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# An algorithm for bilinear fractional problems

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# An algorithm for bilinear fractional problems

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**Abstract** : We consider a special class of nonconvex optimization problems involving a bilinear fractional objective and two convex polyhedrons as constraint sets. From the special structure of these problems it follows that the search for a solution can be restricted to vertices of the polyhedrons. We propose a cutting plane algorithm providing either a minimizing sequence of local solutions or a global minimum.

## 1 Introduction

In the field of global optimization for nonconvex problems, significant results have been obtained by taking advantage of special structures of objectives as well as constraints. For instance Konno [4] (1976) and Gallo & Ülkücü [3] (1977) presented algorithms for bilinear problems with linear constraints. Other algorithms for bilinear programming models have been developed by Al-Khayyal & Falk [1] (1983) and Sherali & Alameddine [10] (1992). In 1985 Tuy [11] proposed a method for global optimization of a concave function over a convex polyhedron. In 1992, Falk & Palocsay [2] presented an algorithm for the sum of fractional functions subject to linear constraints and in 1995, Quesada & Grossmann [9] considered linear fractional and bilinear problems. Here, the problem that we consider have a fractional objective, the numerator and the denominator of which being bilinear functions with respect to the variables x and y. The constraints are given by two convex polyhedrons corresponding respectively to x and y. Such a situation arises for instance in engineering design problems [8] or in bond portfolio optimization techniques [5]. In the bilinear case, that is when the denominator of the objective is constant, two methods have been developed. The first one predicts bounds for the global optimum [1, 10] while an other approach proposed by Konno [4] uses a sequence of cutting planes to generate a sequence of local minima for the problem. In this paper we follow this second approach for the case of a bilinear fractional objective, the search domain being reduced by an appropriate cut at each iteration. When the seach domain becomes empty then a global solution is reached.

The paper is organized as follows. In Section 2, we formulate the problem and study its special structure. We recall in this part a simple algorithm solving linear fractional problems subject to linear constraints. Then, in Section 3, we use this algorithm to obtain local solutions in the bilinear fractional case. In Section 4, a method generating cutting

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planes is proposed. It consists to find a subset of the feasible set as large as possible on which the current local solution obtained in Section 3 is in fact a global one. Then a new constraint is introduced which eliminates from the feasible set some elements which do not improve the current optimal value. Finally a new starting vertex is found in the reduced feasible set and a new iteration occurs. In the last section we solve an example to illustrate the method and we present some numerical experimentations in the programming environment MATLAB, some examples being randomly generated.

# 2 The model

We consider in the following, a nonconvex problem involving a bilinear fractional objective and linear constraints.

$$(\mathcal{P}) \qquad \min F(x,y) = \frac{\phi(x,y)}{\psi(x,y)} = \frac{{}^tc_1x + {}^tc_2y + {}^txCy + s}{{}^td_1x + {}^td_2y + {}^txDy + t}$$

s.t.

$$x \in P_1 = \{x \in \mathbb{R}^{n_1} \ : \ A_1x \leq b_1 \ , x \geq 0\}$$

$$y \in P_2 = \{y \in \mathbb{R}^{n_2} \ : \ A_2y \leq b_2 \ , y \geq 0\}$$

where $c_1, d_1, x \in \mathbb{R}^{n_1}$, $c_2, d_2, y \in \mathbb{R}^{n_2}$, $C, D \in \mathbb{R}^{n_1 \times n_2}$, $s, t \in \mathbb{R}$, $A_i \in \mathbb{R}^{m_i \times n_i}$ and $b_i \in \mathbb{R}^{m_i}_+$, $i = 1, 2$.

When we fixe one variable to a constant value, say $y = \bar{y}$, then

$$F(x,\bar{y}) = \frac{({}^tc_1 + {}^t\bar{y}\ {}^tC)x + ({}^tc_2\bar{y} + s)}{({}^td_1 + {}^t\bar{y}\ {}^tD)x + ({}^td_2\bar{y} + t)}$$

is a linear fractional function of the form studied in [6].
In order to obtain a well-defined problem $(\mathcal{P})$ we suppose in the following that $\psi(x,y)$ is positive on $P_1 \times P_2$.
The next result shows that to solve $(\mathcal{P})$ it is sufficient to consider the vertices of $P_1$ and $P_2$.

**Proposition 2.1** *If the value of $(\mathcal{P})$ is finite then there exists a solution $(x^*, y^*)$ such that $x^*$ and $y^*$ are vertices of $P_1$ and $P_2$.*

**Proof** : Denote by $v \in \mathbb{R}$ the value of $(\mathcal{P})$ and let $(\bar{x}, \bar{y}) \in P_1 \times P_2$ be such that $F(\bar{x}, \bar{y}) = v$. The new objective $F(\cdot, \bar{y})$, where the second variable is fixed to $\bar{y}$, becomes a linear fractional function and it is known [6] that the problem $\min\{F(x,\bar{y}) \ : \ x \in P_1\}$ admits a vertex $\hat{x}$ of $P_1$ as a solution. Thus $F(\hat{x}, \bar{y}) = v$. In the same way, fixing the first variable to $\hat{x}$ we get a vertex $\hat{y}$ of $P_2$ such that $v = F(\hat{x}, \hat{y})$.

As in linear programming we conclude only to the existence and not to the uniqueness of such a couple of vertices. In order to solve a problem of the form

$$(\mathcal{P}_{y=\bar{y}}) \qquad \min_{x \in P_1} F(x, \bar{y})$$

we can use the following algorithm, strongly related with the simplex method, which examine only vertices of $P_1$. Denote by $f(x) = F(x, \bar{y})$.

**Algorithm 1**

**Step 0** : Set $k = 0$ and $x^0$ a basic solution of $P_1$. The corresponding basic and nonbasic variables are denoted by $x_{B^0}$ and $x_{N^0}$.
**Step 1** : Compute $\nabla f(x^k)$.
**Step 2** : If $[\nabla f(x^k)]_{N^k} \geq 0$ stop, $x^k$ is optimal, else perform a pivot operation on the first column $s$ such that $[\nabla f(x^k)]_s < 0$. Denote the new basic solution by $x^{k+1}$ and the new basic and nonbasic variables by $x_{B^{k+1}}$ and $x_{N^{k+1}}$. Set $k = k + 1$ and go to step 1.

