y) dy \leq X^4 V^{\nu-3} e^{-V''} + X^2 \ln\left(\frac{C_1(k)}{2\pi}\right) V^{\nu-2} e^{-V''}.$$

This yields

$$\begin{aligned} I_1 &\leq (1/\pi + q(Y)) X^2 (X^2 V^{\nu-3} + \ln\left(\frac{C_1(k)}{2\pi}\right) V^{\nu-2}) e^{-V''} \\ &\leq (1/\pi - q(T)) e^{-Y} e^{-2X} \left(\frac{kT}{C_1(k)}\right) \left(\frac{X^4}{(X/\nu)^3} + \frac{dX^2}{(X/\nu)^2}\right) \\ &= (1/\pi - q(T)) e^{-Y} e^{-2X} \left(\frac{kT}{C_1(k)}\right) X G_0 \\ &\leq (1/\pi + q(T)) e^{-Y} e^{-2X} \left(\frac{kT}{C_1(k)}\right) X G_0 \end{aligned}$$

where $d = \ln\left(\frac{C_1(k)}{2\pi}\right)$ and $G_0 = \nu^2(\nu + d/X)$. With the help of corollary 1, we write

$$\tilde{A}_1 \leq \tilde{E}(T)/\sqrt{x} + \frac{1}{2} \left\{ (1/\pi + q(T)) e^{-Y} e^{-2X} \left(\frac{kT}{C_1(k)}\right) X G_0 + 2R(T)\phi_0(T) \right\}.$$

2nd part: Let

$$\tilde{A}_2 = \sum_{\substack{\rho \in \wp(x) \\ |\gamma| > T}} \frac{x^{\beta-1}}{|\rho(\rho+1) \cdots (\rho+m)|}.$$

By lemma 4,

$$\tilde{A}_2 \leq \frac{x}{2} \sum_{\substack{\rho \in \wp(x) \\ |\gamma| > T}} \phi_m(\gamma) + \frac{\sqrt{x}}{2} \sum_{\substack{\rho \in \wp(x) \\ |\gamma| > T}} \frac{1}{|\gamma|^{m+1}}.$$

By using corollary 1 ($j = 0$)

$$\sum_{\substack{\rho \in \wp(x) \\ |\gamma| > T}} \phi_m(\gamma) \leq \{1/\pi + q(Y)\} \int_T^\infty \phi_m(y) \ln\left(\frac{ky}{2\pi}\right) dy + B_0(Y, T, \infty).$$

We choose² $m = 2$ and so

$$B_0(T, T, \infty) < 2R(T)\phi_2(T).$$

Moreover,

$$\sum_{\substack{\rho \in \wp(x) \\ |\gamma| > T}} \frac{1}{|\gamma|^{m+1}} \leq \tilde{C}_T + \tilde{D}_T.$$

Let's study more precisely

$$\int_T^\infty \phi_m(y) \ln(ky/2\pi) dy = \frac{z_m^2}{2m^2} \left(\frac{k}{C_1(k)}\right)^m \left(K_2(z_m, U_m) + \frac{2md}{z_m} K_1(z_m, U_m)\right)$$

²In fact, this choice of m is not needed at this stage, but as we'll need it later, we perform it now to simplify the writing.

where $d = \ln\left(\frac{C_1(k)}{2\pi}\right)$ and $U_m = \frac{2m}{z_m} \ln\left(\frac{kT}{C_1(k)}\right) = \nu\sqrt{m} = U'$. Now, by writing $z = z_m$ and using lemma 8,

$$\begin{aligned} K_2(z, U') + \frac{2dm}{z} K_1(z, U') &< (U' + 2/z + 2dm/z) Q_1(z, U') \\ &\leq \sqrt{m} \left(\nu + \frac{1+dm}{mX} \right) \frac{U'^2}{z(U'^2 - 1)} e^{-\frac{z}{2}(U'+1/U')} \end{aligned}$$

But $-\frac{z}{2}(U'+1/U') = X\sqrt{m}(\nu\sqrt{m} + 1/(\nu\sqrt{m})) = m\nu X + X/\nu = m\nu X + (Y + 2X - \nu X)$, where $Y = X(1-\nu)^2/\nu$. Hence

$$K_2(z, U') + \frac{2dm}{z} K_1(z, U') < G_1 e^{-Y} \frac{m}{2(m-1)} X^{-1} e^{-2X} \left(\frac{kT}{C_1(k)} \right)^{-(m-1)}$$

where $G_1 = \frac{m-1}{m} \frac{U'^2}{z(U'^2-1)} \left(\nu + \frac{1+dm}{mX} \right) = \frac{(m-1)\nu^2}{m\nu^2-1} \left(\nu + \frac{1+dm}{mX} \right)$ because

$$e^{\nu X(m-1)} = \left(\frac{kT}{C_1(k)} \right)^{m-1}$$

and $\frac{\sqrt{m}}{z} = \frac{1}{2}X^{-1}$. This yields

$$\int_T^\infty \phi_m(y) \ln(ky/2\pi) dy < \frac{G_1 e^{-Y}}{m-1} \frac{k}{C_1(k)} X e^{-2X} T^{-(m-1)}$$

Let $G_2 = \frac{R_m(\delta)}{2^m} (1 + \pi q(T))$. So, by using lemma 12,

$$\begin{aligned} A(m, \delta) \frac{\varphi(k)}{2} (1/\pi + q(T)) \int_T^\infty \phi_m(y) \ln(ky/2\pi) dy < \\ \left(\frac{2}{\delta} \right)^m \frac{\varphi(k)}{2} \left\{ \frac{G_2}{\pi} \frac{k G_1 e^{-Y}}{(m-1) C_1(k)} X e^{-2X} T^{-(m-1)} \right\} \end{aligned}$$

The results of the first and second parts yield

$$\begin{aligned} A(m, \delta) \frac{\varphi(k)}{2} (\tilde{A}_1 + \tilde{A}_2) \\ < \frac{X G_2 e^{-2X} e^{-Y} \varphi(k)}{2\pi} \left(\frac{k}{C_1(k)} \right) \left\{ \frac{G_1}{m-1} T^{-(m-1)} \left(\frac{2}{\delta} \right)^m + G_0 T \right\} + r \end{aligned}$$

because $1 + m\delta/2 < R_m(\delta)/2^m < G_2/\pi$, with

$$\begin{aligned} r &= \varphi(k) (1 + m\delta/2) R(T) \phi_0(T) + A(m, \delta) \varphi(k) R(T) \phi_2(T) \\ &+ \frac{\varphi(k)}{\sqrt{x}} ((1 + m\delta/2) \tilde{E}(T) + A(m, \delta) (\tilde{C}_T + \tilde{D}_T)/2). \end{aligned}$$

Assume that G_0/G_1 is independent of ν ; the expression between braces can be minimized for $T = (G_1/G_0)^{1/m} \cdot \frac{2}{\delta}$. With this choice,

$$\frac{G_1}{m-1} T^{-(m-1)} \left(\frac{2}{\delta} \right)^m + G_0 T = \frac{m}{m-1} G_1^{1/m} G_0^{1-1/m} \frac{2}{\delta}$$

and we obtain

$$\begin{aligned} \varepsilon_1 &= A(m, \delta) \frac{\varphi(k)}{2} (\tilde{A}_1 + \tilde{A}_2) + \frac{1}{2} m\delta < \\ &\frac{1}{2} m G_2 \left\{ X e^{-2X} e^{-Y} \frac{2k\varphi(k)}{\delta(m-1)\pi C_1(k)} G_1^{1/m} G_0^{1-1/m} + \delta \right\} + r \end{aligned}$$

The expression between braces can be minimized by choosing

$$\delta = \left\{ G_0^{1-1/m} G_1^{1/m} e^{-Y} \frac{2k\varphi(k)}{(m-1)\pi C_1(k)} \right\}^{1/2} X^{1/2} e^{-X}.$$

Hence we write

$$T = \left(\frac{G_1}{G_0} \right)^{1/2m} \left(\frac{2C_1(k)}{k\varphi(k)} (m-1)\pi e^Y / G_0 \right)^{1/2} X^{-1/2} e^X$$

and

$$(12) \quad \varepsilon_1 < G_2 \left(G_0^{1-1/m} G_1^{1/m} e^{-Y} \frac{2k\varphi(k)}{\pi C_1(k)} \right)^{1/2} \frac{m}{\sqrt{m-1}} X^{1/2} e^{-X} + r.$$

Fix $m = 2$ to minimize the expression $\frac{m}{\sqrt{m-1}}$. Is this choice for T suitable? In the first hand,

$$T = \left(\frac{G_1}{G_0^3} \right)^{1/4} e^{Y/2} \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} X^{-1/2} e^X$$

with $Y = X(1-\nu)^2/\nu$ and in the second hand

$$T = \frac{C_1(k)}{k} \exp(\nu X).$$

This two equations are compatible if and only if there exists ν such that $f(\nu) = 1$ where

$$f(\nu) = \frac{C_1(k)\varphi(k)}{2\pi k} \left(\frac{G_0^3}{G_1} \right)^{1/2} X e^{-X(1-\nu)^2/\nu} e^{-2X(1-\nu)}.$$

If $1/\sqrt{2} < \nu \leq 1$, it's easy to see that G_1 decreases when ν increases. Hence, $f(\nu)$ is strictly increasing for $\nu \in]1/\sqrt{2}, 1]$. Moreover, $\lim_{\nu \rightarrow (1/\sqrt{2})^+} f(\nu) = 0$ and $f(1) > 1$

(for all $X \geq 1$). So there exists a unique $\nu \in]1/\sqrt{2}, 1]$ such that $f(\nu) = 1$. For $1/\sqrt{2} < \nu \leq 1$, we have ($m = 2$)

$$H(\nu) = \frac{G_0^3}{G_1} = \frac{[\nu^2(\nu + d/X)]^3}{\frac{\nu^2}{2\nu^2-1}(\nu + \frac{1+2d}{2X})} < (\nu + d/X)^2.$$

Write

$$\nu_0 = 1 - \frac{1}{2X} \ln \left(\frac{C_1(k)\varphi(k)X}{2k\pi} \right).$$

Let's study $H(\nu_0)$.

$$\begin{aligned} H(\nu_0) < 1 & \text{ if } \nu + d/X \\ \text{equivalently} & \quad 1 - \frac{1}{2X} \ln \left(\frac{C_1(k)\varphi(k)X}{2k\pi} \right) + \frac{\ln(C_1(k)/2\pi)}{X} < 1 \\ \text{namely} & \quad X > \frac{kC_1(k)}{2\pi\varphi(k)}. \end{aligned}$$

As

$$\begin{aligned} f(\nu) &= \frac{C_1(k)\varphi(k)}{2k\pi} \left(\frac{G_0^3}{G_1} \right)^{1/2} X \exp(-X(1-\nu)^2/\nu) \exp(-2X(1-\nu)), \\ f(\nu_0) &= \left(\frac{G_0^3}{G_1} \right)^{1/2} \exp \left(-\ln^2 \left(\frac{C_1(k)\varphi(k)X}{2k\pi} \right) / (4\nu_0 X) \right). \end{aligned}$$

For $X > kC_1(k)/(2\pi\varphi(k))$, $f(\nu_0) < 1 = f(\nu)$ and the equalities hold if $T > \frac{C_1(k)}{k} \exp(\nu_0 X) > H_k$. Now $\frac{C_1(k)}{k} \exp(\nu_0 X) = \sqrt{\frac{2\pi C_1(k)}{k\varphi(k)}} e^{X - \frac{1}{2} \ln X}$. Let X_0 verifying

$$e^{X_0 - \frac{1}{2} \ln X_0} = H_k \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}.$$

Choose $X \geq \max(X_0, 10)$.