As in the linear case, it can be seen that if the constraint polyhedron is bounded then the convergence occurs after a finite number of iterations, with eventually a tie breaking rule in the degenerate case. It is important to note that if $\hat{x}$ and $\hat{y}$ are respectively optimal solutions for the two following problems

$$\min\{F(x, \hat{y}) : x \in P_1\} \tag{2.1}$$

and

$$\min\{F(\hat{x}, y) : \, y \in P_2\} \tag{2.2}$$

then $(\hat{x}, \hat{y})$ is not necessarily a solution of $(\mathcal{P})$. Indeed consider for instance the following example [4] :

$$\min x_1 - x_2 - y_1 + (x_1, x_2) \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\text{s.t.} \begin{bmatrix} 1 & 4 \\ 4 & 1 \\ 3 & 4 \end{bmatrix} \begin{matrix} x_1 \\ x_2 \\ \phantom{x} \end{matrix} \begin{matrix} \phantom{\le} \\ \leq \\ \phantom{\le} \end{matrix} \begin{matrix} 8 \\ 12 \\ 12 \end{matrix} \qquad \text{and} \qquad \begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{matrix} y_1 \\ y_2 \\ \phantom{y} \end{matrix} \begin{matrix} \phantom{\le} \\ \leq \\ \phantom{\le} \end{matrix} \begin{matrix} 8 \\ 8 \\ 5 \end{matrix}$$

$$x \geq 0 \qquad\qquad y \geq 0$$

If $\hat{y} = (0, 4)$, the function depending on $x$ is $F(x, \hat{y}) = 5(x_1 - x_2)$ and it is easily seen that the minimum for $x \in P_1$ is obtained at the point $\hat{x} = (0, 2)$ with the value $-10$. Similarly $F(\hat{x}, y) = y_1 - 2y_2 - 2$ and $\hat{y}$ gives the minimum on $P_2$. Thus $\hat{x}$ and $\hat{y}$ solve (2.1) and (2.2), however $F(\hat{x}, \hat{y}) = -10$ whereas the global minimum value is $-15$ for $x^* = (0, 3)$ and $y^* = (0, 4)$.

# 3 Stationary couples

Although the couples $(\hat{x}, \hat{y})$ which are solutions for (2.1) and (2.2) does not necessarily solve $(\mathcal{P})$, they are "good candidates" for such a solution and will play an important role in the final algorithm. We call them "stationary couples".
It follows from [6] that the optimality of $\hat{x}$ for the linear fractional problem (2.1) is

equivalent to say that $\nabla_x F(\hat{x},\hat{y})\cdot(x-\hat{x})\geq 0$ for each $x\in P_1$ or using Kuhn and Tucker multipliers that it exists $\lambda\in\mathbb{R}_+^{m_1}$ such that

$$\nabla_x F(\hat{x},\hat{y}) + {}^tA_1\lambda = 0.$$

In a similar way, if $\hat{y}$ solves (2.2) we have $\nabla_y F(\hat{x},\hat{y})\cdot(y-\hat{y})\geq 0$ for each $y\in P_2$ which is equivalent to the existence of $\mu\in\mathbb{R}_+^{m_2}$ such that

$$\nabla_y F(\hat{x},\hat{y}) + {}^tA_2\mu = 0.$$

In order to get a stationary couple we propose the following algorithm which consists to apply several times Algorithm 1, fixing alternatively the variables $x$ and $y$.

**Algorithm 2**

**Step 0** : Let $x^0$ and $y^0$ be basic solutions for $P_1$ and $P_2$, the corresponding bases being denoted by $B^0$ and $B'^0$. By Algorithm 1 compute a solution $x^1$ of

$$(\mathcal{P}_{y=y^0}) \qquad v^1 = F(x^1,y^0) = \min_{x\in P_1} F(x,y^0)$$

Let $B^1$ be the basis corresponding to $x^1$ and set $k=1$.
**Step 1** : Using Algorithm 1 with the starting basis $B'^{k-1}$ compute $y^k$ a solution of

$$(\mathcal{P}_{x=x^k}) \qquad w^k = F(x^k,y^k) = \min_{y\in P_2} F(x^k,y)$$

If $w^k = v^k$, then $(x^k,y^{k-1})$ is a stationary couple and the algorithm stops. Otherwise denote by $B'^k$ the basis corresponding to $y^k$.
**Step 2** : Using again Algorithm 1, starting from $B^k$, compute $x^{k+1}$ a solution of

$$(\mathcal{P}_{y=y^k}) \qquad v^{k+1} = F(x^{k+1},y^k) = \min_{x\in P_1} F(x,y^k).$$

If $v^{k+1} = w^k$, then $(x^k,y^k)$ is a stationary couple and the algorithm stops. Otherwise denote by $B^{k+1}$ the basis corresponding to $x^{k+1}$, set $k=k+1$ and go to step 1.

**Theorem 3.1** *If $P_1$ and $P_2$ are bounded then Algorithm 2 gives a stationary couple after a finite number of iterations.*

**Proof** : From $v^k = F(x^k,y^{k-1})$ it follows that $w^k\leq v^k$ in step 1 and from $w^k = F(x^k,y^k)$ we have $v^{k+1}\leq w^k$ in step 2. Suppose that Algorithm 2 does not stop. Then we have infinitely many step 1 and step 2, which means that for each $k\geq 1$ we have $w^k < v^k$ and $v^{k+1} < w^k$. It follows that $v^{k+1} < v^k$. The contradiction comes from the fact that the number of vertices of $P_1$ and $P_2$ being finite, the number of possible values for $v^k$ is also finite. Now suppose for instance that Algorithm 2 stops with $w^k = v^k$ in step 1. We have

$$w^k = F(x^k,y^k) = v^k = F(x^k,y^{k-1})$$

and then $y^{k-1}$ is also a solution of $(\mathcal{P}_{x=x^k})$. Since $x^k$ is a solution of $(\mathcal{P}_{y=y^{k-1}})$ it follows that $(x^k,y^{k-1})$ is a stationary couple. In a similar way, if Algorithm 2 stops in step 2, then $(x^k,y^k)$ is a stationary couple.

# 4 Cutting planes

We suppose in this part that we have two polyhedrons $P'_1 = \{x \in \mathbb{R}^{n_1} : A'_1 x \leq b'_1, \, x \geq 0\} \subset P_1$, $P'_2 = \{y \in \mathbb{R}^{n_2} : A'_2 y \leq b'_2, \, y \geq 0\} \subset P_2$ and a current optimal value $v_{opt}$ such that

$$v_{opt} \leq \min_{x \in P_1 \setminus P'_1 \; y \in P_2} F(x, y) \tag{4.1}$$

and

$$v_{opt} \leq \min_{x \in P_1 \; y \in P_2 \setminus P'_2} F(x, y) \tag{4.2}$$

Note that if $P'_1 = \emptyset$ or $P'_2 = \emptyset$, the description above means that the global minimum value of $(\mathcal{P})$ is $v_{opt}$ . Further, it follows from (4.1) and (4.2) that if $(\bar{x}, \bar{y}) \in P_1 \times P_2$ is such that $F(\bar{x}, \bar{y}) < v_{opt}$ then necessarily $(\bar{x}, \bar{y}) \in P'_1 \times P'_2$. In what follows we shrink $P'_1$ and $P'_2$ while preserving (4.1) and (4.2), until one of them becomes empty. At the beginning of the process we set $P'_1 = P_1$, $P'_2 = P_2$ and $v_{opt} = +\infty$.

Consider the problem

$$(\mathcal{P}') \qquad \min_{x \in P'_1 \; y \in P'_2} F(x, y).$$

**Proposition 4.1** *If $(\bar{x}, \bar{y}) \in P'_1 \times P'_2$ is a local ( resp. global) minimum of (P') such that $v_{opt} = F(\bar{x}, \bar{y})$, then it is also a local ( resp. global) minimum of (P).*

**Proof** - Suppose that $(\bar{x}, \bar{y}) \in P'_1 \times P'_2$ is a local minimum of (P'), then it exists $V'_{\bar{x}}$ and $V'_{\bar{y}}$ two neigborhoods of $\bar{x}$ in $P'_1$ and $\bar{y}$ in $P'_2$ such that for all $(x, y) \in V'_{\bar{x}} \times V'_{\bar{y}}$, $F(x, y) \geq F(\bar{x}, \bar{y})$. The subset $V_{\bar{x}} = V'_{\bar{x}} \cup (P_1 \setminus P'_1)$ is a neigborhood of $\bar{x}$ in $P_1$ and $V_{\bar{y}} = V'_{\bar{y}} \cup (P_2 \setminus P'_2)$ is a neigborhood of $\bar{y}$ in $P_2$. Using (4.1) and (4.2), it follows that for all $(x, y) \in V_{\bar{x}} \times V_{\bar{y}}, F(x, y) \geq F(\bar{x}, \bar{y})$ and then $(\bar{x}, \bar{y})$ is a local minimum of $F$ on $P_1 \times P_2$
To get a similar result for a global minimum we replace $V'_{\bar{x}}$ and $V'_{\bar{y}}$ by $P'_1 \times P'_2$ in the previous lines.

The aim of the following is to get a local (or global) minimum for (P'). Applying Algorithm 2 to $(\mathcal{P}')$ we get a stationary couple $(\bar{x}, \bar{y}) \in P'_1 \times P'_2$ . From the definition of a stationary couple we have

$$\bar{v} = F(\bar{x}, \bar{y}) = \min_{y \in P'_2} F(\bar{x}, y) = \min_{x \in P'_1} F(x, \bar{y}).$$

If $\bar{v} < v_{opt}$, then we update $v_{opt}$ by setting $v_{opt} = \bar{v}$ and we set $(x_{opt}, y_{opt}) = (\bar{x}, \bar{y})$ , otherwise $v_{opt}, x_{opt}, y_{opt}$ are kept unchanged, thus we always have $F(\bar{x}, \bar{y}) \geq v_{opt}$. Now we explore the edges emanating from $\bar{x}$ in $P'_1$ and from $\bar{y}$ in $P_2$, to improve $F(\bar{x}, \bar{y})$ if possible. As usual we denote by $B_i, i = 1, 2$ two bases for the systems $A'_1 x \leq b'_1$, $x \geq 0$ and $A'_2 y \leq b'_2$, $y \geq 0$ corresponding to the basic solutions $\bar{x}$ and $\bar{y}$. The elements $x \in P'_1$ and $y \in P'_2$ are characterized by nonbasic components $x_{N_1}$ and $y_{N_2}$ by the formulae

$$(x_{B_1}, x_{N_1}) = (B_1^{-1} b'_1 - B_1^{-1} N_1 x_{N_1}, x_{N_1}) \tag{4.3}$$

and

$$(y_{B_2}, y_{N_2}) = (B_2^{-1} b'_2 - B_2^{-1} N_2 y_{N_2}, y_{N_2}) \tag{4.4}$$

In particular $(\bar{x}_{B_1}, \bar{x}_{N_1}) = (B_1^{-1}b'_1, 0)$ and $(\bar{y}_{B_2}, \bar{y}_{N_2}) = (B_2^{-1}b'_2, 0)$. We suppose that $B_1$ and $B_2$ are such that pivoting on $x_\ell$ (resp. $y_h$) yieds an adjacent vertex to $\bar{x} \in P'_1$ ( resp. $\bar{y} \in P'_2$) denoted by $\hat{x}(\ell)$ (resp. $\hat{y}(h)$). Using (4.3) and (4.4) there exists non negative reals $\hat{x}_\ell$ and $\hat{y}_h$, such that $\hat{x}(\ell)$ and $\hat{y}(h)$ are characterized by

$$\hat{x}(\ell)_{N_1} = (0, \cdots, 0, \hat{x}_\ell, 0, \cdots, 0) \in \mathbb{R}^{n_1} \tag{4.5}$$

and

$$\hat{y}(h)_{N_2} = (0, \cdots, 0, \hat{y}_h, 0, \cdots 0) \in \mathbb{R}^{n_2} \tag{4.6}$$