Assume that $X \geq \frac{2k\pi e}{C_1(k)\varphi(k)}$. Hence ν_0 is an increasing function of X , and since $C_1(k) \leq 32\pi$ then

$$\nu_0 > 0.746 \text{ and } \nu_0 < \nu < 1.$$

We want to evaluate

$$(13) \quad K = G_2(\sqrt{G_0 G_1} e^{-Y})^{1/2}.$$

$$\begin{aligned} G_0 G_1 &< (1 + d/X) \frac{\nu_0}{2\nu_0^2 - 1} \left(\nu_0 + \frac{1 + 2d}{2X} \right) \\ &< 8.995 \end{aligned}$$

The next results are useful in some computations:

1.

$$\begin{aligned} \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} \exp(-X) &\leq \frac{e^{X - \ln \sqrt{X}}}{H \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}}} \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} \exp(-X) \\ &\leq \frac{1}{H} \end{aligned}$$

2.

$$\begin{aligned} \delta &= 2\sqrt[4]{G_0 G_1} \exp(-Y/2) \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} e^{-X} \\ &\leq 2\sqrt{3}/H \end{aligned}$$

3.

$$G_2 = \frac{R_2(\delta)}{2^2} (1 + \pi q(T)) < (1 + 3.006 \cdot \delta/2)^2 (1 + \pi q(T)),$$

because

$$\begin{aligned} \frac{R_2(\delta)}{2^2} &= \left\{ \frac{(1 + \delta)^3 + 1}{2} \right\}^2 \\ &= \left\{ 1 + \frac{1}{2} \delta (3 + 3\delta + \delta^2) \right\}^2 < \left(1 + \frac{3.006}{2} \delta \right)^2 \end{aligned}$$

since $1 + \delta + \delta^2/2 < 1.002$;

4.

$$\begin{aligned} |q(T)| &\leq \left| q \left(\frac{C_1(k)}{k} e^{\nu_0 X} \right) \right| = \frac{C_2}{\frac{C_1(k)}{k} e^{\nu_0 X} \ln \left(\frac{C_1(k)}{2\pi} e^{\nu_0 X} \right)} \\ &< \frac{C_2}{T \ln(kH/(2\pi))} = C_2 \left(\frac{G_0^3}{G_1} \right)^{1/4} e^{-Y/2} \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} e^{-X} / \ln \left(\frac{H}{2\pi} \right). \end{aligned}$$

But $\frac{G_0^3}{G_1} < (\nu + d/X)^2 \leq (1 + \ln(16)/10)^2$, $\exp(-Y/2) \leq 1$ and $H \geq 1000$, this yields

$$\begin{aligned} 1 + \pi q(T) &< 1 + \frac{\pi C_2 \sqrt{1 + \ln(16)/10}}{\ln(1000/(2\pi))} \sqrt{\frac{k\varphi(k)}{2\pi C_1(k)}} X^{1/2} \exp(-X) \\ &< 1 + \frac{\pi C_2}{\ln(1000/(2\pi))} \sqrt{1 + \frac{\ln 16}{10}} \end{aligned}$$

So

$$\begin{aligned} K &< \sqrt{3} G_2 \\ &< \sqrt{3} \left(1 + \frac{\pi C_2}{\ln(1000/(2\pi))} \sqrt{1 + \frac{\ln 16}{10}} \right) \times \left(1 + \frac{3.006}{2} \frac{2\sqrt{3}}{1000} \right)^2 \\ &< 1.75126 \end{aligned}$$

Replacing this upper bound of K (see formula 13) in (12), we obtain

$$\begin{aligned} \varepsilon_1 &< 2\sqrt{\frac{2}{\pi}} K \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r \\ &< 2.79461 \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X) + r \end{aligned}$$

- Upper bound for $(1 + m\delta/2)R(T)\phi_0(T)$ and $A(m, \delta)R(T)\phi_2(T)$. Remember that

$$\begin{aligned} R(T) &= C_2 \ln(kT) + C_3, \\ \phi_0(T) &= \frac{1}{T} \exp(-X^2/\ln(kT/C_1(k))), \\ \phi_m(T) &= \phi_0(T) T^{-m}. \end{aligned}$$

Now

$$\phi_0(T) = \frac{1}{T} \exp(-X^2/(\nu X)) = \frac{1}{T} \exp(-\frac{1}{\nu} X) \leq \frac{1}{T} \exp(-X)$$

and

$$\frac{1}{T} = X^{1/2} \exp(-X) \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left(\frac{G_0}{2\pi e^Y} \right)^{1/2} \left(\frac{G_0}{G_1} \right)^{1/4}$$

hence

$$\begin{aligned} R(T)\phi_0(T) &\leq \frac{C_2 \ln(kT) + C_3}{T} \exp(-X) \\ &\leq X^{1/2} e^{-X} \sqrt{\frac{k\varphi(k)}{C_1(k)}} \left[(C_2 \ln(kT) + C_3) \left(\frac{G_0}{2\pi e^Y} \right)^{1/2} \left(\frac{G_0}{G_1} \right)^{1/4} e^{-X} \right] \end{aligned}$$

But

$$\begin{aligned} G_0 &\leq 1 + \frac{\ln(C_1(k)/2\pi)}{X}, \\ \frac{G_0}{G_1} &\leq 2\nu^2 - 1 < 1 \quad (m = 2), \\ \exp(Y) &> 1, \\ \ln(kT) &= \nu X + \ln(C_1(k)) \leq X + \ln(C_1(k)) \leq X + \ln(32\pi). \end{aligned}$$

So, since $X \geq 10$ and $C_1(k) \leq 32\pi$,

$$\begin{aligned} & (1 + \delta)\varphi(k) \left[(C_2 \ln(kT) + C_3) \left(\frac{G_0}{2\pi e^Y} \right)^{1/2} \left(\frac{G_0}{G_1} \right)^{1/4} \exp(-X) \right] \\ & \leq \varphi(k) \left(1 + \frac{2\sqrt{3}}{1000} \right) [C_2(X + \ln 32\pi) + C_3] \sqrt{\frac{1 + \ln 16/10}{2\pi}} \exp(-X) \\ & \leq 0.933\varphi(k)X \exp(-X) \\ & \leq 0.0933 \text{ if } X \geq X_1. \end{aligned}$$

An upper bound for $\varphi(k)(1 + \delta)R(T)\phi_0(T)$ is

$$0.0933 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

Next,

$$A(2, \delta)/T^2 = 2\sqrt{\frac{G_1}{G_0}} < 2$$

and

$$A(2, \delta)R(T)\phi_2(T) < \frac{R_2(\delta)}{2^2} R(T)\phi_0(T).$$

An upper bound for $A(2, \delta)R(T)\phi_2(T)\varphi(k)$ is

$$0.0943 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

The two terms can be bounded by

$$r \leq 0.2 \cdot \sqrt{\frac{k\varphi(k)}{C_1(k)}} X^{1/2} \exp(-X).$$

- Upper bound for $\tilde{C}_T, \tilde{D}_T, \tilde{E}(T), \tilde{R}/x$.
For $f(k) = \sum_{p|k} \frac{1}{p-1}$ observe that

$$f(k) \leq \frac{\ln k}{\ln 2}.$$

We can rewrite explicitly for $m = 2$, $H \geq 1000$ and $C_1(k) \leq 32\pi$ the following expressions:

$$\begin{aligned} 3\tilde{E}(T) &= 3\left(\frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln\left(\frac{k}{2\pi}\right) \ln T + C_2\right. \\ &\quad \left.+ 2\left(\frac{1}{\pi} \ln\left(\frac{k}{2\pi e}\right) + C_2 \ln k + C_3\right)\right), \\ \tilde{C}_T &= \frac{1}{2\pi T^2} \left(\ln\left(\frac{kT}{2\pi}\right) + 1/2\right), \\ \tilde{D}_T &= (2C_2 \ln(kT) + 2C_3 + C_2/3)/T^3, \\ \tilde{R}/\sqrt{x} &\leq \varphi(k) [(f(k) + 0, 5) \ln x/x + 4 \ln k/x + 13, 4]/\sqrt{x} \\ \text{Sum} &\leq \begin{array}{ll} (\ln T \sqrt{\ln k})^2 & \text{for } k \neq 1 \\ \ln^2 T & \text{for } k = 1 \end{array} \end{aligned}$$

Now find the value c which satisfies

$$A(2, \delta)\varphi(k)\frac{(\ln T\sqrt{\ln k})^2}{\sqrt{x}} \leq c \left(\frac{k\varphi(k)}{C_1(k)}\right)^{1/2} X^{1/2} \exp(-X)$$

with $X = \sqrt{\frac{\ln x}{R}}$.

But $A(2, \delta) \leq \frac{R_2(\delta)}{\delta^2}$ and $T = \left(\frac{G_1}{G_0}\right)^{1/2} \frac{2}{\delta}$ so

$$A(2, \delta) \leq \frac{R_2(\delta)}{2^2} T^2 \frac{G_0}{G_1}.$$

Moreover, $\frac{1}{\sqrt{x}} = \exp(-RX^2/2)$ hence

$$c \geq \frac{R_2(\delta)}{2^2} \frac{G_0}{G_1} T^2 \varphi(k) (\ln T\sqrt{\ln k})^2 \left(\frac{C_1(k)}{k\varphi(k)}\right)^{1/2} X^{-1/2} \exp(X - RX^2/2).$$

As $\frac{G_0}{G_1} < 1$, $T^2 = \frac{C_1^2(k)}{k^2} \exp(2\nu X) \leq \frac{C_1^2(k)}{k^2} \exp(2X)$, hence we can choose

$$c \geq \frac{R_2(\delta)}{2^2} C_1^2(k) \frac{\ln k}{k^2} (\ln(C_1(k)/k) + X)^2 \left(\frac{C_1(k)\varphi(k)}{k}\right)^{1/2} X^{-1/2} \exp(3X - RX^2/2).$$

But $\frac{\varphi(k)}{k} \leq 1$, $\frac{\ln k}{k^2} \leq 1$ and $R_2(\delta) \leq (1 + 3.006\delta/2)^2$ with $\delta \leq \frac{2\sqrt{3}}{H} \leq \frac{2\sqrt{3}}{1000}$. We obtain

$$c \geq \frac{1}{4} \left(1 + \frac{3.006}{2} \frac{2\sqrt{3}}{1000}\right)^2 C_1^2(k) (\ln C_1(k) + X)^2 \sqrt{C_1(k)} X^{-1/2} \exp(3X - RX^2/2).$$

Since $C_1(k) \leq 32\pi$ and $X \geq 10$, we can choose

$$c \geq 0.64286 \cdot 10^{-190}.$$

For $k = 1$, we replace the upper bound $\frac{\ln k}{k^2} \leq 1$ by 1, to obtain the previous computation. \square

5. APPLICATION FOR $k = 3$

Now we are able to compute x_0 and c such that, for $x \geq x_0$,

$$|\theta(x; 3, l) - x/2| < cx/\ln x.$$

This would not have been possible if we had used only the results of [3].

According to theorem 5,

$$\varepsilon(x) = \frac{3}{2} \sqrt{\frac{6}{20.92}} X^{1/2} \exp(-X)$$

for $k = 3$.

Find X_0 such that ($H_3 = 10000$)

$$\exp(X_0 - \frac{1}{2} \ln X_0) \geq 10000 \sqrt{\frac{6}{2\pi \cdot 20.92}} \approx 2136.51.$$

$X_0 \approx 8.76$ holds.

Find X_1 such that

$$\exp(X_1 - \ln X_1) \geq 20.$$

$X_1 \approx 4.5$ holds.

Compute the two others bounds: $X_2 \approx 4.99$, $X_3 \approx 1.22$.