In the same way for every $x_\ell \in \mathbb{R}$ and every $y_h \in \mathbb{R}$ we denote by $x(\ell) \in \mathbb{R}^{n_1}$ and $y(h) \in \mathbb{R}^{n_2}$ the elements obtained from (4.3) and (4.4) by taking $x(\ell)_{N_1} = (0, \cdots, 0, x_\ell, 0, \cdots, 0)$ and $y(h)_{N_2} = (0, \cdots, 0, y_h, 0, \cdots 0)$. Note that if $x_\ell$, for example, is too large, then $x(\ell)$ does not belong necessarily to $P_1$ because $x(\ell)_{B_1}$ can be nonpositive.
Writing $F(x, y)$ as a function of the nonbasic variables $x_{N_1}$ and $y_{N_2}$ we get

$$F(x, y) = \frac{{}^t\bar{c}_1 x_{N_1} + {}^t\bar{c}_2 y_{N_2} + {}^t x_{N_1} \bar{C}\ y_{N_2} + \bar{s}}{{}^t\bar{d}_1 x_{N_1} + {}^t\bar{d}_2 y_{N_2} + {}^t x_{N_1} \bar{D}\ y_{N_2} + \bar{t}} \tag{4.7}$$

where $\bar{c}_1, \bar{d}_1 \in \mathbb{R}^{n_1},\ \bar{c}_2, \bar{d}_2 \in \mathbb{R}^{n_2},\ \bar{C}, \bar{D} \in \mathbb{R}^{n_1 \times n_2},\ \bar{s}, \bar{t} \in \mathbb{R}$. Obviously for $(\bar{x}, \bar{y})$ we have $\bar{x}_{N_1} = 0$ , $\bar{y}_{N_2} = 0$ and then $F(\bar{x}, \bar{y}) = \dfrac{\bar{s}}{\bar{t}}$. Taking $x = x(\ell)$ we get

$$F(x(\ell), y) = \frac{{}^t\bar{c}_{1\ell} x_\ell + {}^t\bar{c}_2 y_{N_2} + x_\ell \bar{C}_\ell\ y_{N_2} + \bar{s}}{{}^t\bar{d}_{1\ell} x_\ell + {}^t\bar{d}_2 y_{N_2} + x_\ell \bar{D}_\ell\ y_{N_2} + \bar{t}} \tag{4.8}$$

where $\bar{C}_\ell$ and $\bar{D}_\ell$ stand for the rows $\ell$ of $\bar{C}$ and $\bar{D}$. In the same way we have

$$F(x, y(h)) = \frac{{}^t\bar{c}_1 x_{N_1} + {}^t\bar{c}_{2h} y_h + x_{N_1} \bar{C}^h\ y_h + \bar{s}}{{}^t\bar{d}_1 x_{N_1} + {}^t\bar{d}_{2h} y_h + x_{N_1} \bar{D}^h\ y_h + \bar{t}} \tag{4.9}$$

where $\bar{C}^h$ and $\bar{D}^h$ stand for the columns $h$ of $\bar{C}$ and $\bar{D}$.
For convenience write denote $F(x_\ell, y)$ instead of $F(x(\ell), y)$ and $F(x, y_h)$ instead of $F(x, y(h))$. Recalling that $F(\bar{x}, \bar{y}) \geq v_{opt}$ the next result shows that if $\tilde{x}(\ell)$ is a point on an adjacent edge to $\bar{x}$ which does not improve $v_{opt}$ then the same fact holds for every point on this edge between $\bar{x}$ and $\tilde{x}(\ell)$.

**Proposition 4.2** *If* $\min_{y \in P'_2} F(\tilde{x}_\ell, y) \geq v_{opt}$, *then for all* $x_\ell \in [0, \tilde{x}_\ell]$, $\min_{y \in P'_2} F(x_\ell, y) \geq v_{opt}$.

**Proof** - For each $y \in P'_2$, $F(\cdot, y)$ is a linear fractional function. From [7], this property implies that for each fixed $y$, $F(\cdot, y)$ is quasi-concave as well as $\min_{y \in P'_2} F(\cdot, y)$. Since we have $x(\ell) = (1-\eta)\bar{x} + \eta\ \tilde{x}(\ell)$ with $\eta \in [0, 1]$, we get $\min_{y \in P'_2} F(x_\ell, y) \geq min\ [\min_{y \in P'_2} F(\bar{x}, y), \min_{y \in P'_2} F(\tilde{x}_\ell, y)]$. As $\min_{y \in P'_2} F(\bar{x}, y) = F(\bar{x}, \bar{y}) \geq v_{opt}$, and $\min_{y \in P'_2} F(\tilde{x}_\ell, y) \geq v_{opt}$ (by assumption) it follows that $\min_{y \in P'_2} F(x_\ell, y) \geq v_{opt}$.

Obviously the same result holds for adjacent edges to $\bar{y}$ in $P_2'$.
Now we consider the following problems :

$$(\mathcal{P}_\ell) \qquad \begin{array}{rcl} \theta_\ell & = & \max\ x_\ell \\ & & s.t.\ \min_{y \in P_2'} F(x_\ell, y) \geq v_{opt} \end{array}$$

and

$$(\mathcal{Q}_h) \qquad \begin{array}{rcl} \eta_h & = & \max\ y_h \\ & & s.t. \min_{x \in P_1'}\ F(x, y_h) \geq v_{opt} \end{array}$$

**Proposition 4.3** *Let $(\bar{x}, \bar{y})$ be a stationary couple. Either it exists an adjacent vertex $\hat{x}(\ell)$ to $\bar{x}$ in $P_1'$ and a vertex $\hat{y} \in P_2'$ such that $F(\hat{x}_\ell, \hat{y}) < v_{opt}$, or, for all $\ell$, $\theta_\ell \geq \hat{x}_\ell$.*

**Proof**-If for some adjacent vertex $\hat{x}(\ell)$, $\theta_\ell < \hat{x}_\ell$ then by definition of $(\mathcal{P}_\ell)$, $\min_{y \in P_2'}\ F(\hat{x}_\ell, y) < v_{opt}$. It follows that there exists a vertex $\hat{y} \in P_2'$ satisfying

$$F(\hat{x}_\ell, \hat{y}) < v_{opt}$$

this vertex $\hat{y}$ can be found using Algorithm 2.

It is clear that a similar procedure can be applied to examine adjacent vertices to $\bar{y}$. In that case we get the following result .