- For $\sqrt{\frac{\ln x}{R}} \geq 10$, write $X = \sqrt{\frac{\ln x}{R}}$, then

$$\varepsilon(X) \ln x = RX^2 \varepsilon(X).$$

Find the value c such that

$$\varepsilon(X) < c / \ln(x).$$

For all x such that $\sqrt{\frac{\ln x}{R}} \geq 10$, $c \leq R \cdot 10^2 \exp(10) \leq 0.12$.

Olivier Ramaré has nicely computed some additional values: we have

$$|\theta(x; 3, l) - x/2| < c \cdot x/2$$

with

$$\begin{aligned} c &= 0.0008464421 \text{ for } \ln x \geq 400 \quad (m = 3, \delta = 0.00042325), \\ c &= 0.0006048271 \text{ for } \ln x \geq 500 \quad (m = 3, \delta = 0.00030250), \\ c &= 0.0004190635 \text{ for } \ln x \geq 600 \quad (m = 2, \delta = 0.00027950). \end{aligned}$$

- For $e^{100} \leq x \leq e^{964.59\dots}$

$$c \leq 0.0004190635 \cdot 964.6 / \varphi(3) \leq 0.203.$$

- for $e^{400} \leq x \leq e^{600}$

$$c \leq 0.0008464421 \cdot 600 / \varphi(3) \leq 0.254.$$

Using the computations of [3],

- For $10^{100} \leq x \leq e^{400}$

$$c \leq 0.001310 \cdot RX^2 / \varphi(3) \leq 1.264/2 \leq 0.632.$$

- For $10^{30} \leq x \leq 10^{100}$

$$c \leq 0.001813 \cdot 100 \ln 10 / \varphi(3) \leq 0.42/2 \leq 0.21.$$

- For $10^{13} \leq x \leq 10^{30}$

$$c \leq 0.001951 \cdot 30 \ln 10 / \varphi(3) \leq 0.14/2 \leq 0.07.$$

- For $10^{10} \leq x \leq 10^{13}$

$$c \leq 0.002238 \cdot 13 \ln 10 / \varphi(3) \leq 0.067/2 \leq 0.00335.$$

- For $0 \leq x \leq 10^{10}$

$$|\theta(x; 3, l) - x/2| < 2.072\sqrt{x} \quad (\text{Th. 5.2.1 of Ramaré \& Rumely})$$

Theorem 6. For $x > 1$,

$$|\theta(x; 3, l) - x/2| < 0.262 \frac{x}{\ln x}.$$

Remark: By using only results published in [2, 3], one obtains

$$|\theta(x; 3, l) - x/2| < 0.393 \frac{x}{\ln x} \quad \text{for } x > 1.$$

6. RESULTS WITH GRH(k, ∞).

With GRH(k, ∞), we obtain more precise results. Under this hypothesis, one can show that function ψ has the following asymptotic behaviour:

Proposition 1 ([8] p.294). *Assume GRH(k, ∞). Then*

$$\psi(x; k, l) = \frac{x}{\varphi(k)} + O(\sqrt{x} \ln^2 x).$$

Theorem 7. *Let $x \geq 10^{10}$. Let k be a positive integer. Assume GRH(k, ∞).*

1) *If $k \leq \frac{4}{5} \ln x$ then*

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \frac{1}{4\pi} \sqrt{x} \ln^2 x.$$

2) *If $k \leq 432$ then*

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \frac{11}{32\pi} \sqrt{x} \ln^2 x.$$

Proof: Let $x_0 = 10^{10}$. Apply theorem 2 in the same way as theorem 3 (Assume that $T \geq 1$),

$$\begin{aligned} & \frac{\varphi(k)}{x} |\psi(x; k, l) - \frac{x}{\varphi(k)}| \\ & < \varepsilon = A(m, \delta) \sum_x \sum_{\gamma > T} \frac{x^{-1/2}}{|\rho(\rho+1) \cdots (\rho+m)|} \\ & \quad + (1 + m\delta/2) \sum_x \sum_{\gamma < T} \frac{x^{-1/2}}{|\rho|} + m\delta/2 + \tilde{R}/x \\ & < A(m, \delta) \frac{\varphi(k)}{\sqrt{x}} \sum_{\gamma > T} \frac{1}{|\gamma|^{m+1}} + (1 + \frac{m\delta}{2}) \frac{\varphi(k)}{\sqrt{x}} \sum_{\gamma < T} \frac{1}{|\rho|} + \frac{m\delta}{2} + \frac{\tilde{R}}{x} \\ & < A(m, \delta) \frac{\varphi(k)}{2\sqrt{x}} (\tilde{C}_T + \tilde{D}_T) + (1 + \frac{m\delta}{2}) \frac{\varphi(k)}{\sqrt{x}} \tilde{E}(T) + \frac{m\delta}{2} + \tilde{R}/x \end{aligned}$$

Taking $m = 1$, we obtain

$$\begin{aligned} \tilde{C}_T &= \frac{1}{\pi T} \left(\ln \left(\frac{kT}{2\pi} \right) + 1 \right) \\ \tilde{D}_T &= \frac{1}{T^2} (2C_2 \ln(kT) + 2C_3 + C_2/2) \\ \tilde{E}(T) &= \frac{1}{2\pi} \ln^2 T + \frac{1}{\pi} \ln(k/(2\pi)) \ln T + C_2 + 2 \left(\frac{1}{\pi} \ln \left(\frac{k}{2\pi e} \right) + C_2 \ln k + C_3 \right) \end{aligned}$$

Choose $T = \frac{2R_1(\delta)}{\delta(2+\delta)}$ and $\delta = \frac{\ln x}{\pi\sqrt{x}}$. So

$$\begin{aligned} \frac{R_1(\delta)}{2\delta}(\tilde{C}_T + \tilde{D}_T) &= \frac{2+\delta}{4\pi} \left[\ln \left(\frac{kR_1(\delta)}{\pi\delta(2+\delta)} \right) + 1 \right. \\ &\quad \left. + \frac{\pi\delta(2+\delta)}{2R_1(\delta)} (2C_2 \ln \left(\frac{2kR_1(\delta)}{\delta(2+\delta)} \right) + 2C_3 + C_2/2) \right] \\ (1+\delta/2)\tilde{E}(T) &= \frac{2+\delta}{4\pi} \left[\ln^2 \left(\frac{2R_1(\delta)}{\delta(2+\delta)} \right) + 2 \ln(k/(2\pi)) \ln \left(\frac{2R_1(\delta)}{\delta(2+\delta)} \right) \right. \\ &\quad \left. + 2\pi C_2 + 4\pi \left(\frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right) \right] \end{aligned}$$

Hence

$$\begin{aligned} \frac{\varepsilon\sqrt{x}}{\varphi(k)} &= \frac{2+\delta}{4\pi} \left[\ln^2 \left(\frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + 2 \ln \left(\frac{k}{2\pi} \right) \ln \left(\frac{2\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) \right. \\ &\quad \left. + \ln \left(\frac{k\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + \frac{\ln x}{\sqrt{x}} \frac{2+\delta}{R_1(\delta)} (A) \right] + \frac{\ln x}{2\pi\varphi(k)} + \frac{\tilde{R}}{\varphi(k)\sqrt{x}} \\ &\quad + \frac{2+\delta}{4\pi} (1 + 2\pi C_2 + 4\pi \left(\frac{1}{\pi} \ln(k/(2\pi e)) + C_2 \ln k + C_3 \right)) \end{aligned}$$

with

$$A = C_2 \ln \left(\frac{2k\pi\sqrt{x}}{\ln x} \cdot \frac{R_1(\delta)}{2+\delta} \right) + C_3 + C_2/4.$$

Let $\delta_1 = \frac{\ln x_0}{\pi\sqrt{x_0}}$. But $\frac{R_1(\delta)}{2+\delta} = \frac{2+2\delta+\delta^2}{2+\delta} = 1 + \frac{\delta^2+\delta}{2+\delta} \leq d_1 := 1 + \frac{\delta_1^2+\delta_1}{2+\delta_1}$ because $x \geq x_0$ and $\frac{2+\delta}{R_1(\delta)} < 1$.

Note that all terms of $\varphi(k)$ simplify except in $\frac{\ln x}{2\pi\varphi(k)}$. We can bound it by 2. We obtain an upper bound ε' for $\varepsilon\sqrt{x}/\varphi(k)$. We will study the sum into brackets for $3 \leq k \leq \frac{4}{5} \ln x$:

$$\begin{aligned} [\dots] &= \left[\frac{1}{4} \ln^2 x + \ln^2 \left(\frac{2\pi d_1}{\ln x} \right) + \ln x \ln \left(\frac{2\pi d_1}{\ln x} \right) + 2 \ln \left(\frac{4 \ln x}{10\pi} \right) \ln \left(\frac{2\pi d_1}{\ln x} \right) \right. \\ &\quad \left. + \ln \left(\frac{4 \ln x}{10\pi} \right) \ln x + \frac{1}{2} \ln x + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}} (A) \right] \\ &= \left[\frac{1}{4} \ln^2 x + \ln x \left(\ln \left(\frac{2\pi d_1}{\ln x} \right) + 1/2 + \ln(4 \ln x / (10\pi)) \right) \right. \\ &\quad \left. + \ln^2 \left(\frac{2\pi d_1}{\ln x} \right) + 2 \ln \left(\frac{4 \ln x}{10\pi} \right) \ln \left(\frac{2\pi d_1}{\ln x} \right) + \ln(4d_1/5) + \frac{\ln x}{\sqrt{x}} (A) \right] \end{aligned}$$

We conclude that

$$\lim_{x \rightarrow +\infty} \frac{\varepsilon'}{\ln^2 x} = \frac{1}{8\pi}.$$

The bound ε' is an increasing function of k . Choose $k = \frac{4}{5} \ln x$. Now $\varepsilon' \ln^2 x$ is a decreasing function of x bounded by 0.0777763 for $x \geq x_0$. Moreover, by direct computation for all k between 1 and 432 of $\varepsilon\sqrt{x}/(\varphi(k) \ln^2 x)$, we find an upper bound 0.1094. \square

Remark: If we take $k = 1$ in theorem 7, our upper bound is twice as bad than the result of ROSSER & SCHOENFELD: for $x > 73.2$,

$$|\psi(x) - x| \leq \frac{1}{8\pi} \sqrt{x} \ln^2 x.$$

These differences are explained by:

- an exact computation of zeros with $\gamma \leq D \approx 158$ (the preponderant ones!) in the sum $\sum \frac{1}{|\rho|}$.
- a better knowledge of $R(T)$ (k fixed, $k = 1$).

Corollary 3. *Assume GRH(k, ∞). For all k used in lemma 2 and $x \geq 224$,*

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \frac{11}{32\pi} \sqrt{x} \ln^2 x.$$

Proof: We use theorem 5.2.1 of [3]: for all k noted in lemma 2 and $224 \leq x \leq 10^{10}$,

$$|\psi(x; k, l) - \frac{x}{\varphi(k)}| \leq \sqrt{x}$$

and $\sqrt{x} < \frac{11}{32\pi} \sqrt{x} \ln^2 x$ for $x \geq 21$. □

7. ESTIMATE OF $\pi(x; 3, l)$

Definition 1. *Let*

$$\pi(x; k, l) = \sum_{\substack{p \leq x \\ p \equiv l \pmod{k}}} 1$$

be the number of primes smaller than x with an Euclidean division's rest equal to l modulo k .

Our aim is to have bounds for $\pi(x; 3, l)$. We show that

Theorem 8. *For $l = 1$ or 2 ,*

- (i)- $\frac{x}{2 \ln x} < \pi(x; 3, l)$ for $x \geq 151$,
- (ii)- $\pi(x; 3, l) < 0.55 \frac{x}{\ln x}$ for $x \geq 229869$.