**Proposition 4.4** *Let $(\bar{x}, \bar{y})$ be a stationary couple. Either it exists an adjacent vertex $\hat{y}(h)$ to $\bar{y}$ in $P_2'$ and a vertex $\hat{x} \in P_1'$ such that $F(\hat{x}, \hat{y}_h) < v_{opt}$ or, for all $h$, $\eta_h \geq \hat{y}_h$.*

Based upon Propositions 4.3 and 4.4, we can derive the following algorithm which stops with a stationary couple $(\bar{x}, \bar{y})$ satisfying the following condition : For every adjacent vertex $\hat{x}(\ell)$ to $\bar{x}$ and every adjacent vertex $\hat{y}(h)$ to $\bar{y}$ we have $\theta_\ell \geq \hat{x}_\ell$, $\eta_h \geq \hat{y}_h$. Note that during this Algorithm the current values of $x_{opt}, y_{opt}$ and $v_{opt}$ can be updated.
The main difficulty in this algorithm is to solve $(\mathcal{P}_\ell)$ and $(\mathcal{Q}_h)$, this question will be discussed at the end of the section.

**Algorithm 3**

**step 0** : Let be given a stationary point $(x_k, y_k)$ in $P_1' \times P_2'$ and the current optimal value $v_{opt} \leq F(x_k, y_k)$. Recall that $P_1' \subset P_1$ and $P_2' \subset P_2$ are two polyhedrons satisfying

$$v_{opt} \leq \min_{x \in P_1 \setminus P_1', y \in P_2} F(x, y) \quad \text{and} \quad v_{opt} \leq \min_{x \in P_1, y \in P_2 \setminus P_2'} F(x, y).$$

**Step 1** : Denote by $L_1$ (resp $L_2$) the list of adjacent vertices to $x_k$ in $P_1'$ (resp. $y_k$ in $P_2'$).
If $L_1 = \emptyset$ go to step 2, else choose $\hat{x}(\ell) \in L_1$ and set $L_1 = L_1 \setminus \hat{x}(\ell)$.
Let $\theta_\ell$ be the value of $(\mathcal{P}_\ell)$.
If $\theta_\ell < \hat{x}_\ell$ then go to step 3, else go to step 1.
**Step 2** : If $L_2 = \emptyset$ stop. Else choose $\hat{y}(h) \in L_2$ and set $L_2 = L_2 \setminus \hat{y}(h)$.
Let $\eta_h$ be the value of $(\mathcal{Q}_h)$.
If $\eta_h < \hat{y}_h$ then go to step 4, else go to step 2.

**Step 3** : Using Proposition 4.3 find a vertex $\hat{y} \in P'_2$ such that $F(\hat{x}(\ell), \hat{y}) = \min_{y \in P_2} F(\hat{x}(\ell), y) < v_{opt}$. Starting from $(\hat{x}(\ell), \hat{y})$, apply Algorithm 2 to find a new stationary couple $(x_{k+1}, y_{k+1})$ and set $x_{opt} = x_{k+1}, y_{opt} = y_{k+1}, v_{opt} = F(x_{k+1}, y_{k+1})$.
Go to step 5
**Step 4** : Using Proposition 4.4 find a vertex $\hat{x} \in P'_1$ such that $F(\hat{x}, \hat{y}(h)) = \min_{x \in P_1} F(x, \hat{y}(h)) < v_{opt}$. Starting from $(\hat{x}, \hat{y}(h))$, apply Algorithm 2 to find a new stationary couple $(x_{k+1}, y_{k+1})$ and set $x_{opt} = x_{k+1}, y_{opt} = y_{k+1}, v_{opt} = F(x_{k+1}, y_{k+1})$.
**Step 5** : $k := k + 1$; go to step 1.

**Proposition 4.5** *Algorithm 3 stops after finitely many iterations with for all $\ell$, $\theta_\ell \geq \hat{x}_\ell$ and for all $h, \eta_h \geq \hat{y}_h$.*

**Proof** - Suppose, on the contrary, that this algorithm does not stop. Then it generates an infinite sequence $(x_k, y_k)$ of stationary couples either by step 3 or by step 4. If $(x_{k+1}, y_{k+1})$ is obtained from $(x_k, y_k)$ through step 3, we have $F(x_{k+1}, y_{k+1}) \leq F(\hat{x}(\ell), \hat{y})$ (by Algorithm 2) and $F(\hat{x}(\ell), \hat{y}) < v_{opt}$ by Proposition 4.3. As $v_{opt} \leq F(x_k, y_k)$ it follows that $F(x_{k+1}, y_{k+1}) < F(x_k, y_k)$. The same result holds if $(x_{k+1}, y_{k+1})$ is obtained from $(x_k, y_k)$ via step 4. Since $P'_1$ and $P'_2$ have a finite number of vertices we have a contradiction.

**Proposition 4.6** *If Algorithm 3 stops with a stationary couple $(\bar{x}, \bar{y})$ such that $v_{opt} = F(\bar{x}, \bar{y})$ then $(\bar{x}, \bar{y})$ is a local minimum of $(\mathcal{P})$.*

**Proof** - From Proposition 4.5 we have for all $\hat{x}(\ell) \in L_1$ and all $\hat{y}(h) \in L_2$, $\theta_\ell \geq \hat{x}_\ell$ and $\eta_h \geq \hat{y}_h$. It follows from Proposition 4.2 that for all $x_\ell \in [0, \theta_\ell]$ we have $\min_{y \in P'_2} F(x_\ell, y) \geq v_{opt}$ and from the formula (4.2), $\min_{y \in P_2} F(x_\ell, y) \geq v_{opt}$. If $V_{\bar{x}}$ denotes the convex hull of $\bar{x}$ and of all adjacent vertices $\hat{x}(\ell), \ell \in L_1$, we have from the quasi-concavity of $\min_{x \in P_2} F(., y)$ : for every $x \in V_{\bar{x}}$ and every $y \in P_2, \quad F(x, y) \geq v_{opt}$. Since $\bar{x}$ is a vertex of $P'_1$, $V_{\bar{x}}$ is a neighborhood of $\bar{x}$ in $P'_1$ and then $(\bar{x}, \bar{y})$ is a local minimum of $F$ on $P'_1 \times P_2$. It follows from Proposition 4.1 that it is also a local minimum for the problem $(\mathcal{P})$.