From this, we can deduce that

$$\frac{x}{\ln x} < \pi(x)$$

because

$$\pi(x) = \pi(x; 3, 1) + \pi(x; 3, 2).$$

7.1. Higher bound. First we write the proof of theorem 8(ii).

Lemma 13. *Let $I_n = \int_a^x \frac{dt}{\ln^n t}$. Then $I_n = \frac{x}{\ln^n x} - \frac{a}{\ln^n a} + nI_{n+1}$. Furthermore,*

$$(x - a) / \ln(x) \leq I_n \leq (x - a) / \ln(a).$$

Theorem 9 (Ramaré & Rumely [3]). *For $1 \leq x \leq 10^{10}$, for all $k \leq 72$, for all l relatively prime with k ,*

$$\max_{1 \leq y \leq x} \left| \theta(y; k, l) - \frac{y}{\varphi(k)} \right| \leq 2.072 \sqrt{x}.$$

Furthermore, for $x \geq 10^{10}$ and $k = 3$ or 4 ,

$$\left| \theta(x; k, l) - \frac{x}{\varphi(k)} \right| \leq 0.002238 \frac{x}{\varphi(k)}.$$

Write first

$$\pi(x; k, l) - \pi(x_0; k, l) = \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(x_0; k, l)}{\ln(x_0)} + \int_{x_0}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt$$

Choose $x_0 = 10^5$.

Preliminary computations :

$$\theta(10^5; 3, 1) = 49753.417198 \dots \quad \pi(10^5; 3, 1) = 4784.$$

$$\theta(10^5; 3, 2) = 49930.873458 \dots \quad \pi(10^5; 3, 2) = 4807.$$

Put $c_0 = \frac{1.002238}{2}$ and $K = \max_l(\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 470$.

- For $10^{20} \leq x$,

$$\begin{aligned} \pi(x; k, l) - \pi(10^5; k, l) &= \frac{\theta(x; k, l)}{\ln(x)} - \frac{\theta(10^5; k, l)}{\ln(10^5)} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt \\ \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt &= \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; k, l)}{t \ln^2 t} dt + \int_{\sqrt{x}}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt \\ \int_{10^5}^{10^{10}} \frac{\theta(t; k, l)}{t \ln^2 t} dt &< M = 1/\varphi(k) \cdot \int_{10^5}^{10^{10}} \frac{dt}{\ln^2 t} + 2.072 \cdot \int_{10^5}^{10^{10}} \frac{dt}{\sqrt{t} \ln^2 t} \\ \int_{10^{10}}^{\sqrt{x}} \frac{\theta(t; 3, l)}{t \ln^2 t} dt &< c_0 \frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}} \\ \int_{\sqrt{x}}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt &< c_0 \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}} \end{aligned}$$

We compute $M = 10381055.54 \dots$. That way

$$\begin{aligned} \pi(x; 3, l) &< c_0 \frac{x}{\ln x} + K + M + c \left(\frac{\sqrt{x} - 10^{10}}{\ln^2 10^{10}} + \frac{x - \sqrt{x}}{\ln^2 \sqrt{x}} \right) \\ &< \frac{x}{\ln x} \left(c_0 + \left(K + M + c \frac{10^{20} - 10^{10}}{\ln^2 10^{10}} \right) \frac{\ln 10^{20}}{10^{20}} \right) \\ &< 0.545 \frac{x}{\ln x}. \end{aligned}$$

- For $10^{10} \leq x \leq 10^{20}$, we have $10^5 \leq \sqrt{x} \leq 10^{10}$.

$$\begin{aligned} \pi(x; 3, l) &< K + \int_{10^5}^{10^{10}} \frac{\theta(t; 3, l)}{t \ln^2 t} dt + \int_{10^{10}}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt \\ &< \frac{x}{\ln x} \left(c_0 + \frac{\ln x}{x} \left(K + M - 10^{10} \frac{c_0}{\ln^2 10^{10}} \right) + \frac{c_0}{\ln^2 10^{10}} \ln x \right) \\ &< 0.5468 \frac{x}{\ln x}. \end{aligned}$$

- For $10^5 \leq x \leq 10^{10}$,

$$\begin{aligned} \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt &< \frac{1}{2} \int_{10^5}^x \frac{dt}{\ln^2 t} + 2.072 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^2 t} \\ &= \frac{1}{2} \left(\frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^x \frac{dt}{\ln^3 t} \right) + 2.072 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^2 t} \end{aligned}$$

$$\text{Now, } \int_a^b \frac{dt}{\sqrt{t} \ln^2 t} = \left[\frac{2\sqrt{t}}{\ln^2 t} \right]_a^b + 4 \int_a^b \frac{dt}{\sqrt{t} \ln^3 t}$$

Therefore

$$\begin{aligned}\pi(x; 3, l) &< \frac{1}{2} \frac{x}{\ln x} + 2.072 \frac{\sqrt{x}}{\ln x} + K \\ &+ \frac{1}{2} \left(\frac{x}{\ln^2 x} - \frac{10^5}{\ln^2 10^5} + 2 \int_{10^5}^x \frac{dt}{\ln^3 t} \right) \\ &+ 2.072 \left(\frac{2\sqrt{x}}{\ln^2 x} - \frac{2\sqrt{10^5}}{\ln^2 10^5} + 4 \int_{10^5}^x \frac{dt}{\sqrt{t} \ln^3 t} \right) \\ &< 0.55 \frac{x}{\ln x} \quad \text{for } x \geq 6 \cdot 10^5\end{aligned}$$

7.2. Lower bound. Let $KK = \min_l(\pi(10^5, 3, l) - \theta(10^5, 3, l)/\ln(10^5)) \approx 462$ and $c = 0.498881 = \frac{1-0.002238}{2}$.

- For $10^{10} \leq x$,

$$\begin{aligned}\pi(x; 3, l) &> KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt \\ &> \frac{cx}{\ln x}\end{aligned}$$

because

$$KK > 0 \quad \text{and} \quad \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt > 0.$$

- For $10^5 \leq x \leq 10^{10}$.

Lemma 14 (McCurley [2]). *For $x \geq 91807$ and $d = 0.49585$, we have $\theta(x; 3, l) \geq dx$.*

Remark: This bound is better than the one given in th.9 for $x \leq 2.5 \cdot 10^5$.

$$\pi(x; 3, l) > KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt$$

That way

$$\begin{aligned}\pi(x; 3, l) &> KK + \frac{\theta(x; 3, l)}{\ln x} + \int_{10^5}^{x_0} \frac{\theta(t; k, l)}{t \ln^2 t} dt \quad \text{for } x \geq x_0 \\ &> \frac{x}{\ln x} \left(d + \left(KK + \int_{10^5}^{x_0} \frac{\theta(t)}{t \ln^2 t} dt \right) \frac{\ln x_1}{x_1} \right) \quad \text{for } x_0 \leq x \leq x_1\end{aligned}$$

Using the previous remark, we find

$$\begin{aligned}\int_{10^5}^x \frac{\theta(t; k, l)}{t \ln^2 t} dt &> d \int_{10^5}^x \frac{dt}{\ln^2 t} \quad \text{if } 10^5 \leq x \leq 2.5 \cdot 10^5 \\ \text{and} & \\ &> d \int_{10^5}^{2.5 \cdot 10^5} \frac{dt}{\ln^2 t} + \int_{2.5 \cdot 10^5}^x \frac{t/2 - 2\sqrt{t}}{t \ln^2 t} dt \quad \text{if } 2.5 \cdot 10^5 \leq x.\end{aligned}$$

x_0	x_1
10^5	$2 \cdot 10^6$
$2 \cdot 10^6$	$3 \cdot 10^7$
$3 \cdot 10^7$	$3 \cdot 10^8$
$3 \cdot 10^8$	$3 \cdot 10^9$
$3 \cdot 10^9$	10^{10}

We use this to make step by step computations with Maple:

We conclude that $\pi(x; 3, l) > 0.499 \frac{x}{\ln x}$ for $10^5 \leq x \leq 10^{10}$.

7.3. Small Values. We now check whether $0.49888 \frac{x}{\ln x} < \pi(x; 3, l) < 0.55 \frac{x}{\ln x}$ for $x < 10^5$. It is sufficient to prove that

$$\pi(p; 3, l) < 0.55 \frac{p}{\ln p} \text{ for } p \equiv l \pmod{3}$$

and if

$$0.49888 \frac{p}{\ln p} < \pi(p-1; 3, l) \text{ for } p \equiv l \pmod{3}.$$

The higher value not satisfying the first inequality is $p = 229849$ and $p = 151$ for the second one. Furthermore, $\pi(229869; 3, l) \leq 10241 < 0.55 \frac{229869}{\ln 229869} \approx 10241.0075$ and $\pi(151; 3, l) \geq 16 > 0.49888 \frac{151}{\ln 151} \approx 15.01$.

The conclusion is:

$$0.49888 \frac{x}{\ln x} \underset{x \geq 151}{<} \pi(x; 3, l) \underset{x \geq 229869}{<} 0.55 \frac{x}{\ln x}.$$

Remark: we can't show that $x/(2 \ln x) < \pi(x; 3, l)$ by using formula $\theta(x) < c \cdot x$. We have obtained other formulas (see theorem 6) which we will use below.

7.4. More precise lower bound of $\pi(x; 3, l)$. Now we will write the proof of theorem 8(i).

Classically,

$$\pi(x; 3, l) - \pi(10^5; 3, l) = \frac{\theta(x; 3, l)}{\ln(x)} - \frac{\theta(10^5; 3, l)}{\ln(10^5)} + \int_{10^5}^x \frac{\theta(t; 3, l)}{t \ln^2 t} dt.$$

Now $\theta(t; 3, l) > \frac{x}{\varphi(3)} \left(1 - \frac{\alpha}{\ln x}\right)$ with $\alpha = \varphi(3) \cdot 0.262$ by use of theorem 6. So we write

$$KK = \min_l \left(\pi(10^5; 3, l) - \frac{\theta(10^5; 3, l)}{\ln(10^5)} \right),$$

$$\pi(x; 3, l) > J(x, \alpha) = KK + \frac{x}{\varphi(k) \ln x} \left(1 - \frac{\alpha}{\ln x}\right) + \frac{1}{\varphi(k)} \int_{10^5}^x \frac{1 - \alpha/\ln t}{\ln^2 t} dt.$$

The derivative of $J(x, \alpha)$ with respect to x equals

$$\frac{1}{\varphi(k)} \left(\frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right).$$

Moreover the derivative of $\frac{x}{\varphi(k) \ln x}$ equals

$$\frac{1}{\varphi(k)} \left(\frac{1}{\ln x} - \frac{1}{\ln^2 x} \right).$$

The inequality

$$\frac{1}{\varphi(k)} \left(\frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) < \frac{1}{\varphi(k)} \left(\frac{1 - \alpha/\ln x}{\ln x} + \frac{\alpha}{\ln^3 x} \right)$$

holds if $\alpha - 1 < \alpha/\ln x$; This holds for all $x > 1$. It remains to find a value x_1 such that

$$J(x_1, \alpha) > \frac{x_1}{\varphi(k) \ln x_1}.$$

For $x_1 = 10^5$, $J(10^5, 0.524) \approx 4607.75$ and $\frac{10^5}{2 \ln 10^5} \approx 4342.94$. We verify by computer that the inequality holds for $x \leq 10^5$ and $l = 1$ or 2 . We conclude that

$$\frac{x}{2 \ln x} < \pi(x; 3, l) \text{ for } x \geq 151.$$

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DÉPARTEMENT DE MATH., LACO, 123 AVENUE ALBERT THOMAS, 87060 LIMOGES CEDEX
E-mail address: dusart@unilim.fr