Now consider the problem $(\mathcal{P}_\ell)$ and observe that it can be solved using linear programming. Indeed with our notations, the condition $y \in P'_2$ can be written $B_2^{-1} N_2 \, y_{N_2} \leq B_2^{-1} b'_2$, $y_{N_2} \geq 0$ and using (4.8)we can reformulate $(\mathcal{P}_\ell)$ in the following way

$$(\mathcal{P}_\ell) \quad \begin{array}{l} \theta_\ell = \max x_\ell \\ \text{s.t.} \\ B_2^{-1} N_2 \, y_{N_2} \leq B_2^{-1} b'_2 \quad , \quad y_{N_2} \geq 0 \;\; \Rightarrow \;\; \dfrac{\bar{c_{1\ell}} x_\ell + ( \, {}^t\bar{c_2} + x_\ell \bar{C_\ell}) y_{N_2} + \bar{s}}{\bar{d_{1\ell}} x_\ell + ( \, {}^t\bar{d_2} + x_\ell \bar{D_\ell}) y_{N_2} + \bar{t}} \geq v_{opt} \end{array}$$

Equivalently we have

$$
(\mathcal{P}_\ell)\quad \begin{array}{l} \theta_\ell = \max x_\ell \\ \text{s.t.} \\ \min\{[\,{}^t\bar{c}_2 + x_\ell \bar{C}_\ell - v_{opt}(\,{}^t\bar{d}_2 + x_\ell \bar{D}_\ell)]y_{N_2} : B_2^{-1}N_2 y_{N_2} \le B_2^{-1}b_2',\ y_{N_2} \ge 0\} \\ \ge (v_{opt}\bar{d}_{1\ell} - \bar{c}_{1\ell})x_\ell + v_{opt}\bar{t} - \bar{s} \end{array}
$$

Taking the dual of the linear problem in the constraint we get

$$
(\mathcal{D}_\ell)\quad \begin{array}{l} \theta_\ell = \max x_\ell \\ \text{s.t.} \\ \max\{-\,{}^t u B_2^{-1} b_2' \ :\ -\,{}^t u B_2^{-1} N_2 \le \,{}^t\bar{c}_2 - v_{opt}\,{}^t\bar{d}_2 + x_\ell(\bar{C}_\ell - v_{opt}\bar{D}_\ell), u \ge 0\} \\ \ge (v_{opt}\bar{d}_{1\ell} - \bar{c}_{1\ell})x_\ell + v_{opt}\bar{t} - \bar{s} \end{array}
$$

or

$$
(\mathcal{D}_\ell)\quad \begin{array}{lcl} \theta_\ell = \max x_\ell & & \\ \text{s.t.} & & \\ {}^t u B_2^{-1} b_2' + x_\ell(v_{opt}\bar{d}_{1\ell} - \bar{c}_{1\ell}) & \le & \bar{s} - v_{opt}\bar{t} \\ -\,{}^t u B_2^{-1} N_2 + x_\ell(v_{opt}\bar{D}_\ell - \bar{C}_\ell) & \le & {}^t\bar{c}_2 - v_{opt}\,{}^t\bar{d}_2 \\ & u \ge 0 \qquad x_\ell \ge 0 & \end{array}
$$

Thus we observe that $(\mathcal{D}_\ell)$ can be solved by linear programming. Obviously the same transformations are also valid for the problem $(Q_h)$. Note that if $v_{opt} = \bar{v} = \dfrac{\bar{s}}{\bar{t}}$ then the right hand sides of the constraints in $(\mathcal{D}_\ell)$ are non negative. Indeed $\bar{c}_2 - v_{opt}\bar{d}_2$ represents the components of $[\nabla_y F(\bar{x}, \bar{y})]_{N_2}$ which are non negative since $\bar{y}$ is a minimal solution of $\min_{y \in P_2'} F(\bar{x}, y)$ .

Now using the real numbers $\theta_\ell$ and $\eta_h$ obtained in Propositions 4.3 and 4.4 we define two cutting planes for the polyhedrons $P_1'$ and $P_2'$ respectively. Denote by $\tilde{x}(\ell)$ the point in $\mathbb{R}^{n_1}$ such that $\tilde{x}(\ell)_{N_1} = (0, \cdots, 0, \theta_\ell, 0, \cdots, 0)$ and by $conv(\tilde{x}(\ell))_{\ell \in L_1}$ the convex hull of all $\tilde{x}(\ell)_{\ell \in L_1}$. As $\bar{x}$ is an extreme point of $P_1'$, $\bar{x} \notin conv(\tilde{x}(\ell))_{\ell \in L_1}$ and we can find an hyperplan which separates $conv(\tilde{x}(\ell))_{\ell \in L_1}$ from $\bar{x}$ of the form :

$$
(H)\ :\ \{x \in \mathbb{R}^{n_1}\ :\ \varphi(x) = \varphi_0\}. \tag{4.10}
$$

We suppose that for every $\ell$, $\varphi(\tilde{x}(\ell)) \ge \varphi_0$ and $\varphi(\bar{x}) < \varphi_0$. Denote by $H^-$ and $H^+$ the half spaces obtained when we replace "=" in (4.10) by "$\le$" and "$\ge$" respectively. In the case where $x$ is a non degenerate vertex, it admits exactly $n_1$ adjacent vertices and one can choose for $(H)$ the hyperplan generated by all $\tilde{x}(\ell)_{\ell \in L_1}$. If $\bar{x}$ is a degenerate vertex a method providing an hyperplan $(H)$ is the following. Firstly we find $\tilde{x} \in conv(\tilde{x}(\ell))_{\ell \in L_1}$ such that for every $\ell$, $(\tilde{x}(\ell) - \bar{x}) \cdot (\tilde{x} - \bar{x}) \ge 1$, for instance by linear programming. Then a convenient hyperplan is defined by

$$
(H)\quad :\quad \{x \in \mathbb{R}^{n_1} : \varphi(x) = (x - \bar{x}) \cdot (\tilde{x} - \bar{x}) = \varphi_0\}
$$

where $\varphi_0 = \min_{\ell \in L_1} (\tilde{x}(\ell) - \bar{x}) \cdot (\tilde{x} - \bar{x}) \ge 1$.

**Proposition 4.7** *For every $x \in P_1' \cap H^-$ and every $y \in P_2$ we have $F(x,y) \geq v_{opt}$.*

**Proof** - For each $\ell \in L_1$, it follows from the definition of $\theta_\ell$ that $\min_{y \in P_2'} F(\tilde{x}(\ell), y) \geq v_{opt}$, and from (4.2) we deduce that $\min_{y \in P_2} F(\tilde{x}(\ell), y) \geq v_{opt}$. Now every $x \in P_1' \cap H^-$ can be written $x = \lambda \bar{x} + (1-\lambda)x_0$ with $\lambda \in [0,1]$ and $x_0 \in conv(\tilde{x}(\ell))_{\ell \in L_1}$. As $\min_{y \in P_2} F(\cdot, y)$ is a quasi-concave function it follows that

$$\min_{y \in P_2} F(x_0, y) \geq \min_{\ell \in L_1} \min_{y \in P_2} F(\tilde{x}(\ell), y) \geq v_{opt}.$$

On the other hand we have $\min_{y \in P_2'} F(\bar{x}, y) = \bar{v} \geq v_{opt}$ and from (4.2), $\min_{y \in P_2} F(\bar{x}, y) = \bar{v} \geq v_{opt}$ Thus, using again the quasi-concavity of $\min_{y \in P_2} F(\cdot, y)$ we have $\min_{y \in P_2} F(x, y) \geq v_{opt}$.

The main consequence of Proposition 4.7 is that $H = \{x \in \mathbb{R}^{n_1} : \varphi(x) \geq \varphi_0\}$ is a valid cut for $P_1'$ since it remove from $P_1'$ only elements which are not candidates to improve the current optimal value $v_{opt}$. Thus if there exists $x^* \in P_1'$ such that $\min_{y \in P_2} F(x^*, y) = v^* < v_{opt}$ then necessarily $x^* \in P_1' \cap H^+$. In other words if $P_1' \cap H^+ = \emptyset$ then $v_{opt}$ is the global optimal value of $(\mathcal{P})$.
In order to test if $P_1' \cap H^+ = \emptyset$, we solve the linear problem

$$\max\{\varphi(x) \ : \ x \in P_1'\} \tag{4.11}$$

Denote by $x_1 \in P_1'$ a solution of (4.11) and $\varphi_1 = \varphi(x_1)$.

**Proposition 4.8** *If the linear problem (4.11) satisfies $\varphi_1 \leq \varphi_0$ then $v_{opt}$ is the global optimal value of $\mathcal{P}$.*

**Proof** - It is clear that $\varphi_1 \leq \varphi_0$ implies $P_1' \cap H^+ = \emptyset$ and then it follows from the previous remark that $v_{opt}$ is optimal.

In the case where $\varphi_1 > \varphi_0$, the hyperplan (H) is a new cut for $P_1'$, which restricts the seach domain, then a new iteration occurs starting with the point $x_1$, solution of (4.11). Naturally the method presented here to generate cuts for the polyhedron $P_1'$ can be applied in the same manner to reduce the polyhedron $P_2'$.
Although in most examples the search domain becomes empty after some iterations, giving a global minimum, theoretically this result is not always true. When the algorithm does not give a global minimum we get a sequence of local minima for which the values of $F$ are decreasing. The complete algorithm for the problem $(\mathcal{P})$ is the following.

**Algorithm 4**

**Step 0** : Let $x_0$ and $y_0$ be vertices of $P_1$ and $P_2$ respectively. Set $x_{opt} = x_0$, $y_{opt} = y_0$, $v_{opt} = F(x_0, y_0)$, $P_1^0 = P_1, P_2^0 = P_2$ and $k = 0$.
**Step 1** : By Algorithm 2 starting from $(x_0, y_0)$ find a stationary point $(x_k, y_k)$ with respect to $P_1^k \times P_2^k$.
If $F(x_k, y_k) < v_{opt}$ then update $x_{opt} = x_k$, $y_{opt} = y_k$, $v_{opt} = F(x_k, y_k)$.

**Step 2** : Apply Algorithm 3 starting from $(x_k, y_k) \in P_1^k \times P_2^k$.
This step terminates with a stationary point $(x_{k+1}, y_{k+1}) \in P_1^k \times P_2^k$ (possibly equal to $(x_k, y_k)$) such that for each adjacent vertex $(\hat{x}(\ell)) \in L_1$ to $x_{k+1}$ in $P_1^k$ and each adjacent vertex $(\hat{y}(h)) \in L_2$ to $y_{k+1}$ in $P_2^k$ we have two positive numbers $(\theta_\ell)$ and $\eta_h$ satisfying $\theta_\ell \geq \hat{x}_\ell$ and $\eta_h \geq \hat{y}_h$.
**Step 3** : Find the equations of two cutting planes $(Hi)$ for $P_i^k, i = 1, 2$ and define for each $i$, $P_i^{k+1} = P_i^k \cap H_i^+$ (use one of the methods presented previously depending on the degeneracy of $x_{k+1}$ and $y_{k+1}$).
**Step 4** : If $P_1^{k+1} = \emptyset$ or $P_2^{k+1} = \emptyset$ then it follows from Proposition 4.8 that $(x_{opt}, y_{opt})$ is a global optimal solution and the algorithm stops. Otherwise let $x_0$ and $y_0$ be vertices of $P_1^{k+1}$ and $P_2^{k+1}$ respectively, set $k := k + 1$ and go to step 1.

Note that in Algorithm 4, each new cutting plane introduces new vertices for $P_1^k$ and $P_2^k$ which are not necessarily vertices of the initial polyhedrons $P_1$ and $P_2$. However, it follows from the definition of the cutting planes that for each of these new vertices denoted $x$ or $y$, we have $\min_{y \in P_2} F(x, y) \geq v_{opt}$ and $\min_{x \in P_1} F(x, y) \geq v_{opt}$. Therefore, these new vertices do not pass the test $F(x_k, y_k) < v_{opt}$ before each updating of $x_{opt}$ and $y_{opt}$. Thus $x_{opt}, y_{opt}$ are always vertices of $P_1$ and $P_2$.
We can obtain a simplified version of Algorithm 4 by adding cuts only to one polyhedron, for instance $P_1$, while the other one remains constant. In that case the algorithm stops when $P_1^k$ becomes empty. This simplified version is illustrated by the following example.

# 5 An example

We consider the following problem, where $x \in RR^2$ and $y \in RR^2$ :

$$\min \frac{(1, -2)x + (-10, 0)y + \ {}^t x \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix} y}{{}^t x \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} y + 70} = f(x, y)$$

$$s.t. \quad \begin{bmatrix} 1 & 3 \\ 2 & 3 \\ 2 & 1 \end{bmatrix} x \leq \begin{bmatrix} 9 \\ 12 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} y \leq \begin{bmatrix} 10 \\ 6 \\ 10 \end{bmatrix}$$

$$x \geq 0, y \geq 0.$$

To solve this problem we use the simplified version of Algorithm 4, reducing only the polyhedron $P_1^0$.(Figures 1 and 2). Setting $x'_0 = (0, 0)$ and $y'_0 = (0, 0)$ in step 0, we apply Algorithm 2 in step 1. We obtain $x_1 = (0, 3)$ as a solution of $(\mathcal{P}_{y=y_0})$ in step 0, then we get $y_2 = (2, 4)$ as a solution of $(\mathcal{P}_{x=x_1})$. As $x_1$ is a solution of $(\mathcal{P}_{y=y_2})$ we have got the first stationary couple $(x_1, y_2)$ with $v_{opt} = f(x_1, y_2) = -0.7308$.
In the second step we apply Algorithm 3 to examine adjacent edges to $x_1$. We calculate $\tilde{x_1}^1 = (0, -0.2655)$ and $\tilde{x_1}^2 = (7.9, 0.36)$ such that for all $(x, y) \in conv(x_1, \tilde{x_1}^1, \tilde{x_1}^2)$ we have $f(x, y) \geq v_{opt}$. In step 3, $P_1^0$ is reduced to $P_1^1$ by adding the cut $[\tilde{x_1}^1, \tilde{x_1}^2]$ and the vertex $x_4 = (5, 0)$ of $P_1^1$ is found. We return to step 1 which produces the new stationary

couple $(x_4, y_4)$ with $y_4 = (5, 0)$. Then step 2 gives the points $\tilde{x_4}^1 = (-3.3846, 0)$ and $\tilde{x_4}^2 = (2.5685, 4.8603)$. After a new reduction of $P_1^1$ we get an empty polyhedron proving that $(x_4, y_4)$ is a global minimum.

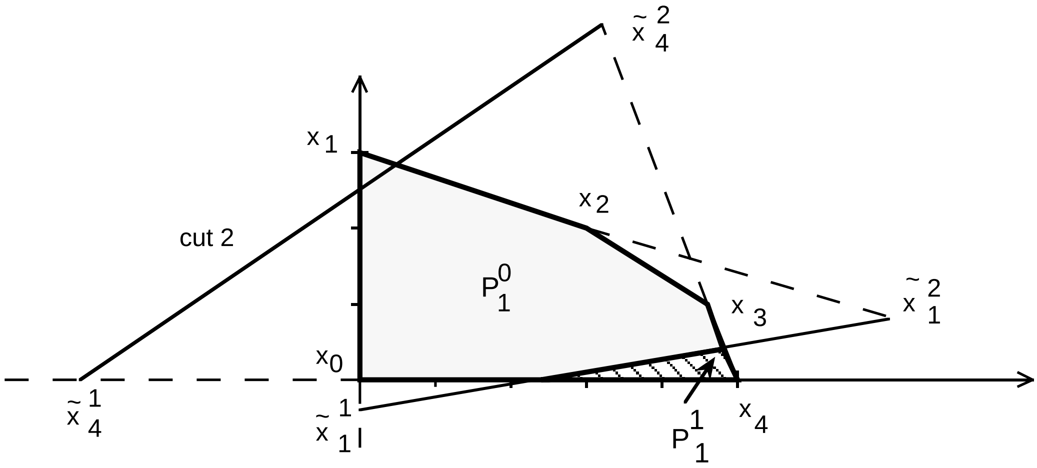

Figure 1 : Constraint polytope for x

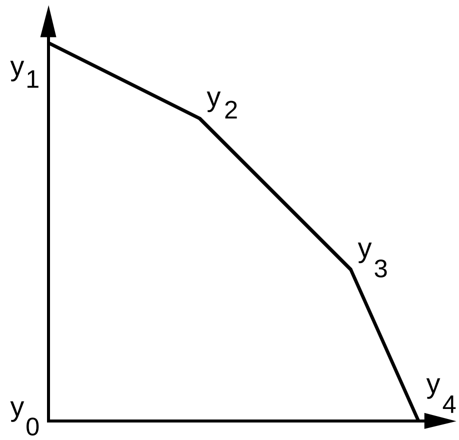

Figure 2 : Constraint polytope for y

# 6 Numerical applications

The algorithm 4 (called BLFP) has been coded in MATLAB and tested for various examples. Problems 1 and 2 are taken from [4] and problem 3 from [3]. The other examples have been randomly generated. For all these examples the optimal values given by BLFP are global minima. These results are compared with the local solutions given by the procedure CONSTR in the OPTIMIZATION TOOBOX of MATLAB. We observe that for several examples the global solution obtained with BLFP is strictly better that the local one given by CONSTR with the same starting point. Obviously this fact is not surprising since CONSTR is aimed to solve general differentiable constrained problems while BLFP adresses only bilinear fractional problems with linear constraints.

| Examples | Size $A_1$ , $A_2$ | Method | value | CPUTIME (sec) |
|---|---|---|---|---|
| Konno 1 | $2\ \times 3, 2\times 3$ | BLFP | -13 | 0,44 |
| | | CONSTR | -10 | 0,88 |
| Konno 2 | $6 \times 6, 6 \times 6$ | BLFP | -24,5 | 17,85 |
| | | CONSTR | 0 | 0,22 |
| Gallo | $2 \times 3, 2 \times 3$ | BLFP | -18 | 1,15 |
| | | CONSTR | -18 | 0,99 |
| Ex 4 | $12 \times 10$ , $12 \times 10$ | BLFP | -3.55 | 7.47 |
| | | CONSTR | -3.55 | 38.28 |
| Ex 5 | $20 \times 15$ , $20 \times 15$ | BLFP | -2.29 | 12.75 |
| | | CONSTR | -2 | 89.74 |
| Ex 6 | $20 \times 20$ , $20 \times 20$ | BLFP | -3.27 | 12.78 |
| | | CONSTR | -3.27 | 103.59 |

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